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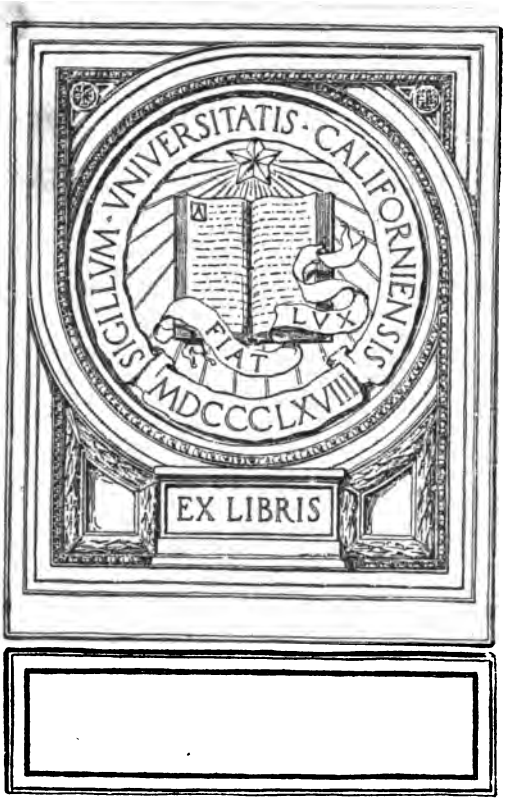
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PREFACE.

THE Authors have endeavoured, in this book, to present in a direct and simple manner the essentials of Practical Geometry and Graphics, and, as far as space has permitted, to apply the principles to the solution of problems constantly met with in practical work.

The ground covered is such as to comprise a two or three years' course of instruction according to the capability of the student.

In designing the book, two objects have been kept in view: (1) to meet the requirements of engineers, architects, surveyors, draughtsmen, and artisans engaged in the constructive arts as a work to guide them in the study of such Geometry and Graphics as they will require in following their profession or craft; and (2) to meet the requirements of those students in technical or other schools who desire to prepare for such examinations as the Board of Education "Lower Stage" examination in Practical Geometry and Graphics.

The course outlined in the syllabus for the "Lower Stage" examination of the Board of Education in Practical Geometry and Graphics comprises nearly all that is necessary for the majority of those engaged in Engineering and Building; hence, in this book, a large number of examples and exercises have been selected from Board of Education examination papers. Many of these examples will be found not only interesting tests of the knowledge possessed by the student, but excellent examples of the practical application of geometrical principles. It will be found that the book covers fully the syllabus above referred to, and in some sections is in excess of the requirements of students preparing for this examination. The note facing p. 1 of the text should therefore be consulted by such students as to which articles they may omit as regards the examination.

The book is divided into three sections: (1) Plane Geometry;

(2) Graphics ; (3) Descriptive Geometry. The first section will be found to include a good deal of Mensuration and calculation relating thereto; an extensive chapter on Loci and Loci Methods of solving problems ; and also a chapter on Conics much in excess of what is usually found in books dealing with practical geometry. In the section on Graphics it will be found that the subject has been developed at greater length than is usual, in order that the practical utility of such methods of solution of everyday problems may be emphasised. Special features of this part of the work are the very extensive chapter dealing with the use of Squared Paper, and the chapter dealing with Velocity and Acceleration Diagrams, Simple Harmonic Motion, and Valve Diagrams. A chapter dealing generally with Vectors is followed by one in which the methods are applied to typical braced structures such as roof trusses, girders, and suspension bridges. The diagrams must be taken as illustrative only of the text, the student being expected, in each case, to reproduce the diagram for himself.

Throughout the book, special care has been taken to present fundamental principles in such a form that a student, when away from his instructor, or who has to work unaided, may obtain such help as he will require.

The Authors cordially thank Mr. F. P. Johns, who has kindly read the MSS. and proofs, and has made many valuable suggestions and criticisms ; also the publisher, who has rendered every possible assistance in the preparation of the book.

E. L. B.
F. C.

LONDON,
September, 1912.

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BOARD OF EDUCATION "LOWER STAGE" EXAMINATION.

The student reading for the above examination may omit the following Articles: 170 to 172; 197 to 204; 259 to 265; 297 to 299; 542 to end of Chapter XXXIII.

NOTE.

The following contractions are sometimes used :—

$\perp$ = perpendicular.	$\angle$ = angle.
$\triangle$ = triangle.	$\parallel$ = parallel to.
$\therefore$ = therefore.	$\because$ = because.
$>$ = greater than.	$<$ = less than.

Exercises marked BE are taken from the Board of Education Examination Papers.

PRACTICAL GEOMETRY AND GRAPHICS.

CHAPTER I.

SOME PRELIMINARY NOTIONS AND DEFINITIONS OF TERMS USED IN GEOMETRY.

1. **Geometry** is the science which accurately proposes and demonstrates the art of measuring. The application of this science to the mechanical art of representing geometric forms in drawings which supply precise information on all points relating to magnitude and position is called **Practical Geometry.**

2. A **geometrical point** has position but not magnitude. In making drawings, it is best represented by two fine lines drawn across each other, thus X.*

A **geometrical line** has length but not breadth. *All lines should, therefore, be as thin as is consistent with distinctness.*

Lines may be either straight or curved.

Intersecting lines are lines which cross each other.

The **plane** is a perfectly straight or flat surface and one on which a straight-edge will lie evenly, no matter how placed.

Plane curves are curves which lie wholly in one plane, such, for example, as any curve traced on a perfectly true drawing-board. See *a*, *b*, *c*, Fig. 1.

3. A **circle** is a plane figure bounded by a curved line called the **circumference**, every point of which is equally distant from a point within called the **centre.** The circumference is sometimes referred to as the circle when no reference is being made to the space enclosed.

A **radius** is a straight line drawn from the centre to the circumference.

*In the diagrams illustrating this book, many particular points are represented by small circles or round dots. This device is employed to make such points conspicuous and more easily located by the reader.

A diameter is a straight line drawn through the centre, and terminated both ways by the circumference.

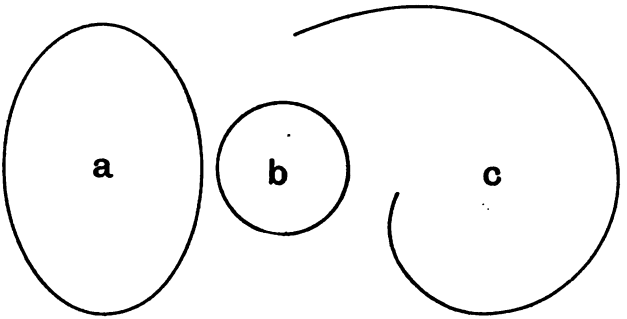


FIG. 1.

Thus, in Fig. 2, O is the centre of the circle; OA, OB, OC, are radii; AOB is a diameter. From the definition of a circle, it is

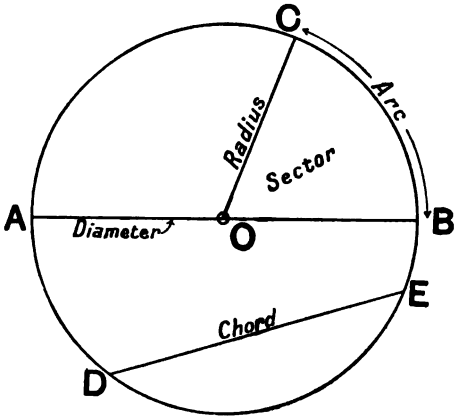


FIG. 2

evident that *all the radii of the same circle are equal;* and *all diameters are equal, and each double the radius.*

An **arc** of a circle is any part of the circumference, as BC.

A **chord** is the straight line which joins any two points in the circumference, as DE. A chord divides a circle into two parts, each of which is called a **segment.**

A **sector** of a circle is the space enclosed by an arc and two radii drawn from the centre of the circle to the extremities of the arc.

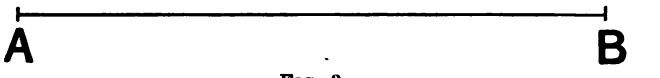


FIG. 3.

4. **Locus.** Every line may be considered as the path traced by a moving point. Thus, the line AB, Fig. 3, traced by a pencil point moving from A to B, marks every position occupied by the point, and is called a locus ; the moving point is said to **generate** its locus.

Constant Direction. If a point situate at A, Fig. 3, move in a

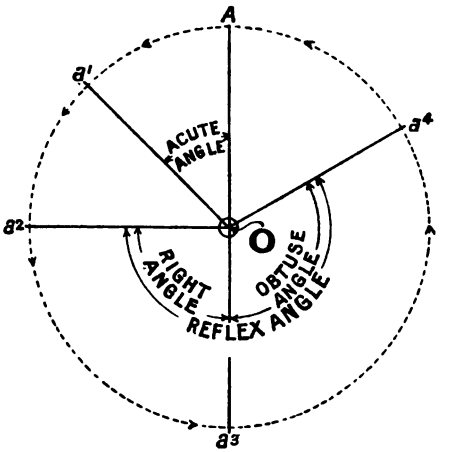


FIG. 4.

straight line to a new position B, it moves all the time in the same direction, viz., along the straight line AB towards B ; and thus the direction of motion of the point is constant. Hence, *a straight line is the locus of a point the direction of which is constant.*

5. **Angles.** Suppose a straight wire OA, Fig. 4, secured to a drawing-board at O by a pin joint so that it can rotate about the

point O. Let OA represent the initial position and direction of the wire. When the wire has turned into the position Oa^1, its direction is Oa^1.

The line Oa^1 is said to be *inclined at an angle* to OA, and the inclination of one line (Oa^1) to the other (OA) is called, in general terms, the *angle* a^1OA. The two lines which form an angle are called the *arms* of the angle, and the point in which the two arms meet is called the *angular point*, or *vertex*.

It must be clearly understood that an angle denotes the inclination, or slope, of one line to another line, but has nothing to do with the length of any line. The size of an angle depends solely on the difference in the directions of its arms.

If the rotating wire makes a complete turn, the point A will describe a circle and return to the starting point. When it has moved through half a complete turn into the position Oa^2, then a^2O, OA are in the same straight line, and divide the circle traced by A into two equal parts. When the wire has made one-quarter of a turn, taking up the position Oa^2, it makes equal angles with OA and Oa^2, in which case the lines are said to be **square** or **perpendicular** to each other, and each of the equal angles is called a **right-angle.**

If the rotating wire OA be supposed to describe a complete turn in 360 equal stages, the circle will be divided into 360 equal parts, and if lines were drawn marking these different positions the angle between adjacent lines would be constant and equal to one-three hundred and sixtieth part of a complete turn. One-three hundred and sixtieth part of a complete turn is called a **degree.** The degree is symbolized by a small circle placed as an index, thus °.

Since there are 360° in a complete turn, it follows that a right-angle contains 90°; one-half of a right-angle, 45°; one-third of a right-angle, 30°; and two-thirds of a right-angle, 60°.

Mathematicians divide the degree into sixty equal parts called **minutes**, symbolized by one dash, thus ′; and the minute is again subdivided into sixty equal parts called **seconds** symbolized by two dashes, thus ″. In Practical Geometry and Practical Mathematics, it is better to divide the degree into tenths and hundredths. Angles of 90°, 60°, 45°, and 30°, are extensively used, and these are the angles to which ordinary set-squares are made.

An **Acute Angle** is an angle less than a right-angle.

An **Obtuse Angle** is an angle greater than a right-angle.

A **Reflex Angle** is an angle greater than two right-angles; it is sometimes called a **re-entrant angle.**

The **complement** of an angle is the difference between the angle and a right-angle.

The **supplement** of an angle is the difference between the angle and two right-angles.

A straight line which intersects two or more lines and makes a right-angle with each is called a **common perpendicular.**

6. **Parallel lines** are lines which have the same direction.

Parallel straight lines. If from two points A, B, Fig. 5, in a

straight line there are drawn straight lines AM, BN, each making the same angle with AB, these two lines have the same direction and are, therefore, parallel. Any two or more parallel lines, whether straight or curved, possess two characteristic properties, viz. :—

(*a*) *A straight line perpendicular to a given line is perpendicular to all lines which are parallel to the given line.*

(*b*) *The distance apart of two parallel lines is constant throughout their entire length,* such distance being always measured along common perpendiculars.

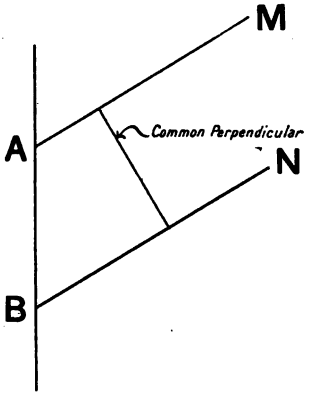


FIG. 5.

7. Drawing-Instruments.

For the immediate wants of the student, the following list of drawing-instruments will suffice: Drawing-board; drawing-paper; tracing-paper; drawing-pins; pencils; T-square; set-squares, one having angles of 45°, and one with angles of 60° and 30°; protractor; clinograph; dividers, provided with pencil point; pricker; scale rule.

8. **Drawing-Boards** are made of pine boards, usually about $\frac{3}{4}$ in. thick and in narrow widths, closely jointed, and secured by screws to two wood battens or ledges. In well-made drawing-boards, the screws pass through slotted metal cups inserted in the ledges, a device designed to give some freedom to the screws and allow them to slide, thus preventing splitting and warping of the board. A straight

narrow slip of hard wood is inset in one edge of the board (see Fig. 6) to serve as a guide against which the T-square is placed when in use. It projects slightly beyond the end of the board to facilitate the moving of the T-square. Drawing-boards are made in standard sizes. "*The Imperial*," 32 in. by 23 in.; and "*Half Imperial*," 23 in. by 16 in., are convenient sizes.

Drawing-Paper. Cartridge paper, with a smooth surface and of medium substance, is suitable for all geometrical drawing in pencil.

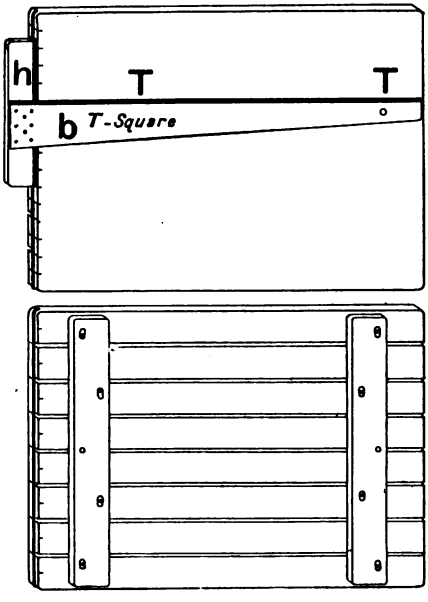


FIG. 6.

Drawing-Pins are used to fasten the paper on to the drawing-board. They should have smooth and thin, slightly rounded heads, so as to offer a minimum obstruction to the T-square and other straight-edges.

9. **Pencils.** Two pencils are required for drawing, one having harder lead than the other. The two degrees of hardness known as "HH" and "H" will be found suitable. Softer pencils are neither necessary nor desirable. A fine point on the pencil is absolutely essential. To obtain this, the wood should be cut away with a sharp

chisel or knife, well back from the lead, as shown in Fig. 7, and the lead pointed by rubbing on a fine flat file. Sandpaper is sometimes used, but a fine file is cleaner, less destructive of the lead, and gives a finer point. Views of what are called the *chisel point* and the *conical point* are shown at *a* and *b*. Both forms have their uses in graphical work. The chisel point is stronger and more durable than

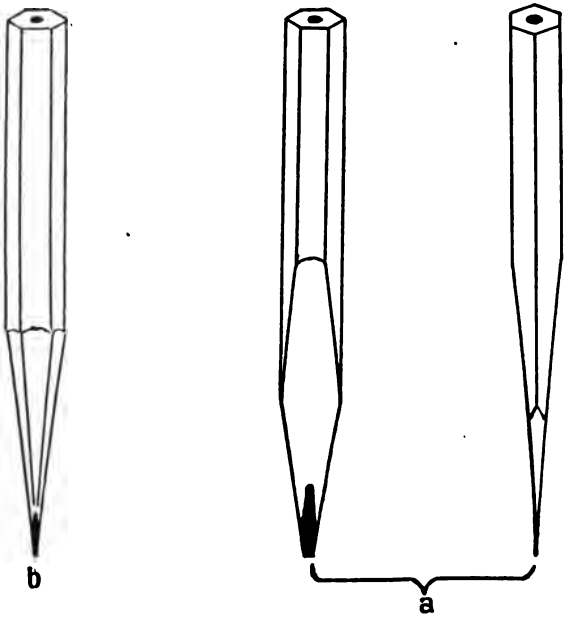


FIG. 7.

the fine conical point, and is well adapted for drawing long straight lines. There is, however, some difficulty in locating the chisel point with exactitude. This difficulty is greatest with beginners, who also find it difficult to prevent a slight rotation of the pencil, resulting in a faulty line. The conical point will be found to be much more conducive to accuracy in marking points and in making constructions. The student is advised to practice drawing with both forms, for only with experience will come that definite knowledge of what one can do, and cannot do, with certain tools.

10. **The T-square** is usually made of mahogany or pear wood,

the former being used for the best instruments. The working edges are frequently of ebony. The long thin portion *b*, Fig. 6, is called the *blade;* the shorter and thicker piece *h* is called the *haft*. The blade is fixed on the top surface of the haft, so that the latter shall

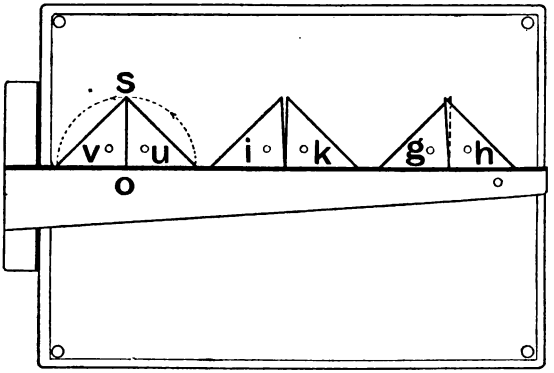


FIG. 8.

be level with the surface of the board, and not interfere with the sliding of set-squares beyond the edge of the board. When in use, the haft is placed in contact with the working edge of the board, as shown in the diagram, and the edge T T of the blade is used as a

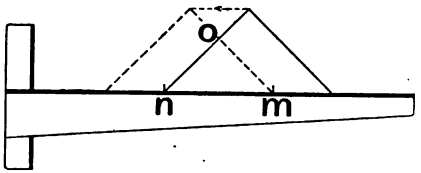


FIG. 9.

straight-edge in drawing straight parallel lines across the board in any position.

11. **Set-squares** are best made of homogeneous material, like vulcanite or celluloid, which does not readily change shape. Two set-squares are necessary, one having angles of 45°, the other angles of 60° and 30°. Set-squares are used as straight-edges to guide the pencil when drawing straight lines in certain definite positions relative to other lines. The fixed angles between the edges of the

squares enable certain lines and points to be determined mechanically and thus render unnecessary some tedious geometrical constructions.

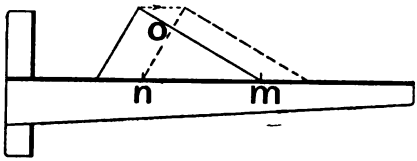


FIG. 9A.

It is, therefore, important that all the edges of the set-square be straight, and the angles correct. The following tests, if carefully applied, will suffice to detect material errors in the form of set-squares.

12. **To test a straight-edge.** Place the tool on the drawing-board and draw a fine pencil line along the edge being tested. Then turn over the tool, bringing the bottom surface to the top, and place the ends of the edge exactly to the line already drawn. If the edge, after turning, wholly coincides with the line, it is straight.

To test the angles. Consider first the right-angle, Fig. 8. With the T-square in position on the board, place the set-square firmly against the working edge of the T-square as at *u*, and draw a line up the edge OS. Then rotate the square in the direction of the arrow so as to bring the edge OS into contact with the T-square, as shown at *v*. If the right-angle is correct, the line OS will coincide with the set-square in its new position. Should the square in the two positions appear as at *i*, *k*, the supposed right-angle is less than 90°; and if as at *g*, *h*, the angle is greater than 90°.

The Angle 45°. Place the set-square on the board with its long side against the T-square in two positions, as shown in Fig. 9, and draw lines up the edges of the set-square in the two positions. If the angle 45° be correct, the lines *on*, *om*, are equal in length.

Angles of 60° and 30°. If these angles are correct, the shortest edge of the set-square is exactly one-half of the longest edge. Place the long edge against the T-square, as in Fig. 9A, and by marking its outline in two positions, determine points *m*, *n*, *o*. If the angles are correct, $nm = 2no$. If *on* is less than $\frac{1}{2}nm$ the supposed angle 60° is too large and the angle 30° is too small, and conversely.

FIG. 10.

13. **The Clinograph** is a set-square with a swivelling arm attached. The moving arm is secured by a friction joint which holds the arm at any desired angle. The edges of the clinograph must be quite straight, and the right-angle ought also to be correct. The

clinograph is used for drawing parallel straight lines in certain positions. See Fig. 21.

14. **Dividers and Compasses.** The former are used for setting off chords on circular and other curves. The latter are used to draw or describe circles. Compasses should be fitted with knee joints, so that the leg points can always be brought into position approximately perpendicular to the drawing-paper. For small and accurate drawings, small instruments fitted with springs and adjusting-screws are employed. These are called *spring bows*.

The pricker, Fig. 10, is a needle inserted in a penholder. It is useful when making a drawing from a tracing, or when taking points off a drawing on to tracing-paper.

15. **The protractor** is a scale of degrees, and is described fully in Art. 55. The best protractors are made of ivory or metal; cheaper qualities are made of celluloid or wood. Low-priced protractors are often inaccurate and worthless.

16. **Scales.** The scales which are necessary for the student are enumerated on p. 40. All these scales are engraved on one rule. If the scale rule is of wood, it should be shaped so as to make the faces on which the scales are marked meet in a line arris or edge. All scales on the rule should be set out along the line edge above referred to, and there ought not to be more than two scales engraved along the same edge on one surface, one scale reading in one direction, and the other in the opposite direction. Some rules are covered with different scales; such rules only mystify the user, and are of no use whatever to the draughtsman.

CHAPTER II.

SOME FUNDAMENTAL CONSTRUCTIONS AND SIMPLE PROBLEMS.

In mathematical drawing, all lines must be drawn as narrow as possible, consistent with distinctness, and must be uniform throughout their entire length.

17. **Problem.** *Fig.* 11. *To draw a straight line through two given points* A *and* B.

Place a finely-sharpened pencil point exactly on the point A, and place a set-square, or other suitable straight-edge, against the pencil. Then rotate the straight-edge until it

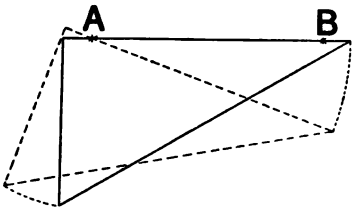


Fig. 11.

is, as near as the eye can determine, the same distance from the point B as from A. Before making any mark with the pencil, run it along the straight-edge and see that its point *does* pass through the position B. Then press the straight-edge on to the drawing-board with the fingers of the left hand so as to secure it against movement, and draw a uniformly fine line from A to B.

18. *Circles* are drawn, or described, with compasses. In Fig. 12 is shown the proper way to hold the compasses when describing a circle. It will be noticed that the head

of the instrument is held between the thumb and fingers, and that the legs are in a plane perpendicular to the drawing-paper. The pressure exerted on the instrument should be just sufficient to maintain the compasses in the correct position. Care must be taken not to close the legs together during the drawing of the circle, nor to force the needle point of the compasses through the drawing-paper.

Problem. *To describe a circle of given radius*—1·6 *in., and to have its centre in a given point* O. (*No Figure.*)

Draw a straight line through O, and make OA = 1·6 in.

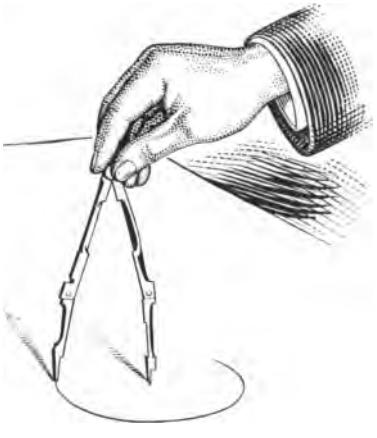


FIG. 12.

(measure OA direct from a scale). Insert the needle point of the compasses at O, and open them out sufficient to bring the leg which holds the pencil into a position about perpendicular to the drawing-board, and make the pencil point coincide with A. Gently rotate the compasses through one revolution, the pencil will then trace out the required circle.

19. **Measuring straight lines.** In measuring straight lines, always apply the scale (*i.e.*, measure) direct to the line, taking care that one end of the line coincides with one

of the unit division marks on the scale. If the line is not exactly measured by inches and tenths of an inch (which may be the smallest graduation on the scale), fractions of the smallest graduation can be measured visually. After a little practice, such fractions as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, etc., of a $\frac{1}{10}$-in. division are easily estimated at sight. In very important work, fractions of a definite length on a scale can be measured with a *vernier*, which is fully described in Art. 51.

Measuring curved lines. The lengths of circular arcs and many other curves may be determined arithmetically by formulæ, and graphically by constructions deduced from certain properties which the curves are known to possess. The draughtsman, however, has frequently to

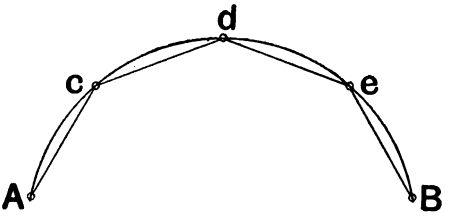


FIG. 13.

measure curves about which he knows nothing beyond what the curves reveal, and often a tedious analysis leads no nearer to accuracy than can be attained much more quickly by approximations which are easily understood without a knowledge of pure mathematics.

20. **Practical method of measuring any plane curve.** Consider the circular arc AB, Fig. 13. Mark a number of points *c*, *d*, *e*, on the arc and join them up by the chords A*c*, *cd*, *de*, *e*B. The length of the curve is greater than the combined lengths of the chords. If more points in the curve are taken, the chords will be more numerous and will approach more nearly to the curve, and if it were possible to take points in the curve indefinitely near to each other, the chain of chords would become indefinitely near to the curve; in fact, the two would coincide. Now similar

reasoning may be applied to the irregular curve AB, Fig. 14, or to any curve. It is possible, therefore, to obtain a near approximation to the exact length of any curve by finding the length of a chain of chords taken of such lengths as the curvature of the line and the importance of exactitude render necessary.

Ex. Measure the length of the irregular curve AB, Fig. 14.

Draw a straight line on a strip of stout tracing-paper. Mark a point *a* on the line and place the tracing-paper over the curve AB with the point *a* exactly over the end A. Insert the pricker at A, and rotate the tracing-paper so as

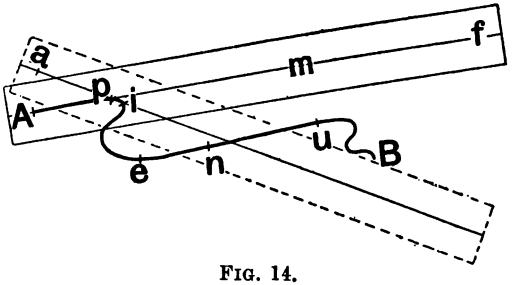


FIG. 14.

to bring a portion of the straight line from *a* into the closest possible agreement with the curve at that end. Now hold the tracing-paper quite still and transfer the pricker to a point *p*—which marks the end of the segment from *A* in agreement with the straight line—and repeat the operation of turning the tracing-paper so as to bring a further portion of the straight line into agreement with the curve at *pi*. Then transfer the pricker to *i*, and continue until the whole line has been measured.

The length of the curve is then shown at A*f*. This method, if applied with care and judgment, will give very good results. In selecting each step along the curve, attention must be paid to the curvature of the line at the particular part being measured. Where the curvature is sharp it is necessary to take short steps; but where the

curve is flat, as between A*p*, *eu*, much longer steps can be taken without increasing the error to an appreciable extent.

Exercise.—Describe a circle, radius 1 in., and measure its circumference by the method just explained.

The length of this circumference is known to be $6\frac{2}{7}$ in.

How near to this is the result you obtain?

21. **Limit of error in graphical work.** An accomplished draughtsman can draw on a suitable white paper a black line which does not exceed one-four hundredth of an inch in width. Narrower lines are not distinctly visible to persons of ordinary eyesight, and it is only by careful manipulation of first-class drawing-instruments that such a near approach to accuracy can be obtained.

Experience has shown that draughtsmen can, with instruments of ordinary quality and the exercise of reasonable care, make drawings of simple geometrical figures which do not differ from the correct forms by more than *one-two hundredth of an inch.* Nearer than that only the expert can go. The "limit of error" in graphical work is now usually taken as **one-two hundredth part of an inch.**

22. **Vertical lines and horizontal lines.** Fig. 15 is a sketch showing three strings, with weights attached, suspended from points in a *flat level* ceiling. We know from experience that these strings hang in vertical lines, and all have the same direction. Hence, *all vertical lines are parallel straight lines.*

A perfectly level line is called a *horizontal line.* The sketch, Fig. 15, shows two parallel straight lines, two concentric circles, and a sinuous curve, all of which lie on the surface of the flat level ceiling. We gather from this diagram that horizontal lines may be either straight or curved, and, further, may be parallel or non-parallel.

In plane geometry, however, we are not much concerned with horizontal lines other than parallel *straight* lines. To facilitate the drawing of such lines, we usually draw them parallel to the base of the drawing-board. Thus we have only to place the T-square in its normal position with the haft against the working edge of the board, Fig. 16, and draw a pencil along the working edge of the T-square. By sliding the T-square up or down as required, parallel lines

may be accurately and quickly drawn to pass through any given points.

Vertical lines are drawn by standing a set-square on the T-square, as shown in Fig. 16.

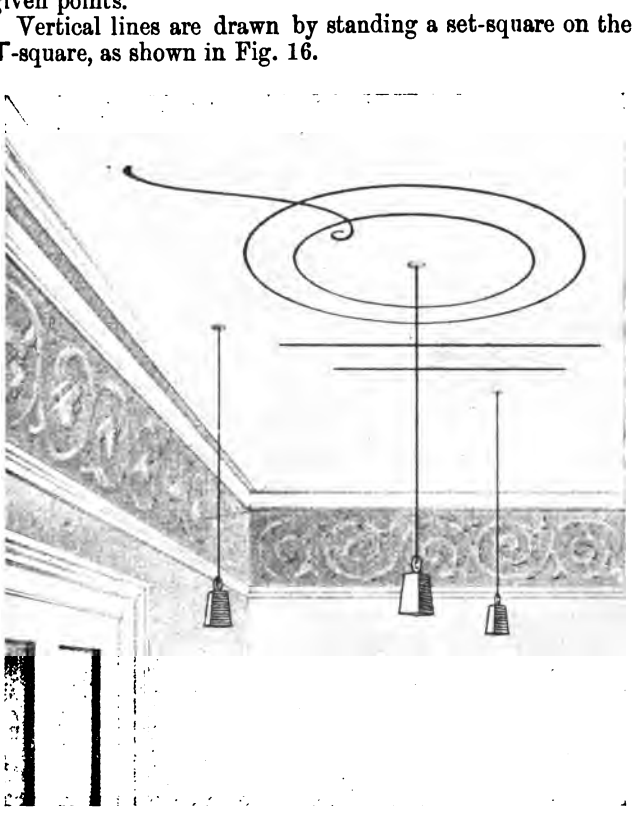


FIG. 15.

23. **Experiment.** Fig. 17. Using the T-square, draw two parallel lines *mm*, *nn*, and draw any straight line LL across the parallels. Make a tracing of the lines *mm*, and LL, and by rotating this tracing, first about the point P, and secondly about the point Q, show that the angles which

are marked by the same Greek letters are equal. In this

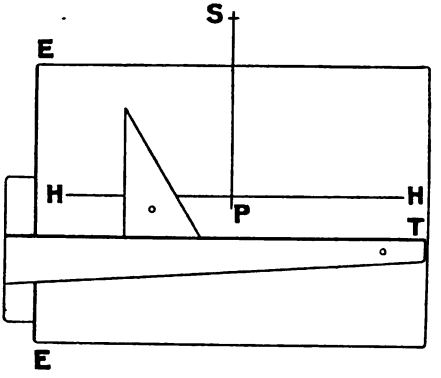


FIG. 16.

way we obtain, by the method of superposition, a proof of the following:—

Theorem. *If a straight line crosses two parallel straight*

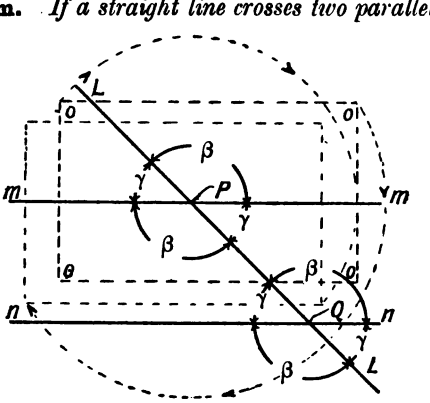


FIG. 17.

lines, it makes (1) *the alternate angles equal;* (2) *the exterior angle equal to the interior opposite angle on the same side of the line.*

24. **Experiment.** Fig. 18. Place a 60° set-square with its long side against the T-square, as at *enm*, and draw lines along the edges *em* and *mn*. Move the set-square along the T-square to a new position *e'm'n'*. Then since *en* and *e'n'* have the same direction, and the set-square remains unaltered in form, therefore *e'm'* is parallel to *em*, and *m'n'* is parallel to *mn*. Hence the following :—

Theorem. *If any figure move in a plane so that one line in it has always the same direction, then all lines of the figure remain parallel to their original directions.*

25. **Experiment.** Fig. 19. Draw a straight line MN, and place a set-square on the line, as at *abc*. Make a tracing of the set-square and the line MN in that position. Rotate the tracing through any angle about any point P,

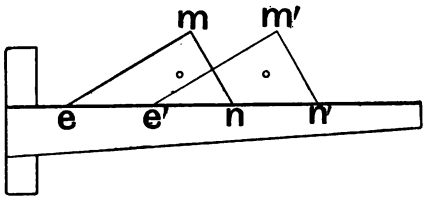


FIG. 18.

into a new position as *a'b'c'*. Prick through the corners *a'*, *b'*, *c'*, and join them up, making a drawing of the set-square in the new position. Make a tracing of the angle between MN and *a'b'* and apply this tracing over all the angles marked ϕ. Observe that all these angles are equal to one another.

From this experiment we conclude that the theorem in Art. 24 may be extended as follows :—

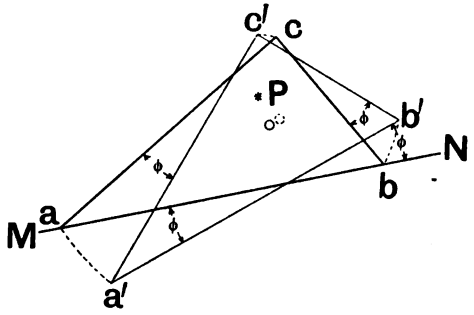
Theorem. *If any figure be moved in its plane so that a line in it makes an angle ϕ with its original direction, then all lines in the figure also make the angle ϕ with their original directions.*

26. The theorem of Art. 25 enables us to use set-squares and the clinograph to determine lines parallel and perpendicular to a given straight line.

Problem. Fig. 20. *Through a given point* P *draw* (1) *a parallel to the given straight line* AB; (2) *a perpendicular to* AB.

(1.) Place two set-squares together, as shown at *a* and *b*, making an edge adjacent to the right angle in one square

FIG. 19.



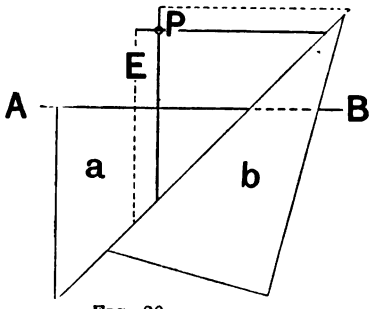


FIG. 20.

coincide with the given line AB. Hold *b* in position and move *a* along the common edge until the edge originally along AB passes through the position P. This edge is now parallel to AB, and the required line can be drawn.

(2.) To obtain the perpendicular to AB, it is only necessary to slide the movable square (*a*) further along until the edge

E passes through P (for this edge is always perpendicular to AB). Proceed in precisely the same way when the given point P is in the given line AB.

27. Lines parallel to a given straight line, as *dd*, Fig. 21, may be drawn with a clinograph, thus: place the T-square

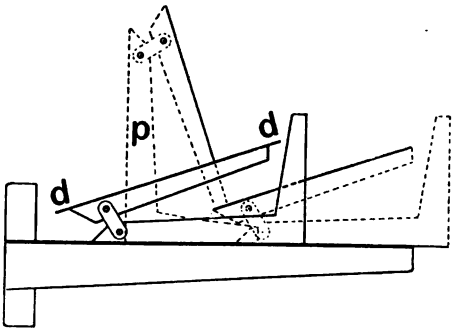


FIG. 21.

in its normal position on the drawing-board, stand the clinograph against the T-square and adjust the movable arm, making it coincide with the given line. This arm will then serve for drawing lines parallel to *dd* in any position by simply sliding the tool along the T-square.

To draw lines perpendicular to *dd*, turn the clinograph through 90°, as shown at *p*.

28. **Experiment.** No Figure. Describe a circle, and draw a diameter AB.

On a piece of tracing-paper draw two straight lines O*a*, O*b*, perpendicular to each other. Place this tracing over the circle with the meeting point (O) of the two lines on the circumference of the circle. Insert a pricker at the point O, and rotate the tracing until the line O*a* passes through A. Then observe that O*b* passes through the opposite end B of the diameter AB. Repeat the experiment, placing O in different positions on the circumference, and observe that when O*a* passes through A, then O*b* passes through B.

In this way we verify the following :—

Theorem. *All angles in a semicircle are equal to one another, and to a right-angle.*

This principle is the basis of a construction employed to set out a right-angle.

29. **Problem.** Fig. 22. *To draw a perpendicular to a given line and to pass through a fixed point.*

(1.) When the fixed point is on the given line.

Let AB be the given line, and B the given point. Assume any convenient point C, not on the line, as centre, and with radius CB describe an arc meeting AB again in F. Draw FC meeting the arc in S. Join SB which is the required perpendicular.

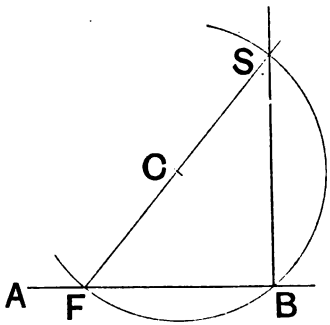


FIG. 22.

(2.) When the fixed point is not on the given line.

Let S be the point and AB the given line. Draw a line from S, as SF, and bisect SF in C. With C as centre, and radius CF describe a semicircle cutting AB in B. Join SB which is the required perpendicular.

30. **Experiment.** Fig. 23. Draw a straight line ER (use the T-square in drawing ER). Place the 45° set-square against the T-square and draw from points E, R lines which intersect in *i*. Next, use the 60° set-square, and draw from the same points (E, R) lines which intersect in pairs at *o* and *v*. Observe that the points *o*, *v* lie on the

line through *i* perpendicular to ER. This perpendicular intersects ER in *n*. Measure E*n*, *n*R and verify that E*n* = *n*R.

Take the compasses, and with any point *p* in *on* as centre, describe a circle of radius *p*E. This circle passes through R. Repeat the test by describing a circle having *q* for centre, and radius *q*E. Hence,

The perpendicular bisector of a straight line is the locus of points equidistant from the ends of the line, and conversely.

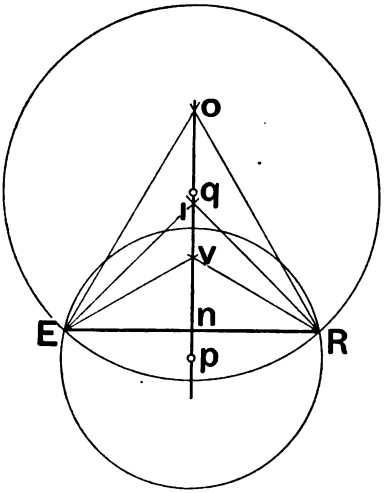


FIG. 23.

31. **Experiment.** No Figure. Describe any circular arc AB ; draw the chord AB and find the perpendicular bisector of this chord as explained in the last experiment. This bisecting line intersects the arc in a point M. Verify by measurements that the chords MA, MB are equal, and also that the arcs MA, MB are equal.

32. **Problem.** Fig. 24. *To bisect a given straight line* CD.

(1.) **With a set-square.** Place a straight-edge parallel to CD, and with the 45° set-square draw lines through C

and D meeting in Z. Draw ZN perpendicular to CD. Then, N is the point of bisection.

(2.) **Intersecting circles.** With C as centre and any radius greater than half CD describe an arc *m*. With D as centre and the *same radius* describe an arc *s*. These two arcs meet in O, O′. Join OO′, cutting CD in N, which is the required point of bisection.

If the radius of the circles *m*, *s* is less than half CD, the circles will not intersect ; if exactly half CD, they will touch each other at the point of bisection N. Any radius greater

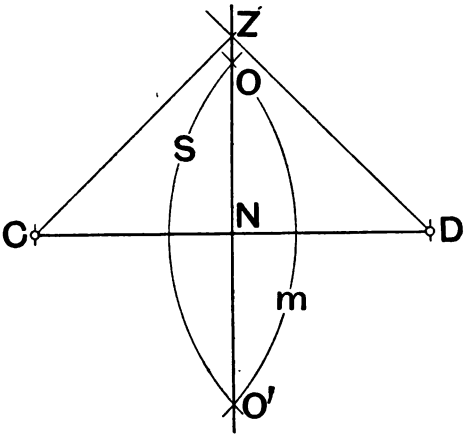


FIG. 24.

than half CD gives two points of intersection, but the radius should be sufficiently long to enable the points of intersection to be precisely located.

33. **The straight line and circle divided by trial.** The only satisfactory way of dividing a circular arc into several equal parts is by the method of repeated trials.

Practical example. A brick arch consisting of fifteen equal voussoirs is shown in Fig. 25. Suppose that we are given the outline *rwxs* of this arch, and are required to set out the given number of voussoirs all exactly the same size.

First, we divide the circular arc *rs* into the given number

of equal parts—in this case 15. As 15 is rather a large number, we take factors 3 and 5 ; and first divide the arc into three equal parts, as follows :—

Open the dividers to one-third of *rs*, guessing this distance, and step them around the curve, as at

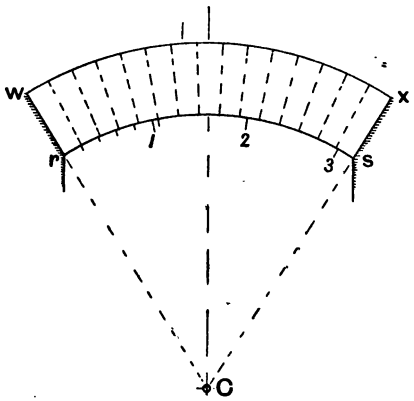


FIG. 25.

1, 2, 3. Suppose our guess to have been inaccurate, and that three steps either go beyond or fall short of the point *s*. In the figure they fall short by the distance 3*s*. Now, since we have three steps, we must add to each one-third of the distance 3*s*. Increase the distance

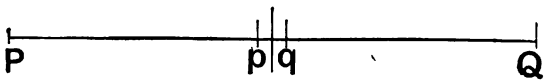


FIG. 26.

between the points of the dividers by one-third of the distance 3*s*, and step over the course again. If necessary, repeat the operation until the desired result is obtained. After a little practice, it should not be necessary to make more than three trials. Now divide each one-third of *rs* into five parts in precisely the same way. Having accurately

divided up the arc *rs*, draw lines through the points of division radiating from *C*.

34. Many draughtsmen employ the visual method to such problems as bisecting straight lines. If the line is long, cut off from each end a substantial segment; the remainder is then much more accurately divided by sight, *e.g.*, in bisecting the straight line PQ, Fig. 26, mark off from the two ends equal lengths P*p*, Q*q*. The middle point of *pq* is then easily guessed.

Another way to divide a straight line is to apply the scale rule directly to the line, and divide the length by 2, *e.g.*,

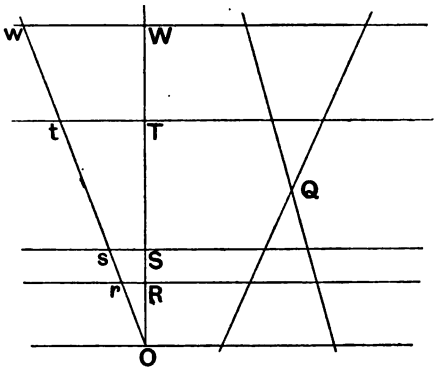


FIG. 27.

if the line measures 12¼ in., the point 6⅛ in. from either end is the central point on the line.

35. Two or more parallel straight lines are sometimes called a **system of parallels.** Any straight line which intersects a system of parallels is called a **transversal.**

Experiment. Fig. 27. Draw any straight line through a point O, and mark off along same, OR = 0·5 in.; RS = 0·25 in.; ST = 1·0 in.; TW = 0·75 in. Draw through these points a series of lines parallel to each other and inclined at any convenient angle to OW.

From O draw any straight line as O*w* across the parallels, intersecting same in *r*, *s*, *t*, *w*.

Then the triangles OR*r*, OS*s*, OT*t*, OW*w* are equiangular. They are also similar. See Art. 80.

i.e., Or = 2rs = $\frac{1}{2}st$ = $\frac{2}{3}tw$,
or, Or : rs : st : tw = OR : RS : ST : TW,
also, Rr : Ss : Tt : Ww = OR : OS : OT : OW.

Verify the above statements by measurements.

Mark any other point as Q, from which draw two or more straight lines across the system of parallels. And verify that the parallels divide all transversals in the ratio, OR : RS : ST : TW.

This principle is extensively used in the solution of problems.

36. **Problem.** Fig. 28. *Divide the given straight line* CD *into a number—say five—of equal parts.*

Through the ends C, D of the line draw parallels Co, Dr (making use of the most convenient angle on your set-squares). Place the zero (0) point of a scale on one of the parallels (Co in the diagrams) and turn the scale until the

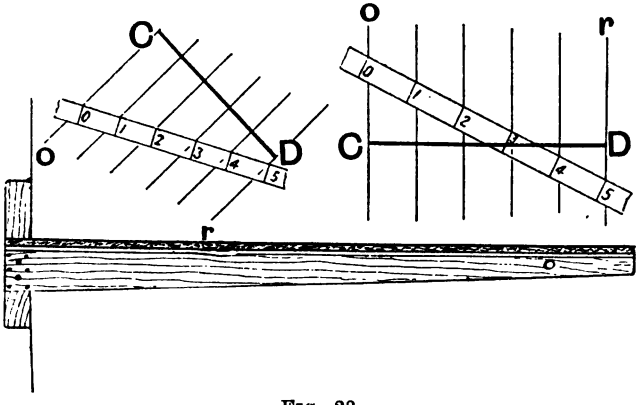


FIG. 28.

unit division which corresponds to the given number of parts required—5 in this case—falls on the other parallel. Mark the positions of the division marks 1, 2, 3, 4 direct from the scale, either with a pencil or pricker. Then draw through these points parallels to Co, Dr, which determine by their intersection with CD the required points of division.

Ex. *To divide a given straight line* AB *into three parts at* C *and* D *such that*

$$AC : CD : DB = 3 : 5 : 7.$$

From A, Fig. 29, draw any convenient line AF, and mark off along AF 15 equal parts (always take the sum of the given numbers in this case $3 + 5 + 7 = 15$). Join B – 15, and draw ‖s through the 3rd and 8th divisions (counting from A) intersecting AB in C and D.

Then the given line is divided as required at C and D.

For, evidently AC : CD : DB = 3 : 5 : 7.

Exercise. Determine a given fractional length, say $\frac{4}{5}$, of a given straight line AB. (Divide AB at D such that AD : AB = 5 : 8.)

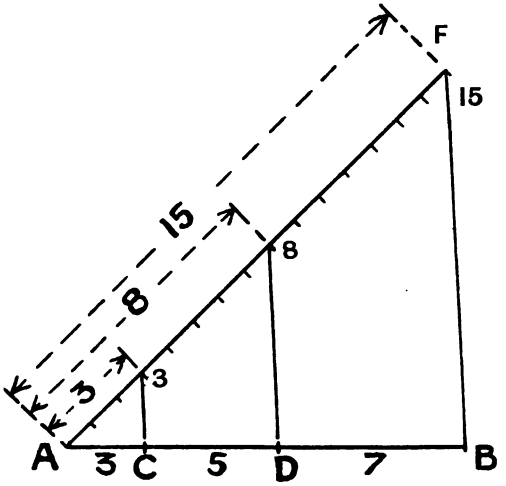


FIG. 29.

Exercises:

(1) Given a line AB 1·75 in. long; produce this line to a point D so that AB is $\frac{4}{5}$ of AD.

(2) Divide a line AB 2·5 in. long at C so that AB = 1·4 of AC.

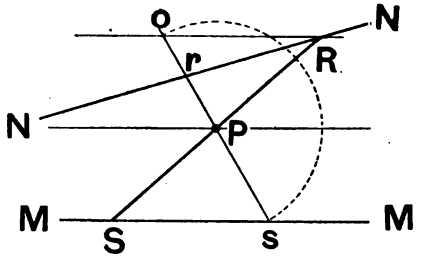
(3) Find a point C in a line AB 3 in. long such that AC = $\frac{2}{7}$ AB.

(4) Divide a line 5 in. long into 10 equal parts.

37. **Problem.** Fig. 30. *Given two straight lines* MM, NN, *and a point* P *not on either line, it is required to draw a straight line through* P *to intersect the given lines* M *and* N *in points* S *and* R *such that* SP = PR.

Draw any line through P intersecting the given lines in *s* and *r*. Make P*o* = *s*P, and draw through *o* a parallel to

FIG. 30.



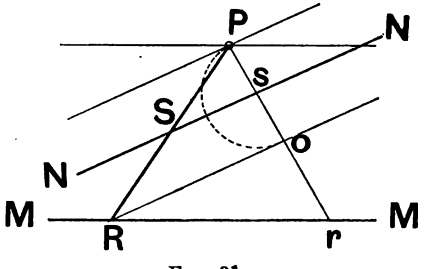


FIG. 31.

MM intersecting NN in R. Draw RP and produce to intersect MM in S. Then RS is the required line, for RS crosses the equidistant parallels drawn through R, S, P.

Fig. 31 shows the construction in a case in which P is not between the given lines. In this case PS = SR, *i.e.*, PS = $\frac{1}{2}$PR.

Exercise.—Solve the above problem when SP : PR = 2·5.

38. **Experiment.** Fig. 32. Draw two straight lines, F*n*, *fm* intersecting in *c*. With centre *c* and any radius describe an arc *ab*. Draw ND, the perpendicular bisector of the chord *ab*. Then ND passes through *c* (Art. 30), and since arc *ao* = arc *bo*, therefore N*c* is equally inclined to *fc* and F*c*, and bisects the angle *fc*F.

From N and some other point as *e* in N*e* draw perpendiculars meeting F*c*, *fc* in F, *f*, and *v*, *r*. By measuring these lines, verify that FN = N*f* and *ve* = *er*. From any point in *oc* draw ⊥rs on to F*c*, *fc* and by measuring these distances verify the

Theorem. *The locus of points equidistant from two straight lines is the bisector of the angle between the two lines.*

39. **Problem.** Fig. 33. *To bisect the angle between two straight lines mc, nc.*

With centre *c* and any radius describe an arc crossing the given lines in *a* and *b*. With *a* and *b*, as centres, and any convenient radius describe arcs which intersect in *o*. Draw *oc*, which is the required bisector.

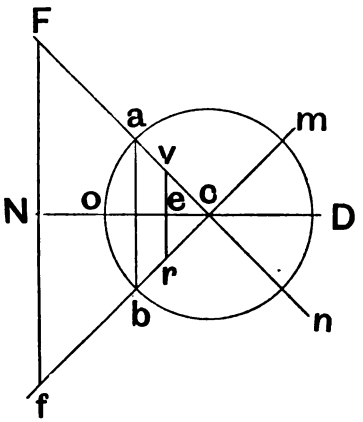


FIG. 32.

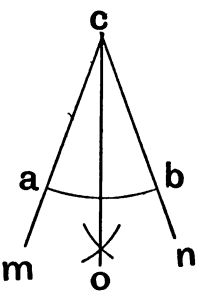


FIG. 33.

When the intersection of the given lines is inaccessible a modification of the construction is necessary. Let *mm*, *nn*, Fig. 34, be the given lines. The point in which they would intersect if produced does not come within the limit of the drawing.

Draw any transversal EF, and bisect the angles EFR and FEQ: these two bisectors intersect in *a*. Next bisect the angles EF*r* and FE*q*: these two bisectors meet in *b*. Draw a straight line through *a* and *b*, which is the required bisector.

Dem.: Denote the distances of the points *a* and *b* from the line *mm* by x and y respectively; their distances from the line EF by x_1 and y_1; and their distance from the line *nn* by x_2 and y_2 respectively.
Then,

$x_1 = x_2$ (for *a* lies on the bisector of angle EFR);
$x_1 = x$ („ „ „ FEQ);
$\therefore x = x_2$.
Again, $y_1 = y_2$ (for *b* lies on the bisector of angle EF*r*);
$y_1 = y$ („ „ „ FE*q*);
$\therefore y = y_2$.
And *ab* is the bisector of the angle between *mm* and *nn*.

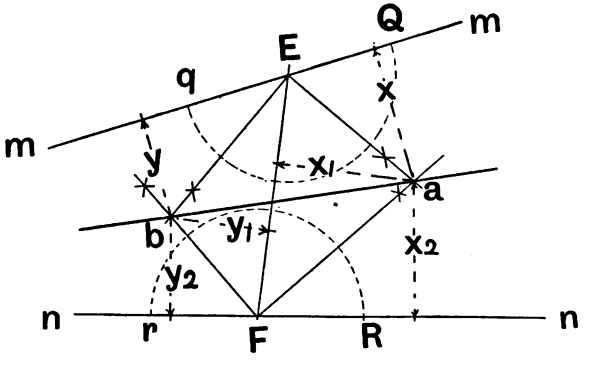


FIG. 34.

Note. If *mm* and *nn* are parallel, then *ab* is also parallel.

40. **Problem.** *Figs.* 35 *and* 36. *Given two lines mm, nn intersecting at an inaccessible point. To draw through a fixed point* P *a line making equal angles with the two given lines.*

Consider first two lines *mc*, *nc*, Fig. 35, which meet at an accessible point *c*.

Let *m*P*n* be equally inclined to both arms of the angle *mcn*, and let *c*A bisect the angle. Then *cm* = *cn*, and *mn* is perpendicular to *c*A (Art. 30).

Let the triangle *cmn* rotate about the point *c* through 90° into position *cm'n'*. Then, angle *n'cn* = angle *m'cm* = angle A'*c*A = 90°; and *mn* is parallel to *c*A'. But

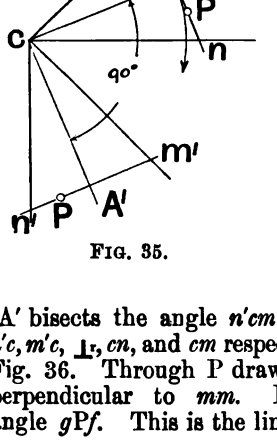


FIG. 35.

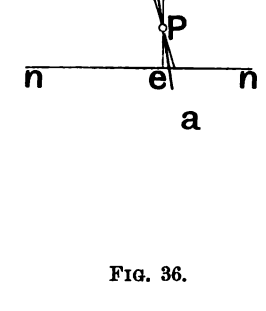


FIG. 36.

*c*A' bisects the angle *n'cm'*, *i.e.*, the angle formed by lines *n'c*, *m'c*, ⊥r, *cn*, and *cm* respectively. Hence the construction, Fig. 36. Through P draw *ef* perpendicular to *nn*, and P*g* perpendicular to *mm*. Draw *ab*, the bisector of the angle *g*P*f*. This is the line required.

Practical application. Bricks for segmental arches are cut to a wooden templet, which is applied to the face of the brick. The two sides of the templet converge to a point usually a long way beyond the templet. Fig. 37 is a

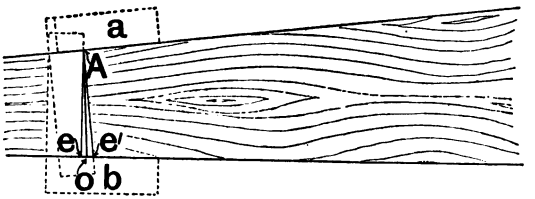


FIG. 37.

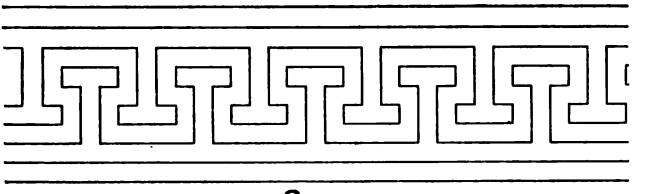
sketch of such a templet. On one edge is marked a point A, which indicates the position of one of the lower corners of the brick. The end of the brick containing the corner A must be square with the centre line of the templet, *i.e.*, be equally inclined to the two sides. This position is obtained thus: Apply a square to one side of the templet as shown at *a*, then transfer the square to the other side as at *b*. Determine A*o*, the bisector of the angle *e*A*e'*. A bevel applied to one side of the templet and set to the line A*o*, will fit the line when reversed and applied to the templet on the other side.

Exercises.

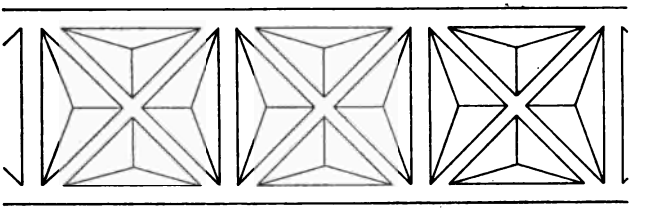
(1) Make a tracing of (or by pricking through transfer) the lines *mm*, *nn*, Fig. 34. Then (*a*) draw a line EF 1 in. long having its ends in, and equally inclined to, *mm*, *nn*. (*b*) Draw a line SP 1·5 in. long parallel to EF and also having its ends in *mm*, *nn*. Then by measurement, or by any other means, verify that the angle SEF is equal to the angle (ESP + the angle between *mm*, *nn*).

(2) From any point O not lying between *mm*, *nn*, draw OP intersecting *mm*, *nn* in points SP such that SP is a minimum.

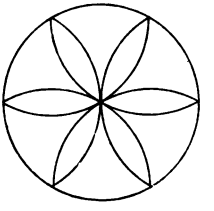
(3) Reproduce the patterns *a*, *b*, *c*, *x*, *y*, *z*, making your drawings three times as large as those in the book. (Linear measure.)



a



b



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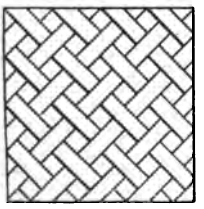
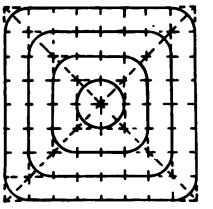
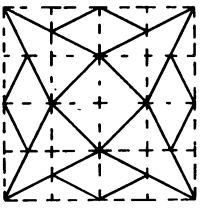


x



y



z

CHAPTER III.

MEASUREMENT OF LENGTH; SCALES.

41. In order to measure the length of any line, straight or curved, we must first select some *definite length* as a *standard* to which we can compare the line we wish to measure. Any definite length can be used as a standard of measure, and such length, whether it be comparatively great or small, must of necessity, so long as it is being used as a standard of measure, be *unity* (or *one*).

When we have a standard of **linear measure**, we can apply it to any line, and see whether such line is less than our measure, or, if greater, how many times our measure is contained in the line; we can express by whole or fractional *numbers* the length of the line in terms of the standard unit.

42. The standard unit of linear measure in Britain is the *yard*. The *yard* is divided into three equal parts, each of which is called a *foot*. The *foot* is divided into twelve equal parts called *inches*. The inch may be subdivided into any number of equal parts. Particular names are also given to certain magnitudes, each of which contains a certain number of yards as shown in the following table :—

Linear Measure Table.

12 inches	= 1 foot.	8 furlongs	= 1 mile.
3 feet	= 1 yard.	3 miles	= 1 league.
5½ yards	= 1 pole.	22 yards	= 1 chain.
40 poles	= 1 furlong.	10 chains	= 1 furlong.

43. In the Metric or French system, the standard unit of length is the **Metre**. By dividing, and multiplying the metre by 10, particular lengths are obtained, as shown in

the table below. Latin prefixes denote division of the metre, Greek denote multiplication.

Metric Table of Linear Measure.

10 millimetres (mm.) = 1 centimetre (cm.)
10 centimetres (cm.) = 1 decimetre (dcm.)
10 decimetres (dcm.) = 1 metre (m.)
10 metres (m.) = 1 dekametre (dkm.)
10 dekametres (dkm.) = 1 hektometre (hkm.)
10 hektometres (hkm.) = 1 kilometre (km.).

The following **Table of Equivalents** will be found useful:—

Linear Measures.

French.		*British.*	*British.*		*French.*
Millimetre	=	0·0394 inch.	Inch	=	2·5399 centimetres.
Centimetre	=	0·3937 „	Foot	=	30·479 „
Decimetre	=	3·937 „	Yard	=	0·914 metre.
Metre	=	39·3708 „	Chain	=	20·116 „
Hektometre	=	0·0621 mile.	Furlong	=	201·164 „
Kilometre	=	0·6214 „	Mile	=	1·609 kilometre.

44. In measuring, we use a *rod, tape,* or *chain,* on which the unit of length is repeated several times. This rod, tape, or chain we call a *graduated measure,* or **scale.** The end of each position of the unit is marked on the measure in some easily distinguishable manner, and a number is placed against each division mark to indicate the number of times the *unit* is contained in the *distance* between that particular division mark and the starting point, or origin, which is denoted by 0 (zero).

In a linear scale, we have already seen that any convenient length may be taken as *unit length.* Fig. 38 shows scales in which the unit is one inch. Three lengths of the unit are shown at A, B, C. Fig. 39 is a scale in which the unit is one centimetre, seven lengths of the unit being shown.

In Fig. 38 (a) the unit A is divided into twelve equal parts; each of these subdivisions is thus $\frac{1}{12}$ in.

Def. A line divided into twelve parts is said to be *duodecimally* divided. Fig. 38 (a) is called a 1-*inch duodecimal scale.*

In Fig. 38 (b), the unit A is divided into ten equal parts; each subdivision is thus $\frac{1}{10}$ in. In Fig. 39, the centimetres

are divided into ten equal parts; each subdivision is thus $\frac{1}{10}$ cm., *i.e.*, 1 mm.

Def. A line divided into ten parts is said to be *decimally*

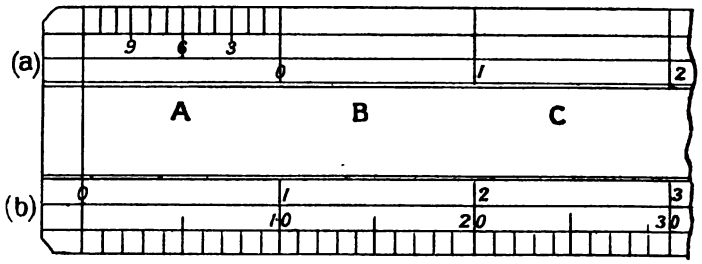


FIG. 38.

divided. Fig. 38 (b) is called a 1-*inch decimal scale*; and Fig. 39 a *decimal scale of centimetres.*

In all geometrical work involving numbers, decimally

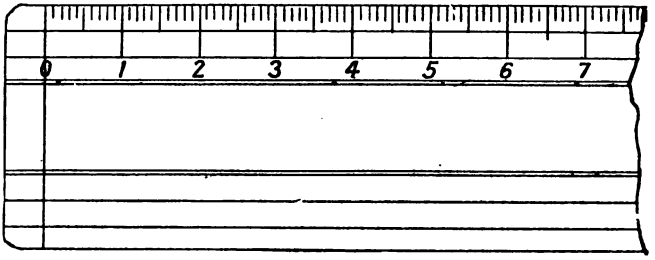


FIG. 39.

divided scales are the most convenient as they enable one to measure directly to one or more decimal places and thus facilitate computation.

45. *Open and close divided scales.* In Fig. 38 (a) only one unit is divided up, whilst in Fig. 38 (b) the whole line is subdivided.

Fig. 38 (b) is called a *close divided scale.* Fig. 38 (a) is called an *open divided scale.*

In close divided scales, the zero (0) point is placed at the end of the scale, whilst in the open type the subdivisions are an extension of the unit scale, and read in the opposite direction, as indicated by the numbers in the diagrams.

All divisions of a scale should extend to the edge of the rule so that the scale can be applied directly to the object, or drawing, and dimensions read off, or transferred thereto, direct.

46. **Full-size scales.** A rod or tape, graded in inches and having the inch subdivided into various parts, such as halves, quarters, eighths, sixteenths, or twelfths or tenths, is sometimes called an *actual* or *full-size scale*, for such a measure, when properly applied to any object on a surface it will fit, furnishes exact information as to the length of

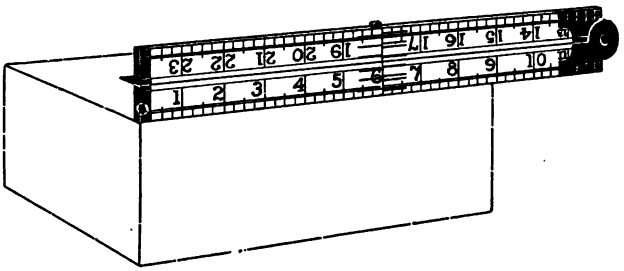


FIG. 40.

any part of the object coinciding with the measure. Thus, in Fig. 40, an ordinary measuring-rule, as used by mechanics, is shown applied to an edge of a building-brick. The zero mark on the rule being placed at one end of a long edge, the opposite end of that edge coincides with a division on the rule which is seen to be one-eighth of the space between the inch divisions marked 9 and 8.

This *reading of the rule* shows that the edge being measured is $8\frac{7}{8}$ in. long.

Again, in Fig. 41, a steel tape measure is shown applied to a circular iron column. The zero end of the tape meets the tape again at a point midway between inch divisions marked 25 and 26, which indicates that the circumference of the column, at the level of the tape, is $25\frac{1}{2}$ in.

Now in both examples, the dimensions are actual measures of real objects, and thus the measures are *actual* or *full-size measures.* Full-size scales are used to measure actual objects and in making full-size drawings of an object, because in such drawings every part of the object is represented in its real size.

47. **Reduced, or small scales.** When preparing a design for a large machine, building, etc., the draughtsman, in order to bring the whole design within convenient

FIG. 41.

limits, has to represent real magnitudes by much smaller magnitudes. Thus, in making drawings of a machine, it may be necessary to represent feet in the actual machine by inches in the drawings; and in making a map of a tract of country, actual miles may be represented by inches, or parts of inches, in the map. Such drawings are called *small scale drawings.* The scale used in making a small scale drawing is a *reduced scale.*

48. **Enlarged scales.** It will be readily understood that the above operation may be reversed. Instead of representing a magnitude, by a smaller magnitude, we may

represent it by a larger magnitude, *e.g.*, in making drawings of the parts of a watch we may represent *millimetre* lengths in the watch by centimetres in the drawing. In this case, the scale employed in making the drawing is an *enlarged scale.*

49. **Scalar unit; representative fraction.** Take two lines, Fig. 42, one 12 in. long, the other 3 in. long (in the figure these lines are reduced to $\frac{1}{4}$). Consider each line as unit length, and let each be divided into twelve equal parts. Then each division of the larger line is $\frac{1}{12}$ of the larger unit, and each division of the short line is $\frac{1}{12}$ of the small unit. Fig. 42 (a) is therefore a miniature of Fig. 42 (b), or Fig. 42 (b) is a **full-size** scale, whilst Fig. 42 (a) is a **small scale.** The unit (3 in.) which

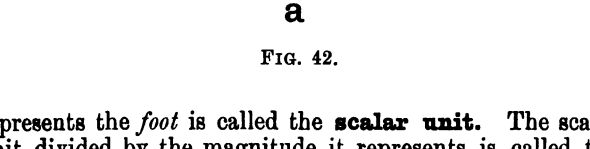


FIG. 42.

represents the *foot* is called the **scalar unit.** The scalar unit divided by the magnitude it represents is called the **representative fraction** of the scale.

A scale is defined either by stating the value of both the actual and the representative units, as 1 inch to a foot, $\frac{1}{2}$ inch to a foot, etc.; or by its representative fraction $\frac{1}{12}$ *full size*, $\frac{1}{24}$ full size, etc.

When a scale representing feet is given in the first form, the representative fraction is determined by dividing the number of inches in the scalar unit by 12 (the number of inches in a foot); and, when the scalar unit is a fraction of an inch, by multiplying the denominator of the fraction by 12.

Ex. 1. A scale of 2 in. to a foot $= \frac{2}{12} = \frac{1}{6}$ full size.

Ex. 2. A scale of $\frac{3}{8}$ in. to a foot $= \frac{3}{8 \times 12} = \frac{1}{32}$ full size.

Some of the scales in common use are given in the following table:—

	Scalar Unit.		*Representative Fraction.*
A scale of	1 inch	to a foot is	$\frac{1}{12}$ full size.
,,	$1\frac{1}{2}$	,, ,,	$\frac{1}{8}$,,
,,	2	,, ,,	$\frac{1}{6}$,,
,,	3	,, ,,	$\frac{1}{4}$,,
,,	6	,, ,,	$\frac{1}{2}$,,
,,	$\frac{3}{4}$	,, ,,	$\frac{1}{16}$,,
,,	$\frac{1}{2}$	,, ,,	$\frac{1}{24}$,,
,,	$\frac{3}{8}$	,, ,,	$\frac{1}{32}$,,
,,	$\frac{1}{4}$	,, ,,	$\frac{1}{48}$,,
,,	$\frac{1}{8}$	,, ,,	$\frac{1}{96}$,,
,,	$\frac{1}{10}$	,, ,,	$\frac{1}{120}$,,

A draughtsman should have the following decimal scales:—1 in., $\frac{1}{2}$ in., $\frac{1}{4}$ in., $\frac{1}{8}$ in., and 1 cm., also inch scales subdivided into eighths and twelfths. This set of scales can be obtained on one boxwood scale rule.

The table above includes only scales which represent a length of *one foot*, but any definite unit length can be used to represent any other known length, *i.e.*, a mile, or a furlong, etc., can be represented by a centimetre, or 0·25 inch, etc. In each case, the representative *unit length* is called the *scalar unit*, and the representative fraction of the scale is ascertained by dividing the scalar unit by the length *represented*.

Ex. A distance known to equal 18 *chains is represented in a drawing by a line which measures* 18 *cm. What is the R. F. of the scale used in the drawing?*

Here we have

$$\text{R. F.} = \frac{18 \text{ cm.}}{18 \text{ chains}} = \frac{1 \text{ cm.}}{1 \text{ chain}} = \frac{0{\cdot}3937 \text{ inch.}}{22 \times 36 \text{ inches.}}$$

$$= \frac{1}{2000} \text{ approx.}$$

50. **Construction of scales.** The problem of constructing a plain scale is that of dividing the straight line into a number of equal parts, and the operations are but a repetition of Problem 36.

The following examples will make the matter clear :—

Problem. Fig. 43. *Construct a scale $\frac{1}{8}$ full size to measure feet and inches.*

Draw a straight line AB 12 in. long, divide AB into eight equal parts (by problem 36) at 0, 1, 2, etc. Then each of these divisions is one-eighth of a foot, and represents a foot on the scale.

To obtain inch measures on the scale, divide A0 into twelve equal parts as shown.

In all scales, the division marks must be sufficiently far apart to ensure distinction. When very close division has to be performed, the only satisfactory method is to place consecutive divisions on different lines as exemplified in the solution of the next Problem.

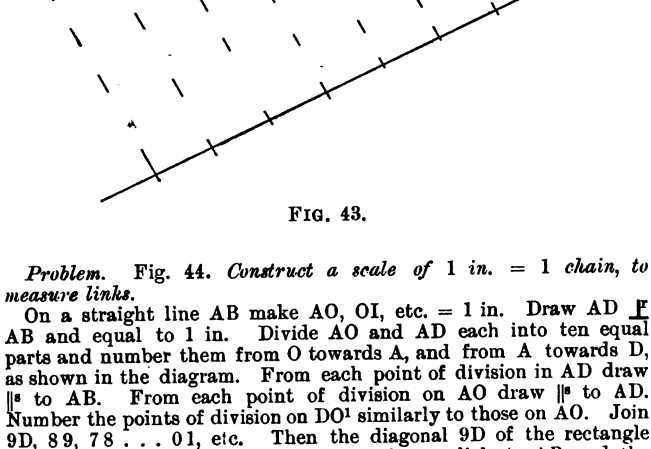


FIG. 43.

Problem. Fig. 44. *Construct a scale of* 1 *in.* = 1 *chain, to measure links.*

On a straight line AB make AO, OI, etc. = 1 in. Draw AD ⊥ AB and equal to 1 in. Divide AO and AD each into ten equal parts and number them from O towards A, and from A towards D, as shown in the diagram. From each point of division in AD draw ∥ to AB. From each point of division on AO draw ∥ to AD. Number the points of division on DO[1] similarly to those on AO. Join 9D, 8 9, 7 8 . . . 0 1, etc. Then the diagonal 9D of the rectangle 9AD9 is divided into ten equal parts by the parallels to AB, and the intercepts of these parallels in the triangle 9D9 increase in length from 9 towards D9 by the constant $\frac{1}{10}$ of A9. And since A9 = $\frac{1}{10}$ of AO, *i.e.*, $\frac{1}{10}$ in., hence the parallels in the triangle 9D9 increase successively by $\frac{1}{100}$ in. from AO towards DN. The same holds for all the transversals of the system of parallels.

Thus, the distance between the two black dots . . is 1·2 in. and a fraction of $\frac{1}{10}$ in. The value of the fraction is found by

counting the number of places the line containing the dots is removed from AO. This is 3; hence the distance between the dots is 1·23 in., and represents a length of 1 chain 23 links.

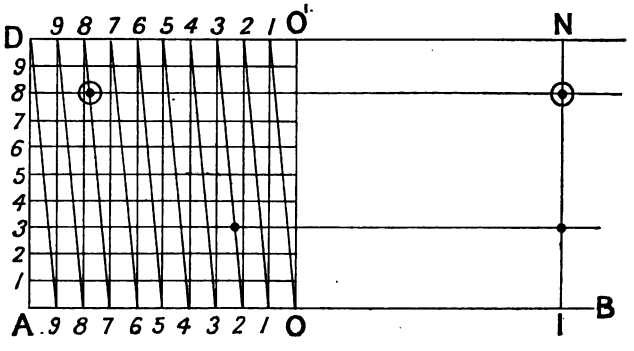


FIG. 44.

Again, the distance between the two dots ⊙ ⊙ is seen to be 1·78 in.

This would represent 1 chain 78 links.

The above type of scale is called a **diagonal scale**. The name is derived from the diagonals of the small rectangles which serve to determine successive divisions of consecutive scalar inches.

Exercises.

(1) (*a*) Divide a line 12″ long into 12 equal parts, then divide one of these parts into 12 equal parts. Express each division as a fraction of the whole line.

(*b*) Having divided the 12″ line into 12 equal parts, divide one of the parts into 4 equal parts, then take 3 of these parts and divide into 12 equal parts. Compare carefully the lengths of the smallest divisions in (*b*) and (*a*). How many small divisions of (*a*) are equal to 12 of (*b*)?

(2) Set out a scale to measure 2 yards making a line $1\frac{7}{8}$″ long represent a foot. What is the representative fraction of the scale?

(3) A line AB $1\frac{3}{4}$″ long represents a length of $4\frac{1}{2}$ ft. Construct a scale on this line showing inches.

(4) The plan of a building is drawn to a scale of $\frac{1}{8}$″ = 1 ft. On this plan a window opening is shown $\frac{7}{16}$″ wide. What is the actual width of this opening?

(5) Two towns are 45 miles apart. On a map they are shown 7 in. apart. Construct a diagonal scale for this map which can be used to measure furlongs.

51. **The vernier** is a small scale attached to an ordinary

scale to enable fractions of small divisions on the ordinary scale to be read at a glance.

Fig. 45 shows a portion of an ordinary decimal scale with a vernier slide attached, the zero readings on the scale and vernier being set together. The principal unit graduation on the main scale is 1 in.; thus each subdivision is $\frac{1}{10}$ (or 0·1) of an inch. The unit on the vernier is made equal

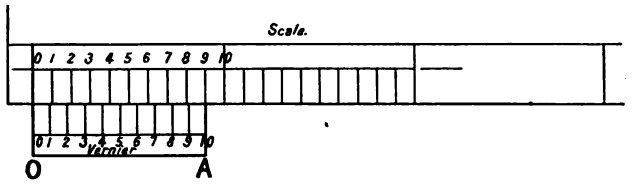


FIG. 45.

to $\frac{9}{10}$ of the principal scalar unit. Thus, in the figure, OA = nine of the subdivisions on the inch scale. The length OA on the vernier is also divided into ten equal parts; hence ten divisions on the vernier are equal to nine

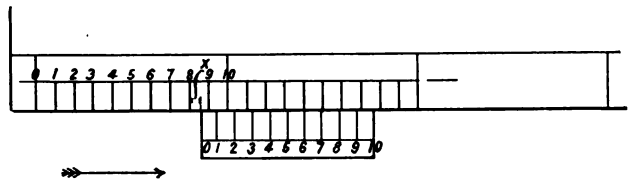


FIG. 46.

scale divisions, and each vernier division is $\frac{9}{10}$ of a scale division. Now, if the vernier slide along the fixed scale in the direction OA into a new position such as to bring the first division mark, (1), on the vernier opposite the first division mark on the fixed scale, the zero mark on the vernier will have moved from the zero mark on the fixed scale a distance equal to $\frac{1}{10}$ of a scale division. If the sliding continues until the second division mark, (2), on the vernier coincides with the second division on the fixed scale, the

zero mark on the vernier will have moved from the zero mark on the fixed scale a distance of $\frac{9}{10}$ of a scale division, and so on. Thus the distance between the zero mark on the vernier and the division next below it on the main scale is always indicated by the number of the particular division line on the vernier which coincides with one of the division lines on the main scale.

Fig. 46 shows the vernier after having been moved in the direction of the arrow into a position in which the sixth

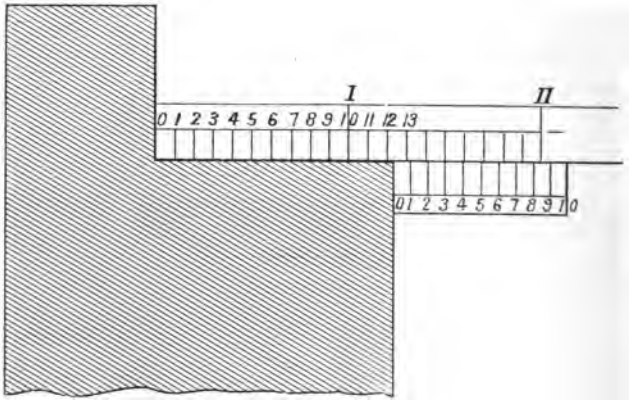


FIG. 47.

division mark, (6), on the vernier (reading from zero) coincides with one of the fixed scale divisions. The measure of the fraction x of the main scale division is therefore $\frac{6}{10}$ of $\frac{1}{10}$ of an inch, or 0·06 in.

The distance between the end of the vernier and the zero mark on the main scale in Fig. 46 is 0·8 in. + 0·06 in. = 0·86 in.

Fig. 47 shows a decimal scale of inches with a vernier slide applied to measure the depth of rebate in a piece of timber. What is the depth as shown by the readings of the rule and vernier?

Answer, 1·23 in.

CHAPTER IV.

ANGLES, MEASUREMENT AND CONSTRUCTION OF.

52. In measuring angles, any magnitude which has *always the same value* may be employed as the unit. The unit employed in *constructive geometry* is the *degree*, which is the angle subtended at the centre of the circle by *one-three-hundred-and-sixtieth part of the circumference*, and is therefore always the same (see Art. 5). The magnitude of an angle is therefore given by the number of degrees contained in the angle.

Another unit of angular measure is the *right-angle*. This has always a fixed value, viz., 90°, and the magnitude of any angle may be expressed in terms of the *unit right-angle*.

If the number of degrees in an angle be given, or determinable, it is only necessary to divide that number by 90 in order to obtain a measure of the angle in terms of the *unit right-angle*. The quotient may, or may not, be a whole number, but it can be obtained correct to any desired number of decimal places.

Ex. Express in terms of the unit right-angle (a) *an angle of* 76·5°, (b) *an angle of* 113°.

$$\text{(a)}\ \frac{76{\cdot}5^\circ}{90} = 0{\cdot}85 \text{ right-angle.}$$

$$\text{(b)}\ \frac{113^\circ}{90} = 1{\cdot}25 \text{ right-angles.}$$

53. Certain angles can be readily constructed by means of intersecting arcs. Thus, an angle of 45° is obtained by first setting out an angle of 90° (Art. 29) and then bisecting this angle. Again, if the angle of 45° is bisected, we obtain angles of $22\frac{1}{2}^\circ$.

Ex. To construct an angle of 60°. Fig. 48.

With the point O as centre, and any radius, describe an arc intersecting OA in N. With N as centre and the *same radius* describe an

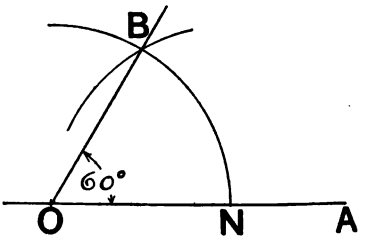


FIG. 48.

arc intersecting the previously drawn arc in B. Join OB, then AOB is the required angle of 60°. (This construction is due to an important fact, viz.: *The chord of* 60° *is equal to the radius of circle.* See Art. 61.)

If the angle AOB be bisected we obtain angles of 30°.

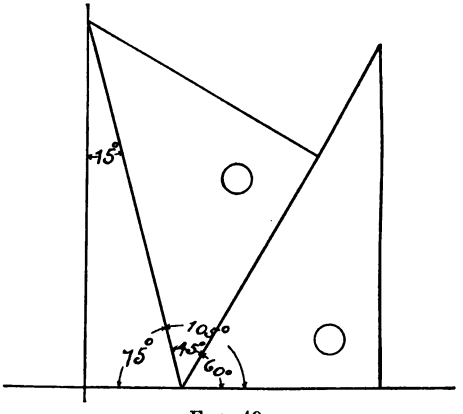


FIG. 49.

54. In practical drawing, we never use a geometrical construction to obtain a result which can be obtained more easily by mechanical means. Thus, angles of 90°, 60°, 45°,

30° are drawn direct by using the usual set-squares ; and by arranging these same tools as shown in Fig. 49, angles of 75°, 105°, and 15° may be drawn.

55. In setting out and measuring angles generally, we use a protractor.

A **protractor** is a scale of degrees ; it is usually either semicircular or oblong in form. See Figs. 50 and 51. In all protractors, lines representing the degrees radiate from a point on, or near to, the base, which is distinguished by a * or some other conspicuous mark.

On good quality protractors, the degrees are numbered in

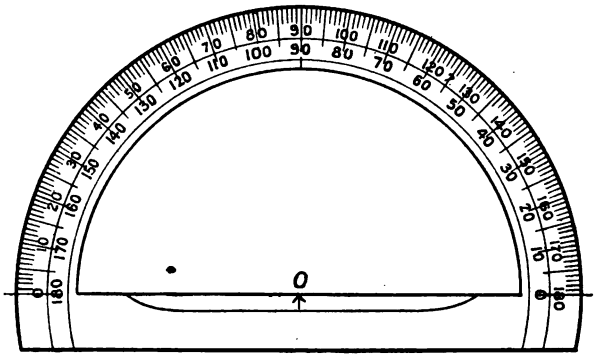


FIG. 50.

both directions, giving to each division two numbers ; this enables the user to read off degrees either from left to right or *vice versâ*.

56. **How to use a protractor** :—

(*a*) to measure a given angle.

(*b*) to set out a given angle at a given point.

(*a*) *To measure the given angle* NOP, Fig. 51.

Place the base of the protractor along the arm NO with the centre of the protractor at the angular point O. The number of degrees in the angle NOP is then indicated by the figures on the protractor at the point where OP crosses the outer edge. In the diagram, OP crosses at the

double reading 45, 135. As the angle NOP is less than a right-angle, the first of these numbers—viz., 45—indicates the number of degrees between ON and OP, and therefore the angle NOP is 45°.

(*b*) Fig. 51. *To draw from a point* O *in a given straight line* ON, *a line inclined at* 65° *to* ON.

Place the base of the protractor on the line NO with the centre of the protractor at the point O. Mark a point on the drawing corresponding to the 65th division on the protractor reading from N in the direction of the arrow,

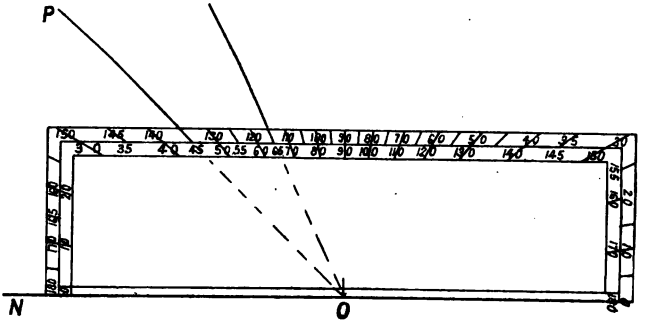


FIG. 51.

and draw a line from O through this point. This line will be inclined to NO at 65°.

57. In the two examples above, we have measured to a number of whole degrees, and on most protractors the degree is the smallest measure shown; but we have frequently to treat of angles containing some fraction of the degree, such as 0·5° or 0·2°, and a very close approximation of these parts of the degree can be *guessed*, or determined *visually*, after a little practice.

Some protractors are wholly unreliable, and many others, even among the higher-priced instruments, have the first degree division inaccurately marked.

58. **A method of testing the protractor** is as follows:—Draw two lines at right-angles to each other and

meeting in a point O. Insert the pricker at O and place the centre of the protractor against the needle, and observe whether there is always a difference of 90° between the readings of the protractor when it is rotated about the point O into various positions. Any variation is evidence of error in the scale. When the *first degree* division only is inaccurate, this division may be dispensed with by placing the line $0-10°$ over an arm of the angle. If this expedient be employed, the magnitude of an angle will be 10° less than the number shown on the scale.

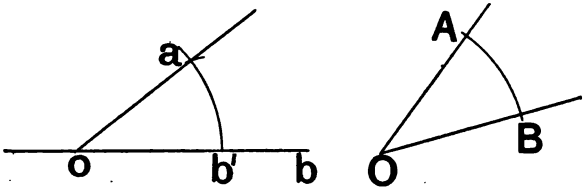


FIG. 52.

59. **Problem.** Fig. 52. *At a point* o *in a line* ob *set out an angle equal to a given angle* AOB.

By far the best way to copy an angle is to place a piece of tracing-paper over the figure and mark the position of the angular point and a point on each arm of the angle. Then transfer the tracing to the desired position, prick through the points, and join up.

The problem may also be done as follows: Make point O the centre of an arc of convenient radius cutting OB, OA in B, A. With centre *o*, and radius OB or OA, describe an arc cutting *ob* in *b'*. With centre *b'*, and radius BA, describe an arc cutting the arc previously drawn in *a*. Join *ao*.

60. **Chords of angles.** Fig. 53. On a straight line OM mark a point A such that OA = 10 units (any unit). With centre O and radius OA describe an arc. With centre A and radius equal 4 units, say, describe an arc intersecting the previously drawn arc in *a*. Draw O*a* and the straight line *a*A. Then the chord A*a* is called the chord of the angle AO*a* ; it is said to subtend that angle.

For the ratio $\frac{\text{length of chord}}{\text{radius}}$ we have $\frac{Aa}{OA} = \frac{4}{10}$ or 0·4.

Now take any points N, M on OA and describe arcs from centre O, to pass through these points intersecting the radial O*a* in *n* and *m*. Draw the chords M*m*, N*n*. Let θ°

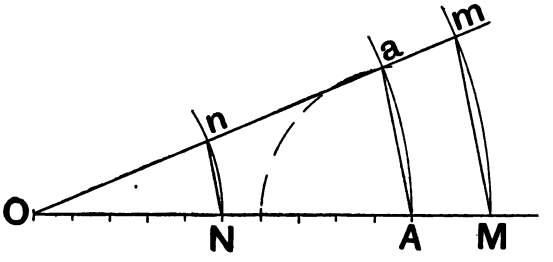


FIG. 53.

denote the angle between the radials OM, O*m*, then, the ratio $\frac{\theta^\circ}{360}$ is the same for the three arcs centred in O.

∴ (i.) Ratio $\frac{\text{length of arc}}{\text{radius of arc}}$ is the same in all the circles.

(ii.) Ratio $\frac{\text{length of chord of angle}}{\text{radius of arc}}$ is the same in all the circles.

Whence,

$$\text{from (i.)} \frac{\text{Arc N}n}{\text{ON}} = \frac{\text{Arc M}m}{\text{OM}} = \frac{\text{Arc A}a}{\text{OA}} = \text{constant};$$

$$\text{from (ii.)} \frac{\text{Chord N}n}{\text{ON}} = \frac{\text{Chord M}m}{\text{OM}} = \frac{\text{Chord A}a}{\text{OA}}$$
$$= 0{\cdot}4 \text{ constant.}$$

61. The value of the ratio $\frac{\text{chord}}{\text{radius}}$ depends solely on the magnitude of the angle at O. For, in an angle less than M*Om*, the chord in the arc through M is less than M*m* whilst the radius OM remains the same ; consequently, the

ratio $\frac{\text{chord}}{\text{radius}}$ is less than $\frac{Mm}{OM}$. Again, in an angle greater than MO*m*, the chord is greater than M*m*, and therefore the ratio is greater than $\frac{Mm}{OM}$.

If a semicircle, diameter OB, were divided into 180 equal parts, and lines drawn from O an extremity of the diameter, to each division on the arc, such lines would be *chords* of angles advancing by single degrees from 1° to 180°.

If all these *chords* were set off along a straight line from the same fixed point O, such line would be a **scale of chords.**

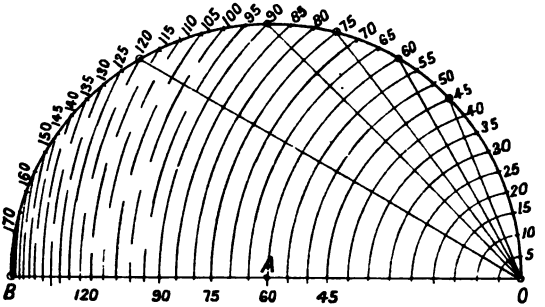


FIG. 54.

In Fig. 54, we show how to construct a *scale of chords.* Draw a straight line BO, and on it describe a semicircle, centre A. Mark off degrees on the circular arc, making use of a *protractor.* Intervals of 5° are shown on the diagram. With centre O describe arcs bringing the degree points on the semicircle down on to diameter BO, and number the divisions from O towards B. Then OB is a scale of chords advancing in steps of 5° from 0° to 180°.

From the Figure, it is evident that—

(i.) In an angle of 60°, $\frac{\text{chord}}{\text{radius}} = 1$ (unity).

(ii.) In an angle greater than 60°, $\frac{\text{chord}}{\text{radius}}$ is *greater than* unity.

(iii.) In an angle less than 60°, $\frac{\text{chord}}{\text{radius}}$ is *less than* unity.

The student, when using a scale of chords, must remember that the chords are chords of a circle having radius equal to 0—60 on the scale.

A scale of chords is engraved on many of the boxwood and other scale rules sold by mathematical instrument makers. It is distinguished by the mark "Cho."

A scale of chords may be used—

(*a*) To measure a given angle.

(*b*) To set out a given angle.

(*a*) Fig. 55. Let MON be the angle we are required to measure. Take the distance 0—60 from your scale of chords in a pair of compasses, and with centre O describe an arc intersecting OM, ON

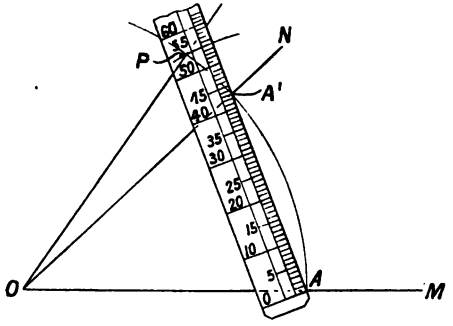


FIG. 55.

(produced if necessary) in A, A′ respectively. Apply your scale to the chord AA′, placing the zero point (0) exactly to the point A. Then the number of degrees in the angle MON is indicated by the reading (43) on the scale at the point A′. Therefore the angle MON is 43°.

(*b*) *From a point O*, Fig. 55, *in a line OM, to set out an angle of* 55°. Make OA on OM = 0—60 from your scale of chords. With centre O and radius OA describe an arc; with centre A and radius 0—55 (taken from the scale) describe an arc intersecting the arc previously drawn in P. Draw OP. Then MOP is the required angle of 55°.

62. In Table 3, Appendix, is given the numerical value of the ratio $\frac{\text{chord}}{\text{radius}}$ for angles advancing by single degrees from 0° to 90°. This Table may be used in measuring angles.

Ex. 1. *Let it be required to measure the given angle* BAC, *Fig.* 56. Select some convenient unit, and with the centre A and radius = 10 units describe an arc intersecting AC, AB (produced if necessary) in E and F respectively. Measure the chord FE, employing the

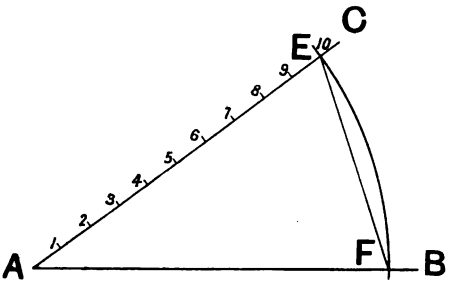


FIG. 56.

same unit as used for the radius AF. Suppose this done, and we find chord EF = 6·2 units. Then ratio $\frac{FE}{AE} = \frac{6{\cdot}2}{10} = 0{\cdot}62$. Hence the numerical value of the ratio is 0·62. Consulting the table, we find the

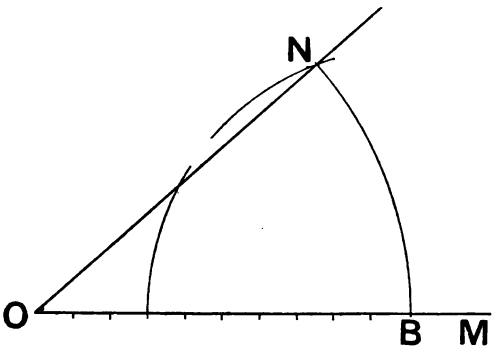


FIG. 57.

number nearest to 0·62 is 0·618, which is opposite the degree number 36. Therefore the nearest measure in whole degrees of the angle BAC is 36°.

We now give an example showing how an angle may be constructed

without the aid of a properly constructed *scale of chords*, or reference to a table of chords.

Ex. 2. *Fig.* 57. *The chord of angle is* 0·7 ; *construct the angle.*

Draw a straight line OM, and take a point B on OM such that OB = 10 cm. (any unit). With centre O and radius OB describe an arc ; with centre B and radius equal 7 units in OB (0·7 being the given ratio) describe an arc intersecting the arc from B in N. Draw ON. Then MON is the required angle.

63. If we take any right-angled triangle ABC, Fig. 58 having a right-angle at B, and call the two sides forming the right-angle respectively the perpendicular and the base of the triangle, and the side opposite to the right-angle the hypotenuse, then whatever the lengths of the three sides are, provided the angles C and A remain unaltered, we have :—

$$\text{Ratio } \frac{AB}{AC} \text{ or } \frac{\text{perpendicular}}{\text{hypotenuse}} \text{ is always the same number.}$$

$$\text{Ratio } \frac{BC}{AC} \text{ or } \frac{\text{base}}{\text{hypotenuse}} \text{ is always the same number.}$$

$$\text{Ratio } \frac{AB}{BC} \text{ or } \frac{\text{perpendicular}}{\text{base}} \text{ is always the same number.}$$

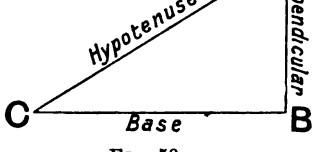


FIG. 58.

If we change the right-angle from B to A, but keep the angle C the same, we do not alter any of the ratios.

$$(1)\ \frac{\text{perpendicular}}{\text{hypotenuse}},\quad (2)\ \frac{\text{base}}{\text{hypotenuse}},\quad (3)\ \frac{\text{perpendicular}}{\text{base}}.$$

When the ratio of two quantities is always equal to the same number, the ratio is said to be *constant.*

64. The ratios of the sides of a right-angled triangle are called **trigonometrical ratios.** Every angle has six definite trigonometrical ratios.

Take any angle C, Fig. 59, bounded by two lines, and consider the six trigonometrical ratios for that angle. From any point A in either of the lines bounding the angle, draw AB perpendicular to the other line. We now have a triangle ABC with a right-angle at B. (Suppose your 60° set-square to be this triangle, and that the angle A is 60°.) The side AC opposite to the right-angle is the hypotenuse (the longest side of the square), the side AB opposite to the given angle C is the perpendicular (the shortest side of the square), and the remaining side BC is the base.

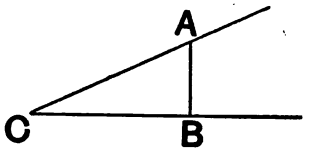


FIG. 59.

The ratio $\frac{\text{perpendicular (AB)}}{\text{hypotenuse (AC)}}$ is called the **sine** of the angle C.

The ratio $\frac{\text{base (BC)}}{\text{hypotenuse (AC)}}$ is called the **cosine** of the angle C.

The ratio $\frac{\text{perpendicular (AB)}}{\text{base (BC)}}$ is called the **tangent** of the angle C.

The ratio $\frac{\text{base (BC)}}{\text{perpendicular (AB)}}$ is called the **cotangent** of the angle C.

The ratio $\frac{\text{hypotenuse (AC)}}{\text{base (BC)}}$ is called the **secant** of the angle C.

The ratio $\frac{\text{hypotenuse (AC)}}{\text{perpendicular (AB)}}$ is called the **cosecant** of the angle C.

These six ratios are abbreviated to *sin.* C, *cos.* C, *tan.* C, *cot.* C, *sec.* C, *cosec.* C; and their numerical value depends *only* upon the angle C. Every angle has a definite value for its sin., cos., etc., and tables are prepared from which these values can be found.

65. **Graphical representation of the change in magnitude of the sin., cos., and tan.** of an angle from

0° to 90°. If we plot on squared paper corresponding values of the angle in degrees, and the sin., cos., or tan. of the angle, the graph so obtained is called a sine, cosine, or a tangent curve. First make out a table of corresponding values of angles increasing by 10°, and values of the sin., cos., and tan., using the trigonometrical table.

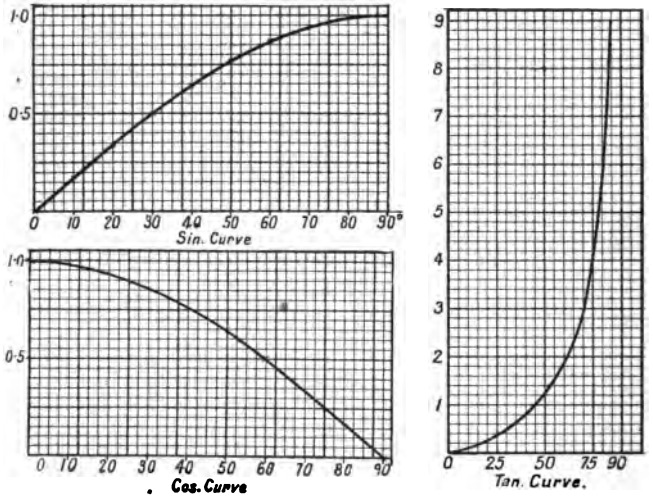


FIG. 60.

Plotting angles horizontally and values of the sin., cos., or tan vertically, we get the graphs shown in Fig. 60. Note, in the tangent graph since tan. 90° is infinite in value, and as we cannot represent infinity by any finite ordinate, we adopt the convention of making the curve parallel to the vertical line through the point at which $x = 90°$ to represent that the value of the tangent becomes infinite for an angle of 90°.

From these curves we can read off values of the sin., cos., and tan. of angles. Thus tan. $45° = 1$, and so on. Similar curves can be constructed for the cot., sec., and cosec. of an angle.

66. In any right-angled triangle, as ABC, Fig. 59, right-angled at B, the angles C and A are complementary, *i.e.*, $C + A = 90°$.

$$\text{And since sin. } C = \frac{AB}{AC}$$

$$\text{,, ,, cos. } A = \frac{AB}{AC}$$

Therefore the sine of an angle is equal to the cosine of the complementary angle.

$$\text{Again, tan. } C = \frac{AB}{BC}$$

$$\text{cot. } A = \frac{AB}{BC}.$$

Therefore the tangent of an angle is equal to the cotangent of the complementary angle.

Consequently, if we set out, as in Table 3, Appendix, the values of the sines and tangents of angles from 0° to 45°, these values will also be the cosines and cotangents of angles from 90° to 45°.

In Table 3, Appendix, we have in col. 1 degrees from 0° to 45°, and in col. 10 degrees from 90° to 45°, as shown in the extract below.

Angle.		Chord.	Sin.	Tan.	Cot.	Cos.			
Degrees.	Radians								
0°	—	—	—	—	—	—	—	—	90°
1°	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89°
43°	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47°
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cos.	Cot.	Tan.	Sin.	Chord.	Radians.	Degrees.
								Angle.	

If any value of an angle be read off in col. 1, the value (90° − angle) is found in col. 10. Thus, 43° in col. 1 and (90° − 43°) = 47° in col. 10 are in the same line.

At the top of col. 4 we find sin. and at the bottom cos.

,,	,,	5	,,	tan.	,,	,,	cot.
,,	,,	6	,,	cot.	,,	,,	tan.
,,	,,	7	,,	cos.	,,	,,	sin.

Suppose we require to know the value of sin. 43°. Look for 43° in col. 1 and in col. 4 with sin. at top, read "sin. 43° = 0·6820." If we require the value of cos. 47°, look in col. 10 for 47°, and read in col. 4 which has cos. at the bottom, the value "cos. 47° = 0·6820."

Thus we see from the Table, sin. 43 = cos. 47 = 0·6820. Above we found that sin. 43 = cos. (90 − 43) = cos. 47.

In reading any ratio from the Table, if the angle is in col. 1 we take the column with sin., cos., etc., at the top agreeing with the particular ratio we require; whereas, if the angle is found in col. 10, take the column with sin., cos., etc., at the bottom.

Ex. 1. *Find tan.* 15°.

15° is in col. 1, hence read in col. 5 "tan. 15° = 0·2679."

Ex. 2. *Find tan.* 78°.

78° is in col. 10, hence read in col. 6 "tan. 78° = 4·7046."

67. The trigonometrical table is very useful in calculation, particularly in calculating distances or heights which are found inconvenient to measure directly..

Ex. We wish to know the height of a church tower, or a cliff, or hill, etc. Measure a distance AB, Fig. 61, say 300 ft. from the

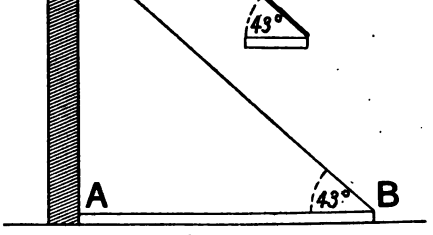


FIG. 61.

foot of the tower. At the point B we take a box with a hinged lid and holding the box horizontally we open the lid until looking along it we just see the top C of the tower. Now measure the angle the lid makes with the box. Suppose this to be 43°.

We now have $\frac{AC}{AB}$ = tan. 43°.

We know AB = 300 ft., and from the table we find tan. 43° = 0·9325, hence $\frac{AC}{300} = \frac{0·9325}{1}$

or AC = 300 × 0·9325
= 279·75 ft.

The point B is at the level of the observer's eye.
Suppose this is 5 ft. 3 in. above the ground.
Hence height of tower = (AC + 5·25) ft.
= 279·75 + 5·25 ft.
= 285 ft. high.

68. Each trigonometrical ratio is a measure of one side of a right-angled triangle in terms of another side, and all these ratios are interdependent, their values depending on the *magnitude* of the angle; thus, when the magnitude of the angle is fixed, the values of all the ratios are fixed. Conversely, if the value of either ratio be given, then each of the other ratios has, at the same time, been definitely fixed; and the magnitude of the angle also fixed. It is therefore a determinate problem to construct an angle when given the value of one of the above ratios. The solution of this problem involves the drawing of a right-angled triangle.

Ex. 1. *The sine of an angle* ($\theta°$) *is* 0·39; *construct the angle.*

Sin. $\theta = \frac{\text{perpendicular}}{\text{hypotenuse}}$ ∴ perpendicular = $\begin{cases} \text{sin. } \theta \times \text{hypotenuse} \\ 0·39 \times \text{hypotenuse.} \end{cases}$

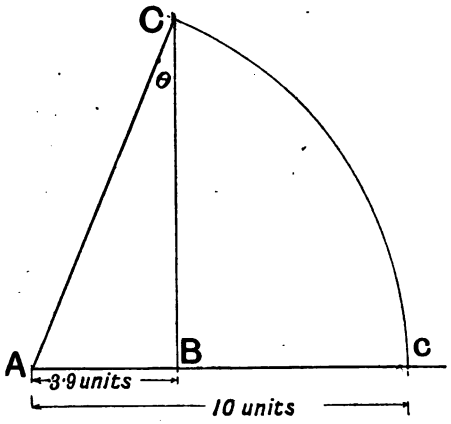


FIG. 62.

Draw a straight line Ac, Fig. 62, 10 units—say centimetres—in length. Take a point B on Ac, such that $AB = 3\frac{9}{10}$ cm. (the unit being 1 cm.).

At B draw a perpendicular to Ac. With centre A and radius Ac describe an arc meeting the perpendicular from B in C. Join CA. Then $\frac{AB}{AC} = \text{sin. } \theta = \frac{3{\cdot}9}{10} = 0{\cdot}39$. $\therefore$ ACB is the required angle.

Ex. 2. *The cosine of an angle* ($\theta°$) *is* 0·6 ; *construct the angle.*

$$\text{Cos. } \theta = \frac{\text{base}}{\text{hypotenuse}} \quad \therefore \text{ base} = \begin{cases} \cos.\ \theta \times \text{hypotenuse.} \\ 0{\cdot}6 \quad \times \text{hypotenuse.} \end{cases}$$

Draw a straight line Ac, Fig. 63, 10 (units) cm. in length. Take a

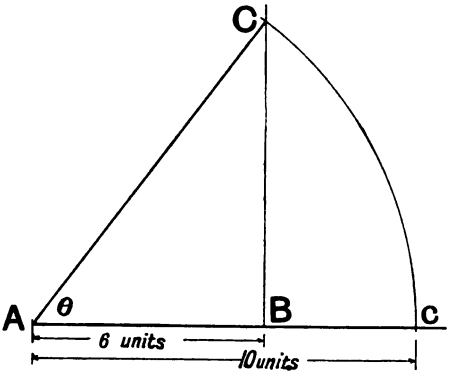


FIG. 63.

point B on Ac such that AB = 6 cm. At B draw a perpendicular to Ac. With centre A and radius Ac describe an arc meeting the perpendicular from B in C. Join CA. Then $\frac{AB}{AC} = \text{cos. } \theta = \frac{6}{10} = 0{\cdot}6$. $\therefore$ BAC is the required angle.

Ex. 3. *The tangent of an angle* ($\theta°$) *is* 0·42 ; *construct the angle.*

$$\text{Tan. } \theta = \frac{\text{perpendicular}}{\text{base}} \quad \therefore \text{ perpendicular} = \begin{cases} \text{tan. } \times \text{ base.} \\ 0{\cdot}42 \times \text{ base.} \end{cases}$$

Draw AC, Fig. 64, 10 (units) cm. in length. At A draw AB perpendicular to AC ; and make $AB = 4\frac{1}{2}$ cm. Draw BC. Then ACB is the required angle.

69. In problems similar to the three preceding, angles may be plotted direct from a protractor, if we have a table

of values similar to Table 3, Appendix. Thus, being given sin. $\theta^\circ = 0{\cdot}39$, we search the trigonometrical table for the measure in degrees of the angle whose sine is 0·39. This we find is 23, hence the required angle is 23°, and can be drawn as explained on p. 48.

Similarly, being given cos. $\theta = 0{\cdot}6$, reference to the table shows $\theta = 37^\circ$.

Again, given tan. $\theta = 0{\cdot}42$, the table shows $\theta = 23^\circ$.

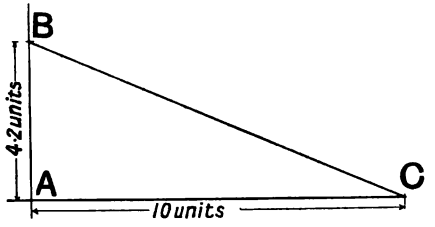


FIG. 64.

70. **Important ratios connected with the circle and angles.** *The length of the circumference of the circle.* Draw two straight lines. Take a centimetre as a unit, and make AB on one line equal 7 units, *i.e.*, 7 cm. Take $\frac{1}{4}$ in. as a unit, and make CD on the other line equal 7 units, *i.e.*, $7 \times \frac{1}{4}$ in. $= 1\frac{3}{4}$ in. Describe circles on diameters AB, CD, and by the method of Art. 20 carefully measure the circumference of each. The length of circumference of the circle on AB will be found approximately 22 cm., *i.e.*, $\frac{22}{7} \times$ AB.

The length of circumference of circle CD will be found approximately $5\frac{1}{2}$ ins., *i.e.*, $\frac{22}{7} \times$ CD. Thus, in each case the circumference is approximately $3\frac{1}{7}$ times the diameter.

The student may draw other circles, and by careful measurements satisfy himself that the circumference of a circle is always approximately $3\frac{1}{7}$ times its diameter.

Hence, the ratio $\frac{\text{length of circumference}}{\text{length of diameter}}$ is the same for

all circles, and it is universally denoted by the letter π (*pi*). The value of this ratio we have given above as $\frac{22}{7}$. This is the simplest approximate value of π; the exact value cannot be stated by any number. More nearly the ratio is 3·141592, etc., so that $\frac{22}{7}$ is too large. The error, however, is only about $\frac{1}{2500}$ of the value.

For graphical work, the value $\frac{22}{7}$ may be used. In arithmetical work, the number 3·1416 is usually taken.

In this book π is taken as 3·1416. Whence we have :—

(1) length of circumference of circle $= \pi \times$ diameter $= 2\pi \times$ radius $= 2 \times 3{\cdot}1416 \times r$ (where $r =$ radius of circle) ;

(2) semicircumference $= \pi \times r$;

(3) $\frac{1}{6}$ of circumference $= \frac{1}{6} \times 2\pi r$;
and generally where n equals any number,

(4) $\frac{1}{n}$ of circumference $= \frac{1}{n} \times 2\pi r$.

Ex. 1. *In order to ascertain the diameter at the base of a circular chimney shaft, a steel tape measure is folded around the chimney at the base level. The tape shows the circumference of the shaft to be* 39 *ft.* 7 *in. What is its diameter?*

Here we have $\pi \times$ diameter = circumference = 39·58 ft.

$$\therefore \text{diameter} = \frac{39{\cdot}58}{3{\cdot}1416}\text{ft.}$$

$$= 12{\cdot}5 \text{ ft.} = 12 \text{ ft. } 6 \text{ in.}$$

Ex. 2. *The diameter of a bicycle wheel is* 22½ *in. How far does the bicycle travel in one revolution of the wheel?*

(*Note.*—In one revolution the wheel measures its circumference along the track.)

$$\text{Circumference} = \pi \times 22{\cdot}5 \text{ in.}$$
$$= 3{\cdot}1416 \times 22{\cdot}5 \text{ in.}$$
$$= 70{\cdot}686 \text{ in.}$$

$\therefore$ Bicycle travels 5 ft. 10·7 in. during one revolution of the wheel.

71. The length of the circumference of a circle may be determined graphically by the following construction :—

Fig. 65. Draw a diameter BCA ; and a tangent to the circle at A. Draw CO making 30° with BA meeting

the tangent in O. Make OD = $3r$ (r being the radius of circle). Join BD. Then we have:—

$$\begin{aligned} BD^2 &= BA^2 + AD^2 \\ &= BA^2 + (OD - OA)^2 \\ &= (2r)^2 + (3r - r \tan. 30)^2, \text{ for } OA = AC \tan. 30^c. \\ &= 4r^2 + \left(3r - r\frac{1}{\sqrt{3}}\right)^2 \\ &= 4r^2 + r^2\left(3 - \frac{1}{\sqrt{3}}\right)^2 \\ &= r^2(4 + 5{\cdot}869) \\ &= 9{\cdot}869\, r^2. \end{aligned}$$

Hence BD = $3{\cdot}1415 \times r$; it is therefore an exceedingly

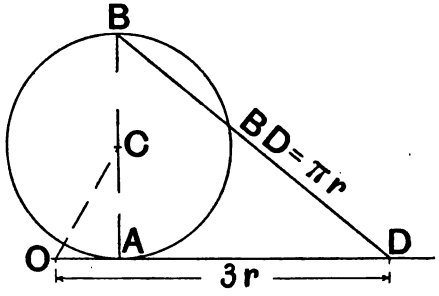


FIG. 65.

near approximation to the correct value πr. Hence, 2BD = $2\pi r$ = length of circumference of circle. And $\frac{1}{2}$BD = length of a quadrant.

72. We already know that the magnitudes of angles may be measured in units called degrees. If we choose as a unit the angle subtended at the centre of a circle by an arc equal in length to the radius, such an angle or unit is called a **radian**, and the magnitude of an angle expressed in radians is called the **circular measure of the angle.**

The comparison between the magnitudes of angles, when expressed in degrees, and when expressed in radians, is an example in proportion.

Ex. 1. *Find the number of degrees in one radian.*

Consider a complete circle. The angle subtended by the circumference is 360°. Now, since the circumference is equal to 2π times the

radius, there will be 2π arcs in the circumference, each equal to the radius, and hence if the extremities of each arc equal in length to the radius be joined to the centre of the circle we should have 2π angles, each of which would be 1 radian. It follows that 360 degrees is equal to 2π radians, hence the number of degrees in one radian is $\frac{360}{2\pi} =$ 57·3 degrees nearly, or $\frac{360 \text{ degrees}}{2\pi \text{ radians}} = \frac{\text{number of degrees in a radian}}{\text{one radian}}$.

We have already seen that the ratio $\frac{\text{circumference}}{\text{radius}}$ is *constant*, and since the circular measure of an angle is the ratio $\frac{\text{arc subtending the angle}}{\text{radius}}$, the circular measure of 360°, or four right-angles, will be expressed by 2π; the circular measure of one right-angle by $\frac{1}{2}\pi$; and of $\frac{1}{n}$ of a circumference by $\frac{2\pi}{n}$.

Exercises.

(1) A line OB 10 in. long rotates in a plane about the fixed point O; through what angle does it turn in (i.) one revolution, (ii.) 0·5 revolution, (iii.) 0·4 revolution? Give these answers in degrees, right-angles, and radians. What is the length of the chord, and of the circular path of B in each case?

(2) From a point O draw OB, OC each three inches long, and including an angle of 62°. By measurement and calculation determine (a) the length of BC; the sin., cos., and tan. of angle BOC. Compare the results you obtain with the tables in Appendix.

(3) Two lines *mm*, *nn*, Fig. 34, meet at an inaccessible point P. Take any point Q and by construction, measurement, and calculation determine how far P is from Q.

Hint.—Draw a ⊥r from Q on to either line—say *mm*—meeting same in B, and crossing the other line *nn* in A. Then if P were joined to Q, the triangles QBP, ABP would each have a right-angle at B. Hence the ∠BPQ = (90° − ∠BQP). The ∠BQP may be measured with a protractor, and in this way the magnitude of ∠BPQ is determined. Denote ∠BPQ by θ, then $BP = \frac{AB}{\tan.\theta}$. We now know length of BP and QB, and since ∠QBP = 90, we have $\frac{BP}{BQ} = \tan.\angle BQP$. Denote ∠BQP by ϕ. Then $QP = \frac{BQ}{\cos\phi}$. Alternatively $QP = \sqrt{BQ^2 + BP^2}$. Refer to Art. 84.

73. **Points of the compass.** On the compass there are four cardinal points, viz., North, East, South, and West, and a number of intermediate points at regular intervals. The directions North, East, South, and West are indicated in Fig. 66 by ON, OE, OS, and OW

respectively, which form 4 rt $\angle$s at the point O. The direction of any intermediate point we may denote by specifying the angle which that direction makes with the direction of either cardinal point. Thus, the direction E.N.E. is completely given as $22\frac{1}{2}°$ N.E. (read "North of East"); or as $67\frac{1}{2}°$ E.N. (read "East of North").

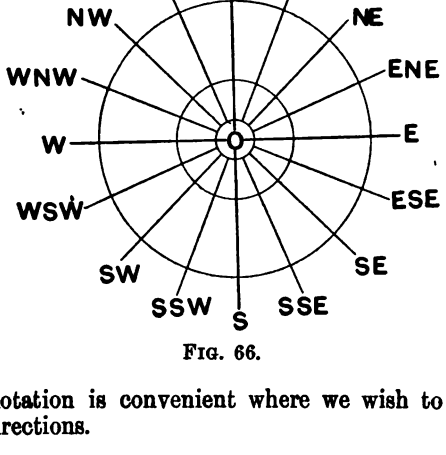


FIG. 66.

This notation is convenient where we wish to indicate several directions.

Ex. A road runs due North from a place A *to a place* B; *from* B *to a place* C *its direction is* 32° *East of North. At* C *its direction is again changed by an anticlockwise turn of* 48°. *What angle does the direction of the road from* C *make with the direction* AB; *and what with the direction due East?*

Draw a line AN, Fig. 67, representing the direction due North.

Draw AE $\perp$ to AN representing the direction due East.

From any point B in AN draw BC making 32° with AN and to the right. From any point C in this line draw CO making 48° with BC and to the left, and let this line extend in both directions so as to intersect AN in O and AE in O. Then α and ϕ are the angles which CO makes with the directions AN and AE respectively.

For angle α we have $48° - 32° = 16°$.

For angle ϕ we have $90° - 32° + 48° = 106°$.

Ex. A surveyor is making a map on which he wishes to locate two inaccessible objects H *and* K *situated towards the North. He lays*

off a base line DE, 20 *chains long, going due East. When stationed at* D, *he measures the angles* EDK, KDH, *by means of a sextant, and finds them to be* 51° *and* 55°. *When stationed at* E *the angles* DEH *and* HEK *measure* 48° *and* 62°.

Plot the points D, E, H, K *to a scale of* 8 *in. to the mile. Measure the distance and direction of* HK. (B.E. 1904.)

Fig. 68. Draw the base line DE 2 in. long. Draw D*k* making

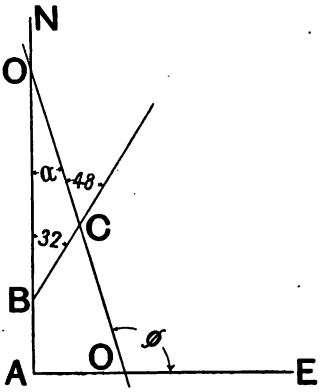


FIG. 67.

the given angle 51° with DE. Draw D*k* making the given angle 55° with D*k*. Draw EH making the given angle 48° with DE and produce to meet D*k* in H. Draw EK making the given angle 62° with HE and produce to meet D*k* in K. Then H and K mark the position of the two objects. Draw through H a parallel to DE and measure HK and the angle ϕ. HK measures 4·75 in. = 47·5 chains; ϕ measures 15°. Hence K is 47·5 chains from H in direction E. 15° N.

74. The positions of places or fixed objects relatively to one another are conveniently given by specifying the distance of each place or object from some fixed point, and the magnitude of the angle which the line drawn from the fixed point to the place or object makes with a fixed direction.*

* If the fixed direction be regarded as an axis of reference, and the fixed point as the origin, and if the distance be denoted by r and the angle by θ, the measurements r, θ locate the point, and are called the **polar** or r, θ co-ordinates of the point.

Thus, for example, in Fig. 68, the position of the place K relatively to H is completely defined as 47·5 chains in direction E. 15° N. Similarly, the position of K relatively to D may be given as DK distance in direction E. 51° N.

75. In practice, it is usual to take the fixed direction or direction of reference due East, and instead of writing

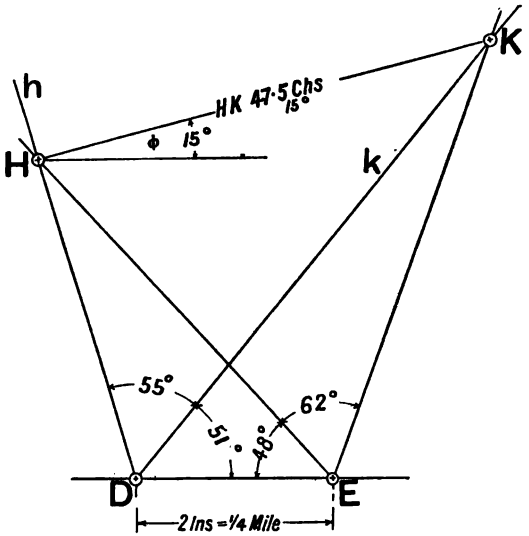


FIG. 68.

$HK = 47{\cdot}5$ chains in direction E. 15° N.
we write $HK = 47{\cdot}5_{15°}$ chains,
the direction of reference being understood to be due East, the first numbers indicating the distance, and the smaller numbers, written as a subscript, indicating the angle which the direction of the object makes with the direction of reference.

76. Angles when measured in an anticlockwise direction are considered **positive**; when measured in the opposite,

or clockwise direction, they are considered **negative**. Thus, the position of K is given as :—

$$HK = 47{\cdot}5_{15^\circ} \text{ chains, or } HK = 47{\cdot}5_{-345^\circ} \text{ chains.}$$

Note carefully that the *minus sign is given to the angle measure only ;* the *distance* is positive in both cases.

From the above illustration, we see that when a direction is given by a positive angle the same direction may be expressed as a minus angle by subtracting the number of

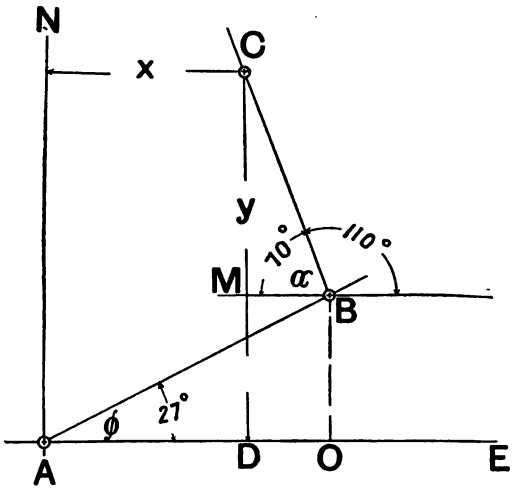


FIG. 69.

degrees in the positive angle from 360° ; the remainder is the number of degrees in the minus angle, *e.g.*, 360 — 15 = 345 ; and since 15 is + we have — 345.

Ex. In a traverse survey, lengths and directions are measured by using a chain and a magnetic compass. A, B, C *being three trees, the following measurements are made :—*

AB = 8·35 *chains in direction* E. 27° N. = $8{\cdot}35_{27^\circ}$ *chains.*

BC = 6·24 *chains in direction* E. 110° N. = $6{\cdot}24_{110^\circ}$ *chains.*

Plot AB *and* BC *to a scale of* 1 *cm. to* 1 *chain.*

Measure and calculate *how much* C *is to the East of* A, *and how much to the North of* A. (B. E. 1905.)

Draw lines of reference AE, AN, Fig. 69, at right-angles. Draw AB 8·35 cm. long, and making the given angle 27° with AE. Draw BC 6·24 cm. long and making 110° with the direction East. From C draw x and y perpendicular to AN and AE.

By measurement x = 5·3 cm. ∴ C is 5·3 chains East of A.

By measurement y = 9·7 cm. nearly ∴ C is nearly 9·7 chains North of A.

By calculation :—

$$\begin{aligned} x &= AD = AO - DO = AO - MB \\ &= (AB \cos. 27°) - (BC \cos. 70°) \\ &= (8·35 \times 0·891) - (6·24 \times 0·342) \\ &= 5·306 \text{ chains.} \\ y &= CD = CM + MD = CM + BO \\ &= (BC \sin. 70°) + (AB \sin. 27°) \\ &= (6·24 \times 0·9397) + (8·35 \times 0·454) \\ &= 9·65 \text{ chains.} \end{aligned}$$

Exercises.

(1) A ship S is observed from a station O to be at a distance of 3·2 miles in the direction 16° northwards of East; after ¼ hour it has moved to the position S′ distant 5·6 miles from O, in the direction 23° N. of E. Plot the points O, S, S′ to the scale of 1 in. to 1 mile. Measure the magnitude and direction of SS′, the change of position of the ship. What is the average velocity of the ship during this time? (B.E. 1904.) *Ans.* 2·45 miles E. 22° N.; 9·8 miles per hour.

(2) The following are the field notes of the survey of a triangular plot of ground ABC. Bearing means the direction from one station to the next succeeding station, and N. 31½° W. indicates the direction 31½° West of North :

Stations.	*Bearings.*	*Distances.*
A	N. 31½° W.	10 chains.
B	N. 62¾° E.	8·25 chains.
C		

Plot the survey to a scale of 1 cm. to 1 chain. Find and measure the bearing and distance from C to A. What is the enclosed area in square chains? (B.E. 1908.) *Ans.* 12·65 chains W. 79° S.; 82·25 square chains.

Symmetry.

77. Describe a circle on a piece of tracing-paper and draw a diameter DD. Fold the paper along DD so as to form a double page, press the two leaves together, and observe that the semicircle on the top sheet falls exactly

over the semicircle on the bottom sheet; and hence, the crease line DD divides the circle into *halves*, and the extremities of any chord perpendicular to DD are equidistant from that diameter and on opposite sides of the same.

Any figure which can be divided by a straight line into two equal parts such that, provided the figure were free to

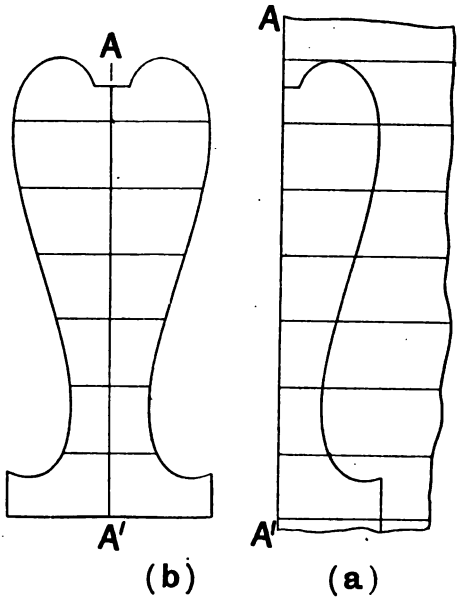


FIG. 70.

fold along the dividing line, one part could be made to coincide with the other part, is called a *symmetrical figure.*

The line about which the figure is symmetrical is called an *axis of symmetry.*

The circle is symmetrical about any diameter.

Many figures are symmetrical about only *one* or *two* lines.

As an example of symmetry about one line : Take a piece of ordinary ruled note-paper and draw a line AA′, at right-angles to the ruled line ; fold the paper along AA′ so as to make a double sheet in close contact, and cut the paper to any shape with a pair of scissors. See Fig. 70 (a). Now, open out the paper into a flat position, see Fig. 70 (b), and

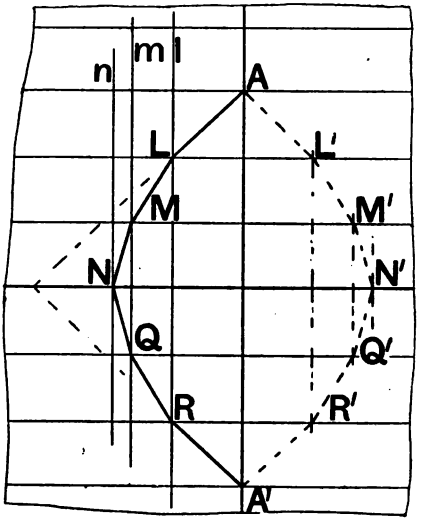


FIG. 71.

by measuring the ruled lines on the paper, verify that the figure is symmetrical about the axis AA′.

As an example of symmetry about two lines : Take a piece of ruled note-paper, as described above ; draw a perpendicular to the ruled lines, and mark thereon points A and A′, the distance between these points being six ruled spaces. See Fig. 71. Draw any three lines *l*, *m*, and *n*, parallel to AA′. Denote the points in which the line *l* intersects the ruled lines next to A and A′ respectively by L and R ; and the point in which *n* intersects the middle ruling by N ;

and the points in which m intersects the other rulings, by M and Q respectively. Join AL, LM, MN, NQ, QR, RA′; fold the paper about AA′, as explained above, and when close folded cut along the lines AL, LM, etc., with scissors. Open out flat the cut paper, and by measurements verify that the figure is symmetrical about both AA′ and NN′.

In Fig. 71, the points L, L′; M, M′, etc., are said to be symmetrical about the line AA′; the points A, A′; L, R, etc., are symmetrical about the line NN′. Also, the lines AL, A′R; LM, RQ, etc., are symmetrical about NN′; the lines AL, AL′; LM′, L′M′, etc., and LR, L′R′, etc., are symmetrical about AA′.

Observe, that any two straight lines which are symmetrical about a given axis and are inclined thereto, intersect if produced on that axis.

When two lines or figures are symmetrical about a given line, one line or figure is sometimes called the *image of the other*.

This notation is taken from Optics, where it is shown that any object placed in front of a plane mirror appears in the mirror as an object of precisely the same form but occupying a position behind the mirror at a distance equal to that of the object in front.

Exercises.

(1) Measure as accurately as you can the given angle AOB in degrees and decimals of a degree. Measure also OA in inches and decimals of an inch.

From A draw AM perpendicular to OB, and measure OM. By ordinary arithmetic divide OM by OA, and give the quotient. (B.E. 1901.) *Ans.* 72°; OA = 1·7 in.; OM = ·531 in.

(2) Draw two lines OA, OA, each 10 in. long, and containing an angle AOA of 88°. Measure accurately and state the distance apart of A, A. (B.E. 1903.) *Ans.* 6·925 in.

(3) Two steamers leave a pierhead together, one going in a direction 22° N. of E. at the rate of 8 miles an hour, the other going in a direction 10° South of East at the rate of 11 miles an hour. How far are the ships apart 1½ hours after the start? *Ans.* 9 miles.

(4) At a point A in a straight road which runs due North, a road branches off in the direction N.W. Find the distance apart of two places, one being 7 miles on the main road North of A; the other being on the branch road 3½ miles from A. *Ans.* 5·3 miles.

(5) The point B is 4 miles North and 2 miles East of the point A.

What is the distance from A to B, and what angle does the line AB make with the due East direction? (B.E. 1906.) *Ans.* $4{\cdot}5_{68^\circ}$ miles.

(6) A and B are two places 55 miles apart, the direction from A to B being 80° Northwards of East (*i.e.*, AB = 55_{80° miles). How much is B to the East, and how much to the North of A? Adopt a scale of 1 in. to 10 miles.

On a certain day in the year the time of sunset at A is 7 hrs. 56 min. after noon; and at neighbouring places sunset is 1 min. later for every 10 miles North, and also 1 min. later for every 20 miles West of A. At what time does sunset occur at B on this day? (B.E. 1903.) *Ans.* 9·55 miles; 54·16 miles; 8 hrs. 0·5 min. P.M.

(7) In a survey the following measurements were made, lengths being in chains and angles being reckoned anticlockwise from the East:—

$$AB = 32_{16^\circ},\ BC = 51{\cdot}3_{142^\circ},\ CD = 67_{41{\cdot}4^\circ}.$$

Plot the points A, B, C, D to the scale of $\frac{1}{2}$ in. to 10 chains. Measure from your drawing the length and direction of AD. Also give the easterly and northerly components of AD. (B.E. 1907.) *Ans.* 94·2 chains; 85·2; 41.

(8) In the plan of an estate, drawn to a scale of 1 in. to 100 yds., three trees, A, B, C, are marked, and their relative positions measure, in yards,

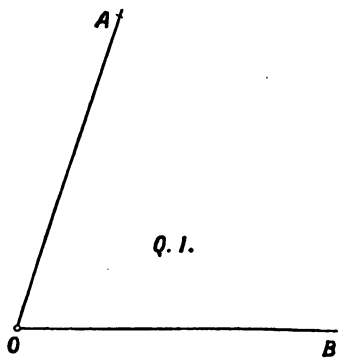
$$AB = 232_{79{\cdot}6^\circ},\ BC = 358_{6{\cdot}8^\circ},$$

angles being measured anticlockwise from the East.

In order to locate two places P and Q on the estate, lying on the North side of BC, the angles subtended by the trees at P and Q are measured with a sextant and are as follows:—

$$APB = 36{\cdot}2^\circ,\ BPC = 22{\cdot}3^\circ,\ APC = 58{\cdot}5^\circ$$
$$BQA = 18{\cdot}1^\circ,\ BQC = 78{\cdot}8^\circ,\ AQC = 60{\cdot}7^\circ.$$

Plot the five points to scale, and measure the length and direction of PQ. (B.E. 1907.) *Ans.* PQ = 500_{10° yds.



CHAPTER V.

PLANE RECTILINEAR FIGURES.

78. **Definitions and properties.** Plane figures bounded by straight lines are called **rectilinear figures.** The lines which bound the figure are called **sides.** If the figure has 3 sides, it is called a **triangle;** if 4, a **quadrilateral;** if more than 4, it is called a **polygon.**

Particular names are given to polygons according to the number of their sides: a *pentagon* has 5 sides, a *hexagon* 6,

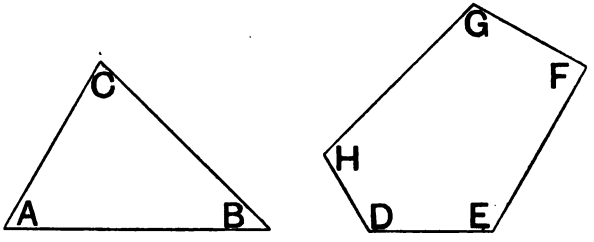


FIG. 72.

a *heptagon* 7, an *octagon* 8, a *nonagon* 9, a *decagon* 10, etc.

The sum of the sides or length of outline of a figure is called the **perimeter.**

A line joining any two non-consecutive corners of a rectilinear figure is called a **diagonal.**

In order to refer definitely to a particular figure or part of a figure, we denote each angular point or vertex by a distinct letter, and call the figure by these letters; thus, the triangle in Fig. 72, we call the *triangle ABC;* and the polygon we refer to as the *figure DEFGH.*

Greek letters are sometimes used to represent angles; thus, the angle EDH may be denoted by β (beta). Sometimes the angle is represented by the letter attached to the angular point; thus, the angle A would mean the angle BAC.

If any side, say AB, in the triangle, Fig. 73, be produced, this line and the adjacent side BC form two angles at B; one (B) an *interior* angle, and one (ϕ) an *exterior* angle. When the term *angle* is applied to a figure, an internal angle is always understood, in the absence of any statement to the contrary.

If every angle in a figure is less than 180°, it is called a **convex figure.** If all the angles are equal to one another, it is said to be **equiangular.** If all the sides are

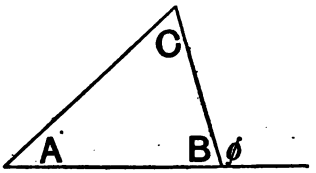


FIG. 73.

equal in length, it is called **equilateral.** A figure having all its sides equal and all its angles equal is called **a regular figure.**

Regular figures have a definite centre O, equidistant from the corners and sides, so that (*a*) all the angular points of a regular figure lie on the circumference of a circle centred in O; (*b*) a circle centred in O and touching one side touches every side of the figure.

79. **Experiment.** Draw any triangle ABC, Fig. 74, and extend each side, as shown in the diagram. Denote the exterior angles by α, β, ϕ. From any point O, draw O*b* parallel to the side AB; draw O*c* parallel to the side BC; draw O*a* parallel to the side CA. Then

angle $bOc = \alpha$; angle $cOa = \beta$; angle $aOb = \phi$.

And since the sum of the angles bOc, cOa, aOb is 360°,

$$\therefore \alpha + \beta + \phi = 360°.$$

Now draw any *convex* polygon, as DEFGH, Fig. 75. Extend each side as above described and denote the exterior angles by α, β, γ, θ, ϕ. From any point O, draw O*e* parallel to the side DE ; draw O*f* parallel to the side EF ; . . and O*d* parallel to the side HD. Then, the angles which are marked by the same letters in Fig. 74 are equal to one another, and the *sum of the exterior angles* is equal to the sum of the angles about the point O, viz., 360°, and the same is true of Fig. 75.

Hence the **Theorem.** *The sum of the exterior angles of any convex rectilinear figure is* 360°, *or* 4 *right-angles.*

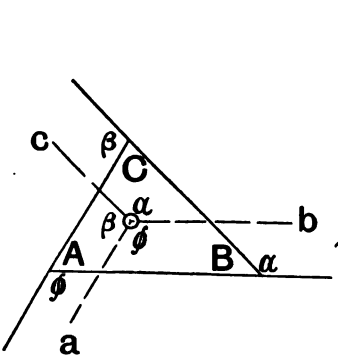


FIG. 74.

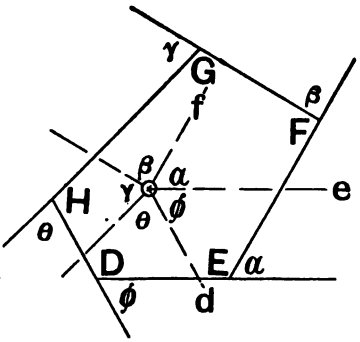


FIG. 75.

If we denote the interior angles of the triangle, Fig. 74, by A, B, C, we have—

A + ϕ = 2 right-angles ; B + α = 2 right-angles ;
C + β = 2 right-angles ;

and, by addition,

A + B + C + ϕ + α + β = 6 right-angles.

Again, if the interior angles of the polygon in Fig. 75 be denoted by D, E, F, G, H, we have

D + ϕ = 2 right-angles ; E + α = 2 right-angles;
F + β = 2 right-angles, etc. ;

and, by addition,

D + E + F + G + H + ϕ + α + β + γ + θ =
10 right-angles.

Comparing these results we find: *The sum of the interior and exterior angles of any rectilinear figure is = twice as many right-angles as the figure has sides.*

And, since the exterior angles always equal 4 right-angles, we have: (a) *The interior angles of a convex rectilinear figure of n sides* = $(2n - 4)$ *right-angles.* (b) *The sum of the interior angles of a triangle is* = 2 *right-angles.*

In Fig. 76, we show how this last important theorem may be verified by folding paper.

Draw any triangle ABC and bisect AC, CB in E and F Join EF and draw perpendiculars EM, FN on to AB. Cut out the triangle

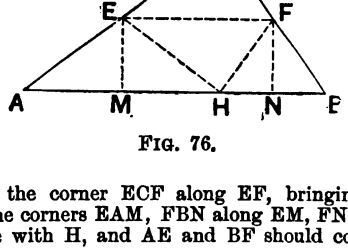


FIG. 76.

ABC, and fold the corner ECF along EF, bringing C on to AB in H. Fold the corners EAM, FBN along EM, FN; then A and B should coincide with H, and AE and BF should coincide with HE and HF.

When we have two figures, as, for example, the triangles ABC, *abc*, Fig. 77, or the polygons DEFGH, *defgh*, which we know are alike in *some* respects, and we wish to compare them, we call the points which are denoted by the *same* letters *corresponding points.* Lines which join two points in one figure and the corresponding points in the other figure we call *corresponding lines.* Thus, for example, in the triangles, AB, *ab*; BC, *bc*; and in the polygons, DE, *de*; DF, *df* are *corresponding lines.* The angles between corresponding lines in the two figures are *corresponding angles.*

If any two figures, as, for example, the triangles or polygons in Fig. 77, have their corresponding angles equal, *i.e.*, A = *a*, B = *b*, C = *c*, and D = *d*, E = *e*, . . H = *h*, the *figures are equiangular.*

(It must be noticed that in this case we are not comparing *one angle* with *another angle in the same figure*, but with an angle in *another figure*. The angles in the same figure may, or *may not*, be equal.)

If two or more equiangular figures are such that every line in one figure is equal in length to the corresponding line in the other figure, the figures are alike in all respects, and are said to be **congruent.** Congruent figures can always be superposed one on the other so as to make the two wholly coincide; and the practical test of **congruency** in plane figures is to make a tracing of one figure and

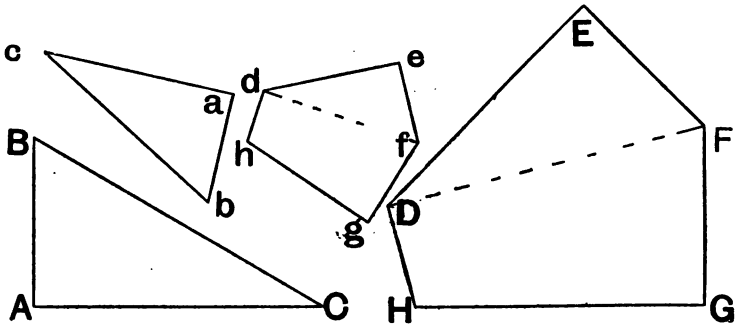


FIG. 77.

place this tracing over the other figure and see whether all lines on the tracing coincide with corresponding lines on the figure beneath.

80. **Similar figures.** When two or more figures have the angles in one figure equal to the corresponding angles in the other, and corresponding lines in the figures are proportionals, the figures are called similar figures.

The triangles ABC, *abc*, Fig. 77, and the polygons DEFGH, *defgh*, are examples of *similar figures.* In each pair of figures, corresponding angles are equal, and, in the triangles, linear dimensions are in the ratio $\frac{3}{5}$. In the polygons, the ratio is $\frac{2}{1}$. Test this statement by

measuring the angles and sides of each pair of figures. You will find—

$$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CA}{ac} = \frac{3}{2}$$

and $$\frac{DE}{de} = \frac{EF}{ef} = \frac{FG}{fg} = \frac{GH}{gh} = \frac{HD}{hd} = \frac{FD}{fd} = \frac{2}{1}.$$

This constant ratio is called the **ratio of similitude** of the two triangles (or polygons).

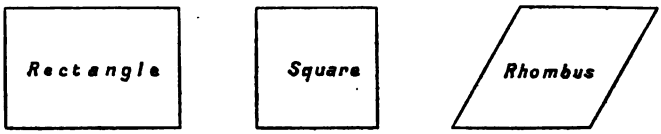


FIG. 78.

Circles, squares, and equiangular triangles are always similar figures, because in these figures corresponding lines are in a constant ratio. But rectilinear figures of more than *three* sides may be equiangular without having sides in a constant ratio, *e.g.*, the *square* and the *rectangle*, Fig. 78.

Also, two figures may have sides in a constant ratio without being equiangular, *e.g.*, the square and the rhombus, Fig. 78.

Exercises.

(1) *In a triangle ABC, the exterior angle at A is double that at B while the exterior angle at C is* 150°. *What are the exterior angles at A and B, and the three interior angles?* *Ans.* 140°; 70°; 40°; 110°; 30°.

(2) *Six lines radiate from a point O; the angles between them are, respectively,* 60, 40, 20, 70, 50, *and* θ *degrees. What is the value of* θ*? A six-sided polygon has sides parallel to the six radial lines. What are the values of the six interior angles of the polygon?* *Ans.* $\theta = 120°$.

(3) *What is the magnitude of each of the interior angles of a regular pentagon, heptagon, nonagon, and decagon?* *Ans.* 108°; 128·6°; 140°; 144°.

(4) *You are required to set out two consecutive sides only of a regular octagon of* 25 *ft. side. How would you proceed to do this?*

CHAPTER VI.

RIGHT-ANGLED TRIANGLE.

81. A triangle has 3 *sides* and 3 *angles;* these are called *elements* of the figure. Thus a triangle has 6 elements.

When two sides of a triangle are of equal length, the triangle is called **isosceles.** In an *isosceles* triangle, the unequal side is called the *base.* In other forms, any side may be called the base. If we call one side the base, the

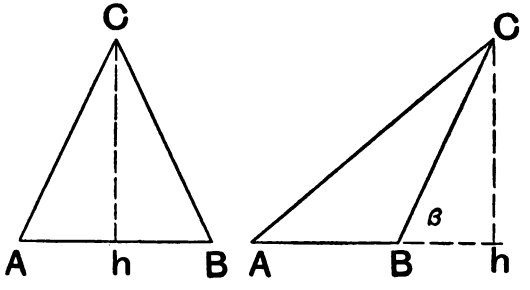


FIG. 79.

angles adjacent to that side are called *base angles*, and only the angle opposite to the base is called the *vertex.*

The perpendicular distance of the vertex of a triangle from the base line is called the *height or altitude.* Thus, in Fig. 79, C*h* is the height of each of the triangles on the base AB.

If A denotes the angle between a side AC and the base line AB, we have—

$$\text{Height } Ch = AC \times \sin. A.$$

82. A triangle having one side perpendicular to another

side is called a **right-angled triangle.** The side opposite the right-angle is called the **hypotenuse.** The sides forming the right-angle are sometimes called legs; either of these sides may be called the base.

Experiment. Draw any triangle, and by carefully measuring the 3 sides and angles verify the following

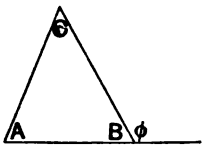


FIG. 80.

Theorem. *The shortest side of a triangle is opposite the smallest angle.*

Hence, we see—(*a*) in an equilateral triangle, the three angles are equal; (*b*) in an isosceles triangle, the angles opposite to the equal sides are equal.

If a side AB, Fig. 80, of a triangle ABC be produced, the adjacent interior and exterior angles at B, when added together, = 2 right-angles. And, since the 3 interior angles together = 2 right angles (Art. 79), we have angle B + angle ϕ = angle B + angle C + angle A. Subtracting common angle B, angle ϕ = angle C + angle A, *i.e.*, *An exterior angle of a triangle is*

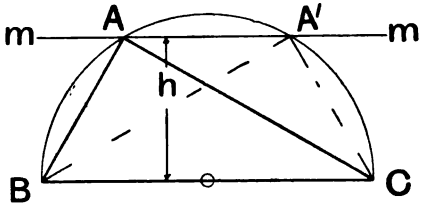


FIG. 81.

equal to the two interior and opposite angles. In a right-angled triangle, the angles adjacent to the hypotenuse are complementary.

83. We have learned from experiment, Art. 28, that any right-angled triangle has its right-angle on the circumference of the circle drawn on the hypotenuse as diameter. It follows, therefore, that when we know the hypotenuse of a triangle we also know the *locus* of the right-angle.

Construct right-angled triangles having given the following data:—

(i.) Fig. 81. *Given the hypotenuse a and its distance h from the opposite angle* A, *say* 3·5 *in. and* 1·5 *in.*

Draw side BC of given length $a = 3{\cdot}5$ in. Describe a circle on diameter BC. Draw *mm* parallel to, and at the given distance *h* above, BC, intersecting the semicircle in two points A and A′. Join AB, AC, or A′B, A′C. There are thus two solutions in this case. For limitations, see note to problem, Art. 96.

(ii.) No Figure. *Given the two sides including the right-angle, say* b *and* c, 3 *in. and* 7 *in.*

Draw AC, AB at right-angles and equal to the given lengths, viz., 3 in.; 7 in. Join BC.

(iii.) *Given one side* b *and the hypotenuse* a, *say* 3 *in. and* $4\frac{1}{2}$ *in.*

In this case, we may draw the hypotenuse of given length *a*, describe a semicircle on same and cut off a chord of given length *b*. The drawing of the semicircle is, however, unnecessary if we use the following method:—

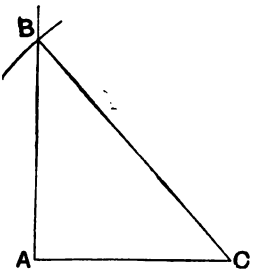


FIG. 82.

Fig. 82. Draw AC of given length $b = 3$ in. Erect a perpendicular at A. With C as centre and radius of length $a = 4{\cdot}5$ in. describe an arc intersecting the perpendicular from A in B. Join BC.

(iv.) No Figure. *Given the hypotenuse a and one adjacent angle, say* 5 *in. and* 57° *respectively.*

Draw side BC of given length $a = 5$ in. At C set out the given angle 57°. From B draw BA perpendicular on to the line from C.

84. All right-angled triangles possess the following property:—

If squares be constructed on each of the three sides, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This is a very useful theorem. In Fig. 83, we show how

the square on the hypotenuse may be divided and rearranged to form the squares on the other two sides.

Draw any triangle ABC with a right-angle at A. Draw squares on each of the 3 sides of the triangle. Through the mid-points of the sides of the square on the hypotenuse draw parallels to the sides of the triangle, *i.e.*, S*a*, Q*k* parallel to BA; P*c*, R*h* parallel to AC. Through the centre O of the square on long side AC draw *qv* parallel to BD, *rs* parallel to BC. Make a tracing of DS*c*P, and by superposing this

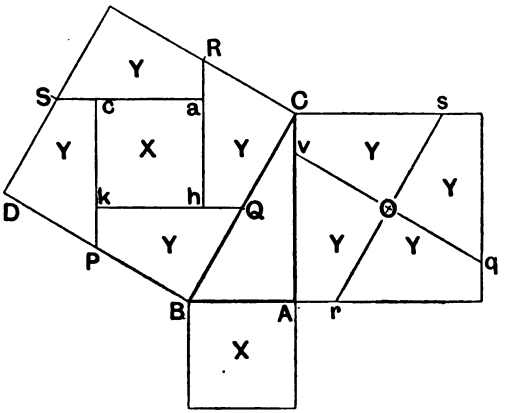


FIG. 83.

tracing show that each of the figures lettered Y are equal, and that the figures lettered X are equal.

If we denote the two legs of the right-angled triangle by *b* and *c*, and the hypotenuse by *a*, the theorem may be expressed thus:—

$a^2 = b^2 + c^2$; or $b^2 = a^2 - c^2$; or $c^2 = a^2 - b^2$.

These equations enable us to determine the length of either side when the other two sides are given.

Ex. 1. *Given* $b = 3$ in., $c = 4$ in. Then

$$a^2 = 3^2 + 4^2$$
$$a = \sqrt{9 + 16}$$
$$= \sqrt{25} = 5 \text{ in.}$$

Ex. 2. *Given* $a = 9$ in., $c = 7$ in. Then

$$b = \sqrt{9^2 - 7^2}$$
$$= 5{\cdot}657 \text{ in.}$$

Ex. 3. *Given* $a = 5$ in., $b = 3$ in.

$$c = \sqrt{5^2 - 3^2}$$
$$= 4 \text{ in.}$$

85. In the practical work of building and surveying, right-angles have frequently to be set out, and it is an advantage to have all three sides of the triangle whole numbers, so that the foot, or the *link*, will serve to measure

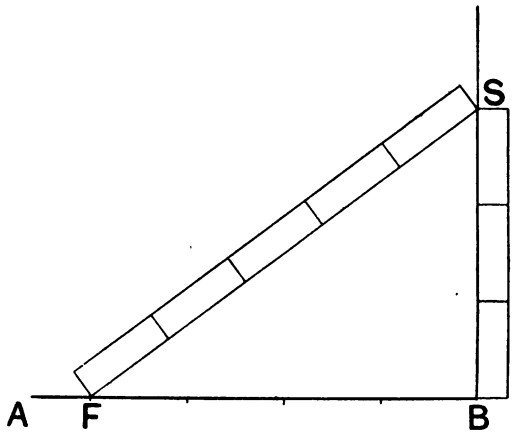


FIG. 84.

the figure. In example 1 above, we have a useful series, viz., 3, 4, 5, which can be used to set out a right-angle.

Mark off from B, Fig. 84, along a straight line AB the point F 4 ft.; place at B one end of a rod 3 ft. long, and at F one end of a rod 5 ft. long; the other ends of the rods will meet in S, a point in the perpendicular from B.

There are other sets of three integers which form the sides of right-angled triangles, but in many of these two of the dimensions are almost equal, whilst the third is very much smaller, and consequently are not so convenient in practice as the series 3, 4, 5. If larger numbers are

desired, it is only necessary to multiply each number by some other number; we then obtain a new series as convenient and well-proportioned as the first set.

Thus, multiply by 2 and we have sides 6 . 8 . 10
" 3 " " 9 . 12 . 15
" 4 " " 12 . 16 . 20

and so on.

86. **Practical applications.** *Fig.* 85 *is a dimensioned sketch of a portion of a hipped roof; the span of the roof, and the height of the ridge above the level of supporting walls are given by figures; the slope of the end portion is given in degrees. Determine* (*a*) *the length of the common rafters for the main roof and the bevels to which they must be cut to make them fit against the ridge at top, and*

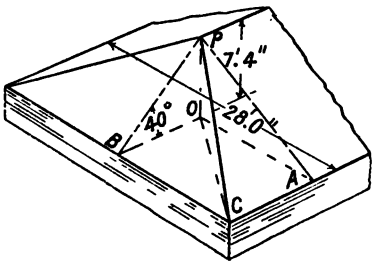


FIG. 85.

the top of supporting wall at the bottom (i.e., *the length of the hypotenuse of the triangle having* OA, OP *for legs, and the angles at* P *and* A); (*b*) *the length of the rafter* BP; (*c*) *the length of the hip rafter* CP.

(1) **Graphical method.** Select a convenient scale and construct the right-angled triangle POA, making PO = the given length 7 ft. 4 in., and OA = ½ of 28 ft. = 14 ft. by problem 81 (ii.).

Then measure the hypotenuse and the angles it makes with the sides OP, OA.

To find the length of BP, construct the right-angled triangle having one side = PO and the angle OPB = (90 — 40) = 50°.

To find the length of the hip CP, first find the length of OC, which can be determined as the hypotenuse of a right-angled triangle having legs respectively equal to AO, BO. With OC and OP known, the length of CP is found in the manner explained in Art. 81 (ii.).

(2) **By calculation.**

$$AP = \sqrt{OP^2 + OA^2}$$
$$= \sqrt{88^2 + 168^2} \text{ in.}$$
$$= 15 \text{ ft. } 10 \text{ in.}$$
$$BP = \frac{OP}{\sin. 40^\circ}$$
$$= \frac{88}{0{\cdot}643}$$
$$= 11 \text{ ft. } 5 \text{ in.}$$
$$CP = \sqrt{OP^2 + OC^2}$$
$$= \sqrt{OP^2 + (OA^2 + AC^2)}$$
$$= \sqrt{OP^2 + (OA^2 + OB^2)}$$
$$= \sqrt{OP^2 + \{OA^2 + (BP^2 - OP^2)\}}$$
$$= \sqrt{OA^2 + BP^2}$$
$$= \sqrt{168^2 + 137^2} \text{ in.}$$
$$= 18 \text{ ft. } 0\tfrac{1}{2} \text{ in.}$$

Exercises.

(1) The height of a wall is 20 ft.; required the length of a ladder which will reach to the top, when the foot of the ladder is 5 ft. from the wall. Draw a figure and measure the ladder and its inclination to the ground.

Answer. 20·6 ft.; 80·5°.

(2) A flagstaff on a tower is kept in a vertical position by three wire cable stays fastened to the staff at a height of 18 ft. above the

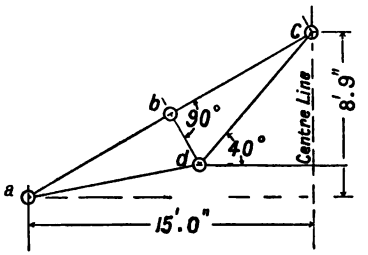


FIG. 86.

level of parapet walls to which the cables are secured. The horizontal distances of the staff from points convenient for securing the cables are respectively 12 ft., 17 ft., and 9 ft. What is the length of each stay?

Answer. 21 ft. 7½ in.; 24 ft. 9 in.; 20 ft. 3 in.

(3) Fig. 86 is a line diagram of one-half of a roof truss which is

symmetrical about the centre line (shown dotted). The small circles indicate the positions of pin joints connecting different bars in the framework. The bars *ab*, *bc* are equal in length. Set out this truss to scale and measure the length of each bar from centre to centre of the pins. Also calculate the length of each bar, making use of the figured data given on the diagram.

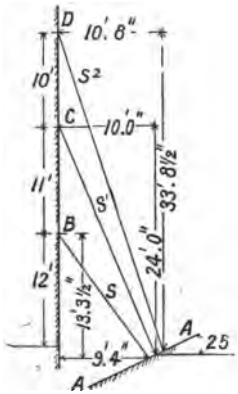


FIG. 87.

Answer. $ab = bc = 8{\cdot}65$ft.; $da = dc = 8{\cdot}8$ ft.; $bd = 1{\cdot}65$ ft.

(4) Fig. 87 shows the centre lines of a system of raking shores supporting a wall BCD. The heights of the needles BCD against which the heads of the shores abut are shown by figures, and the position of the foot of each shore on the sole-piece AA is also given by figures.

What length timbers are required for the three shores S, S^1, S^2?

Answer. 16 ft. 3 in.; 26 ft. 0 in.; 35 ft. 4 in.

Set out the shores above to scale and measure the bevels to which the head and foot of each shore must be cut to fit the wall and sole-piece AA.

(5) A tower stands on a rock; a man on the sea at 100 yds. from the foot of the rock finds the angle of elevation of the foot of the tower 15°; he then rows 100 yds. further off and finds the elevation of the top of the tower 15°. Find the heights of the rock and of the tower.

The term **angle of elevation** is applied to the angle through which a horizontal line level with the spectator's eye must be rotated upwards to come into line with the elevated object.

Answer. 26 yds. 2 ft. 3 in.; 53 yds. 1 ft. 6 in.

(6) At the top of a mountain, the angle of depression of the peak of a neighbouring mountain is 5°; if the difference of the heights of the two peaks be known to be 500 ft., find the distance from the centre of the base of one mountain to that of the other. Scale ½ in. = 500 ft.

The term **angle of depression** is applied to the angle through which a horizontal line level with the spectator's eye must be rotated downwards to come into line with the object below.

Answer. 5,715 ft.

(7) From two points A, B on a line of earthworks, the bearings of a gun are respectively N. 16° E. If B is 2,000 yds. from A in direction E. S. E., how far is the gun from A?

Answer. 6,700 yds.

CHAPTER VII.

CONSTRUCTION OF TRIANGLES FROM GIVEN DATA.

87. A triangle can in general be drawn to satisfy three geometrical conditions. The given conditions, or data, must comprise three *independent elements.*

Only two angles of a triangle can be chosen arbitrarily, since the three angles added together = 180°. If we are

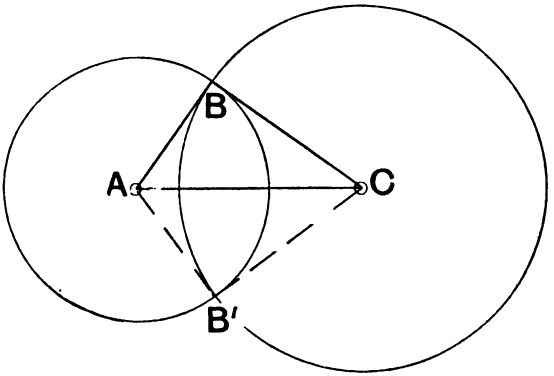


FIG. 88.

asked to draw a triangle ABC having given angle A = 46°; angle B = 64°; angle C = 70°, this problem is possible, because 46° + 64° + 70° = 180°. But we find we can draw innumerable triangles having angles of the given magnitude, and some other condition is necessary to make the problem definite.

Sometimes the given data is sufficient to fix the size of the figure and the position of one side, and yet not wholly fix its position, consequently more than one figure can be found. Fig. 88 is an example of this type.

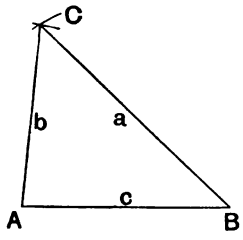


FIG. 89.

Again, three independent magnitudes chosen arbitrarily may, and sometimes do, constitute a problem to which two or more solutions can be given. Such problems are called *ambiguous*. Examples of this type will be found in problems 82 and 88.

We will now consider some of the conditions which suffice to determine a triangle. We denote the sides of the triangle by a small italic letter corresponding to the capital letter denoting the angle opposite. Thus, a denotes the side BC which is opposite the angle A. And the three sides of a triangle ABC are denoted by the letters a, b, c.

Experiment. Fig. 88. Draw any triangle ABC. With centres A and C describe circles through the corner B. These circles intersect in a second point B′; and evidently AB′ = AB, CB′ = CB, *i.e.*, the triangles AB′C, ABC are congruent. We learn from this diagram two things, viz., (i.) *a triangle is given in magnitude when its three sides are given;* (ii.) *how to construct a triangle when its three sides are given.*

88. **Problem.** Fig. 89. *Construct a* △ *when given a, b, c, say* 4 *in.*, 2·8 *in.*, 3·2 *in.* Draw one side, say, AB, of given length c = 3·2 in. With centre A, and radius b = 2·8 in., describe an arc. With centre B and radius a = 4 in. describe an arc cutting the arc previously drawn in C. Join CA, CB.

The above problem is always possible if the sum of any two sides is greater than the third side.

89. **Problem.** To copy a given triangle in a new position. This may be done as explained in the last problem, but a much better way is to lay a piece of tracing-paper over the triangle, and mark on this tracing-paper the positions of the corners of the triangle; then lay the tracing in the desired position and prick through the corners, and join up the points thus found. Adopt this method for copying any rectilinear figure.

90. If we mark the positions of the three corners of a triangle ABC on a piece of tracing-paper, as explained in the last problem, and join AB, AC, we see that these two sides completely fix the size of the triangle. From this we conclude that a triangle is given in magnitude if two sides and the included angle are given. It is also evident that one side and two angles suffice to fix the magnitude of a triangle. When two sides and an angle opposite one side are given the triangle is determined, except when the given angle is opposite the smaller of the given sides, in which case two triangles can be drawn.

To illustrate these cases, we show in the following problems how to construct triangles from given data :—

91. **Problem.** Fig. 90. Given two sides a, b and the included angle C. Let $a = 3$ in., $b = 4{\cdot}2$ in., $C = 55°$.

Draw side BC of given length $a = 3$ in. From C draw side CA of given length $b = 4{\cdot}2$ in., and making $C° = 55°$ with BC. Join BA.

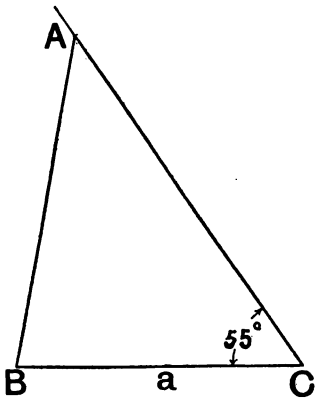


FIG. 90.

92. **Problem.** Fig. 91. Given two sides a, b and an angle B opposite one side. Let a, b, $B = 2{\cdot}5$ in., $2{\cdot}2$ in., $45°$ respectively.

Draw side BC of given length $a = 2{\cdot}5$ in. From B draw a straight

line making B° = 45° with CB. With C as centre and radius = b (2·2 in.) describe a circle. In the diagram the circle intersects the line from B in two points; either point may be taken as vertex A; and there are thus two solutions.

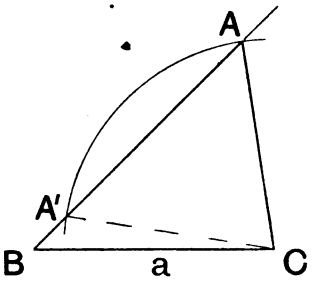


FIG. 91.

If side b be reduced so that the circle centre C only *touches* the line from B, there will be but one solution. And if b be further reduced, the line from B will have no point in common with the circle, and in this event the problem is impossible.

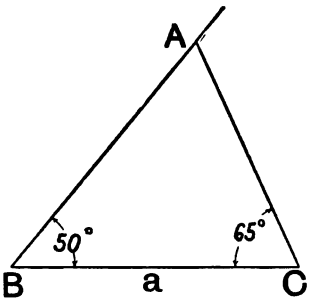


FIG. 92.

Observe, also, that when a is less than b the circle of radius b intersects the line from B in one point only on the same side of BC, and there is but one solution.

93. **Problem.** Fig. 92. Given one side a and the two adjacent angles B, C. Let a, B, C = 3 in., 50°, 65° respectively.

Draw side BC of given length $a = 3$ in. From B, C draw lines making the given angles B°, C° with BC intersecting in A.

94. **Problem.** Fig. 93. Given one side a, the opposite angle A and one adjacent angle C. Let a, A, C = 2·5 in., 50°, 70° respectively.

In this case, since the three angles B + C + A = 180°

$\therefore$ angle B = 180 − (A + C)
= 180 − (50° + 70°)
= 60°.

With the angle A known, the problem is reduced to that immediately preceding.

Or, proceed as follows:—

Draw side BC of given length $a = 2{\cdot}5$ in. From C draw CS making C° = 70° with BC. From any point S in this line draw SR making A° = 50° with CS. Draw BA ∥ RS meeting CS in the point A.

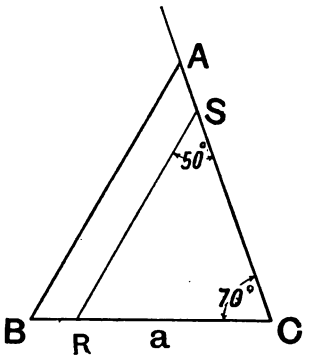


FIG. 93.

95. **Problem.** Fig. 94. Given the base a, one base angle C and the height h. Let a, C, h = 2·5 in., 42°, 1·7 in. respectively.

Draw base BC of given length $a = 2{\cdot}5$ in. From C draw CA making C = 42° with BC. Draw a parallel to BC at given distance $h = 1{\cdot}7$ in. above same, meeting CA in A. Join AB.

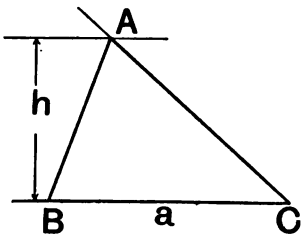


FIG. 94.

96. **Problem.** Fig. 95. Given the base a, the angle opposite A, and the height h. Let a, A, and h = 2·5 in., 70°, 1·7 in. respectively.

From Art. 142 the locus of the vertex A is seen to be a segment of the circle which stands on the given base a and contains the given angle A. It has also a second locus parallel to the base at the given distance h above the base.

These two loci may intersect in two points, in one point, or not at all.

Draw base BC of given length $a = 2{\cdot}5$ in. Describe a segment of a circle on BC to contain the given angle A = 70° by problem 149.

Draw a line parallel to, and at the given distance $h = 1{\cdot}7$ in., above, BC.

In the diagram, this line cuts the circle in two points A, A′, either of which may be taken as the vertex of the required triangle.

Had the line been tangent to the circle there would have been but

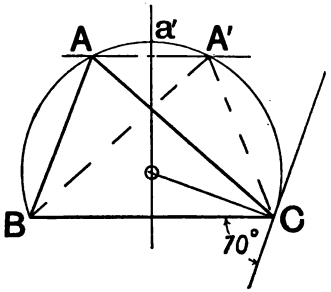


FIG. 95.

one solution, *i.e.*, the triangle a'BC, which is *isosceles*. Again, if the two loci have no point in common, there is evidently *no solution*.

97. **Experiment.** Fig. 96. Draw any straight line and on it take four points A^1, B, C, A^2. With centre B, and radius BA^1, describe

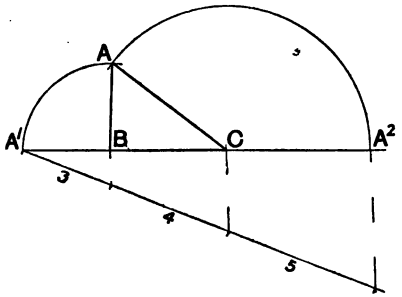


FIG. 96.

a circle. With centre C, and radius CA^2, describe a circle meeting the circle centre B in A. Join AB, AC. Then the perimeter of the triangle ABC is equal to A^1A^2. We have thus available a construction for drawing a triangle when given its perimeter and the ratio of its sides.

Problem. Fig. 96. The perimeter of a triangle is 5·5 in.; the sides c, a, b are in the ratio of 3 : 4 : 5. Lay off on a straight line the

perimeter A^1A^2 of given length = 5·5 in. Divide A^1A^2 at B and C in the given ratio (see prob. 36), and complete as above.

98. **Problem.** *Given one side* (*a*), *one angle adjacent to that side, and* (i.) *the sum,* (ii.) *the difference, of the other two sides.*

(i.) Fig. 97 (a). Let BC be the given side a, and β the given angle at B. Make BA′ = the sum of b and c. Join A′C and draw its $\perp^r$ bisector which intersects BA′ in A. Join AC; then ABC is the required triangle, for, ACA′ is isosceles ∴ AC = AA′; and BA + AC = BA′.

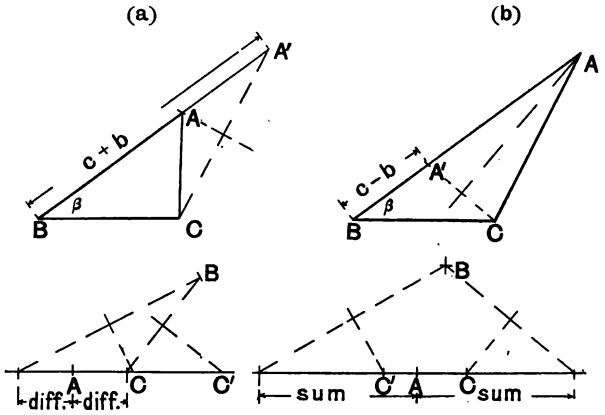


FIG. 97.

(ii.) Fig. 97 (b). Let BC be the given side a, and β the given angle at B. Make BA′ = the difference of b and c. Join A′C and draw its $\perp^r$ bisector which intersects BA′ prolonged in A. Join AC; then ABC is the required triangle, for, ACA′ is isosceles ∴ AB − AC = BA′.

Problem. *Given a point* A *in a straight line and an external point* B. *Determine a point* C *in the line such that* (i.) AC + BC, (ii.) AC − BC, *shall be equal to a given length.*

Solutions are given in Fig. 97 (c) and (d) respectively.

There are two points, C, C′, *which satisfy the conditions.*

99. **Area of a Triangle.** When the base and height are given, the area = ½ base × height, or ½ *bh*.

As the height $h = c \times \sin. A$, the area = ½ *bc* sin. A, or generally, area = ½ product of two sides and the sine of included angle (as *b*, *c*, A).

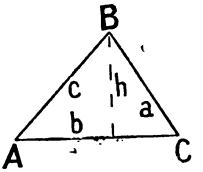


FIG. 98.

When the three sides are given, the area is found from the formula.

$$\text{Area} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}, \text{ in which}$$

$$s = \frac{a+b+c}{2}, \text{ or half the sum of the three sides.}$$

Ex. *Find the area of a triangle having sides* 12, 10, *and* 8 *ft.*

$$s = \frac{12+10+8}{2} = 15. \qquad \begin{matrix}(s-a) = (15-12) = 3\\ (s-b) = (15-10) = 5\\ (s-c) = (15-8) = 7.\end{matrix}$$

$$\therefore \text{ Area} = \sqrt{15 \times 3 \times 5 \times 7} = \sqrt{1575} = 39{\cdot}7 \text{ sq. ft.}$$

Exercises.

(1) A triangle has sides respectively 3, 5, and 6 in. long. Construct the triangle, measure each angle in degrees, find the length of the perpendicular falling on the longest side, and calculate the area.

(*Ans.* Angles 29°, 56°, 95°; perp., 2·5 in.; area, 7·5 sq. in.)

(2) Construct a triangle having base 2 in. long, height 1¾ in., and one base angle 75°. Measure and write down the lengths of the other two sides, and calculate the area. *Ans.* 2·32 in.; 1·82 in.; 1·75 sq. in.

(3) Draw a triangle ABC, base AB = 3 in., AC = 2·5 in., BC = 2·75 in. Measure the height of the triangle and calculate its area; also measure the angle ABC and take out its sine from the tables. Calculate the area of the triangle making use of the formula,

$$\text{area} = \tfrac{1}{2}\,AB \,.\, AC \sin. BAC.$$

Find the mean of the two answers.

(4) In any triangle, if A, B, C be the angles, and *a*, *b*, *c* the opposite sides, then,

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Calculate cos. C when *a* = 6 cm., *b* = 5 cm., *c* = 4 cm.

Now verify the formula (and your answer) by construction, measurement, and, if you like, the use of the tables. (B.E. 1910.)

(5) A person, 6 ft. high, walking along a road with the sun directly behind him, observes that at a place where the road is level his shadow measures 7½ ft., and at a place where the road has an upward gradient his shadow measures 5¼ ft. What is the angle which the sun's rays make with the horizontal? What is the inclination of the road? (B.E. 1910.)

(*Ans.* 38°, 40′; 24°, 32′.)

(6) Calculate the area of a triangle which has sides 12, 10, and 8 ft. long. *Ans.* 39·7.

CHAPTER VIII.

RECTILINEAR FIGURES OF MORE THAN THREE SIDES.

100. **Quadrilaterals.** Any figure enclosed by four straight lines is called a **quadrilateral.**

A quadrilateral in which two sides only are parallel is called a **trapezium.**

A quadrilateral having pairs of adjacent sides equal is called a **kite.**

A quadrilateral having its opposite sides parallel is a **parallelogram.**

It is evident from Expt. Art. 23, that the opposite angles

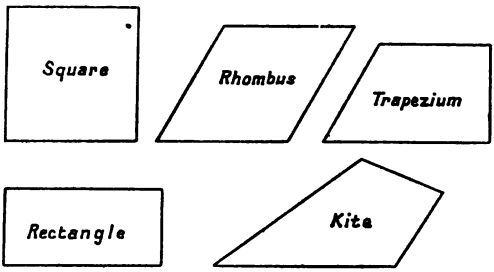


FIG. 99.

in a parallelogram are equal, and any two consecutive angles are supplementary. Hence, *if one angle in a parallelogram is a right-angle, then every angle is a right-angle.*

If the angles in a quadrilateral are right-angles, the figure is a **rectangle.**

If all sides of a rectangle are equal, the figure is a **square.**

A parallelogram having all sides equal, but not rectangular, is called a **rhombus.**

Some Properties of Parallelograms.

101. **Experiment.** Fig. 100. Draw any parallelogram ABCD. Draw a diagonal DB. Then, because AB = CD and AD = CB, and the angles at A, C are equal, the triangles DAB, BCD are congruent.

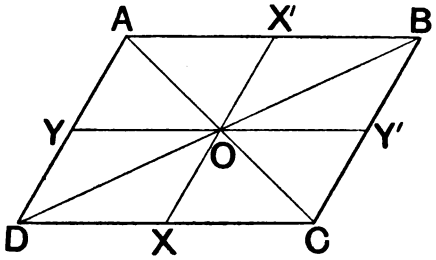


FIG. 100.

This may be verified with tracing-paper. Draw the diagonal AC, and satisfy yourself that this line also divides the parallelogram into two equal parts. The two diagonals meet in O. Verify also that

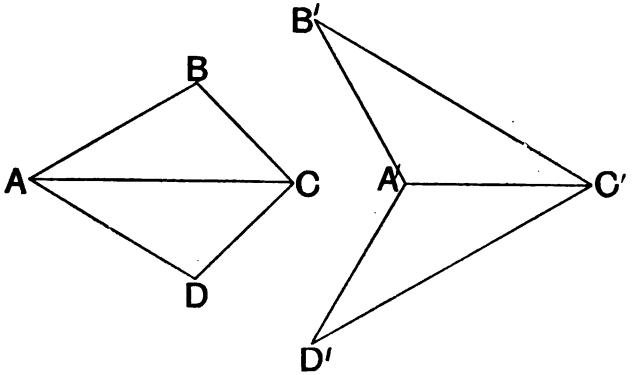


FIG. 101.

the two diagonals are bisected in O, and that the triangles AOB COD are congruent; also AOD, COB are congruent. Next, through O, draw XX′, YY′, respectively, parallel to AD, AB. Show that XX′, YY′, divide ABCD into four parallelograms exactly alike. Also verify the following statements: The triangles YAO, Y′CO, X′OA, XOC are congruent, also YDO, Y′BO, XOD, X′OB are congruent.

Draw any *rhombus*, a rectangle, a square, and apply the tests described above. Also verify—

(1) In a *rhombus*, and a square, the diagonals are at right-angles, and each diagonal is an axis of symmetry.

(2) A circle can be inscribed in a *rhombus*.

(3) The four corners of a rectangle are on the circumference of a circle which has its centre in the meeting point of diagonals, *i.e.*, a rectangle can be inscribed in a circle.

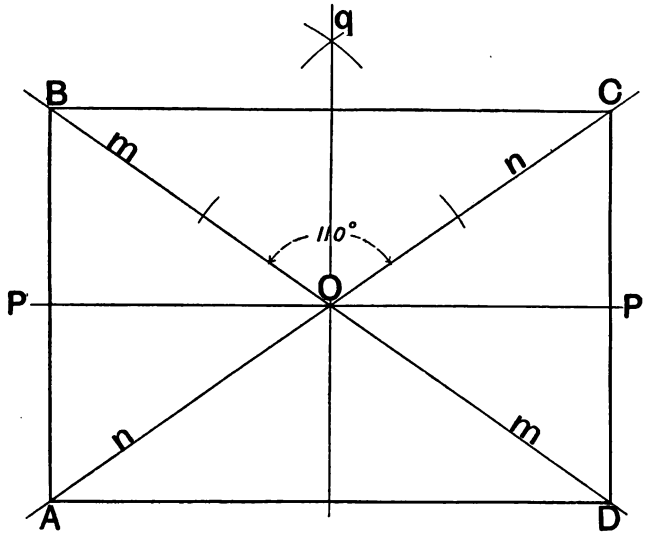


FIG. 102.

(4) A square can be inscribed and circumscribed by circles centered in the meeting point of diagonals.

102. **Experiment.** Fig. 101. Draw two figures like ABCD, A′B′C′D′, using set-squares of 45°, 60°, and 30° to do this. Then show that the diagonal AC (A′C′) is an *axis of symmetry*.

The Construction of Parallelograms, the Kite, Trapeziums, and Regular Polygons.

103. Consider a square and a rectangle. Evidently a rectangle can be drawn when we know (*a*) the lengths of its adjacent sides; (*b*) one side and a diagonal; (*c*) one side and the angle between the diagonals. The data under (*a*) and (*b*) above involve the construction of right-angled triangles only, and may be solved as in previous work. To illustrate case (*c*), we give an example.

Problem. The longer side of a rectangle is 3 in., the diagonals include an angle of 110°. Construct the figure.

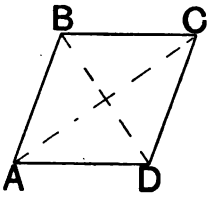


FIG. 103.

Draw On, Om, Fig. 102, making the given angle 110°. Draw Oq, the bisector of angle mOn. Draw P'OP perpendicular to Oq, and make OP, OP' equal half length of the given side, *i.e.*, $1\frac{1}{2}$ in. Through PP' draw parallels to Oq meeting Om, On, in AB, CD. Then ABCD is the required rectangle.

103*a*. Consider the rhombus, Fig. 103. A rhombus can be drawn if we know (*a*) the lengths of its diagonals; (*b*) one diagonal and the angle opposite; (*c*) length of side and one angle.

In case (*a*), the diagonals bisect each other at right-angles.

In (*b*), place two isosceles triangles base to base; the bases of the triangles being equal to the given diagonal of rhombus, and apex angle the given angle.

In (*c*), place two isosceles triangles base to base, the equal sides of

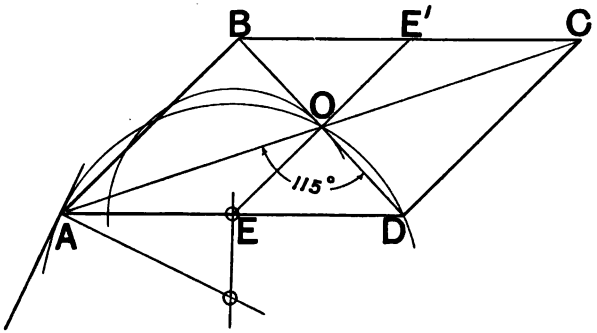


FIG. 104.

the triangles being equal to the given side of rhombus, and apex angle the given angle.

Consider the kite, Fig. 101. A kite can be drawn if we know (*a*) the two diagonals and one side; (*b*) two sides and one diagonal; (*c*) two sides and the angle between them.

104. Consider the parallelogram ABCD, Fig. 104. This figure can be drawn when we know—

(*a*) two sides and the included angle;
(*b*) two diagonals and angle between them;
(*c*) a diagonal, the angle opposite, and one side;
(*d*) two sides and the angle between the diagonals;

(*e*) one side, one diagonal, and angle between the diagonals.

Examples. In cases (*a*), (*b*), *and* (*c*), *apply the solution in problems* 91 *and* 92 *respectively.*

Case (*d*).—Construct a parallelogram ABCD, Fig. 104, on the given side AD, such that the angle AOD between the diagonals shall measure 115° and the side AB shall be $\frac{5}{7}$AD.

Let ABCD be a figure satisfying the given condition. Then, since the angle AOD = 115°, the point O lies on the segment which contains 115° and stands on the chord AD.

Draw OE parallel to BA. Then

AE = $\frac{1}{2}$AD, and OE = $\frac{1}{2}$AB = $\frac{5}{7}$AE. Hence the following construction.

Bisect AD in E.

On AD describe a circular segment to contain the given angle 115° by problem 149. With E as centre, and radius equal to $\frac{5}{7}$AE, describe a circle intersecting the arc on AD in O. Then O is the centre of the parallelogram.

Draw EO and produce to E′ making OE′ = OE. Through A, D draw parallels to EO. Through E′ draw parallel to AD meeting the lines from A, D in B, C.

Case (*e*). No figure. Construct a parallelogram ABCD having given side AB = 3 in., diagonal AC = 4 in., and the angle between diagonals opposite AB = 110°.

A slight modification of the previous construction will solve this problem.

Draw side AB of given length 3 in. On AB as a chord construct a circular segment to contain the angle 110°. With A as centre and radius equals $\frac{1}{2}$ given diagonal, viz., 2 in., describe arc to intersect the arc on AB in O. Draw AO and produce to C, making OC = OA. Draw CD parallel to BA. Join CB and draw AD parallel to BC.

105. The **area of a parallelogram** is the product of base and altitude; symbolically A = bh where A = area, b = base, and h = altitude of the Fig. Again if a and b represent adjacent sides of the parallelogram and θ the angle between them then A = ab sin. θ. The **area of a rhombus** may be found, if the lengths of the diagonals are known.

In a rhombus, the diagonals bisect each other at right-angles. See Fig. 103.

Hence the area of rhombus ABCD = sum of areas of triangles ABD and CBD.

Area of triangle ABD = $\frac{1}{2}$BD × AE (since AE is ⊥ to BD).

Similarly, area of triangle BCD = $\frac{1}{2}$BD × CE.

Hence area of rhombus $= (\frac{1}{2}BD \times AE) + (\frac{1}{2}BD \times CE)$
$= \frac{1}{2}BD\,(AE + CE)$
$= \frac{1}{2}BD \times AC$

or, *area of rhombus* $= \frac{1}{2}$ *product of diagonals.*

Exercise. The diagonals of a parallelogram are respectively 4 and 3 in. long, and one side is $1\frac{1}{2}$ in. long. Construct the figure and find its area.

Ans. 5 sq. in.

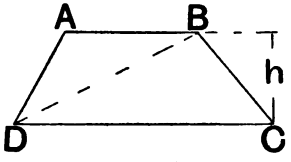


FIG. 105.

106. **Area of a trapezium** ABCD, Fig. 105. Divide it into two triangles ABD, DCB, by drawing a diagonal BD.

The height of each triangle $= h =$ height of the trapezium.

Area of triangle DCB $= \frac{1}{2}$ DC $\times h$.
Area of triangle ABD $= \frac{1}{2}$ AB $\times h$.
Then total area of trapezium $= (\frac{1}{2}DC \times h) + (\frac{1}{2}AB \times h)$.
$= \frac{1}{2}(DC + AB)h$.
$= \frac{1}{2}$ sum of parallel sides $\times$ perpendicular distance between them.

Ex. In a trapezium, the parallel sides are 10 *and* 8 *ft. long respectively, and they are* 5 *ft. apart. Find its area.*

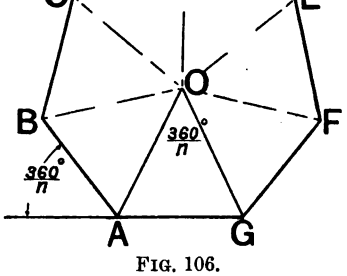


FIG. 106.

Area $= \frac{1}{2}$ sum of parallel sides $\times$ perpendicular distance between them.
$= \frac{1}{2}(10 + 8) \times 5$.
$= 9 \times 5$.
$= 45$ sq. ft.

107. Consider the regular polygon ABCDEFG, Fig. 106. If we join the angular points to the centre of the figure, these lines divide the angle at the centre O into a number of equal parts. Consequently, the angle between the lines radiating to consecutive corners of the polygon is equal to $\frac{360}{n}$, where n = the number of sides. In our example, the angle AOB $= \frac{360}{7} = 51{\cdot}428°$.

Now we know from Art. 79 that the exterior angle of any regular polygon of n sides is equal to $\frac{360}{n}$, and *therefore* the angle subtended at the centre by the side of a

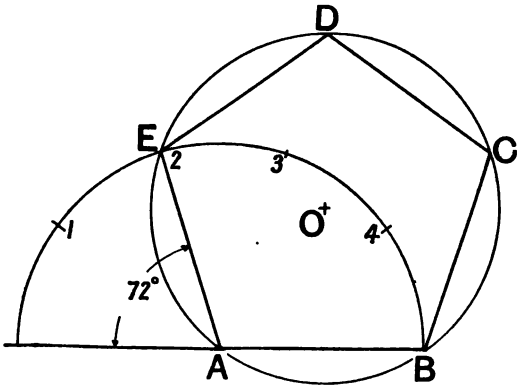


FIG. 107.

regular polygon is equal to each of the exterior angles of the polygon.

108. **Problem.** *Fig.* 107. *Construct a regular polygon of* 1·75 *in. side, say, a pentagon.*

Lay off AB given length 1·75 in. on a straight line. At A set out an angle of $\frac{360}{5} = 72°$ (making use of a protractor). Make AE = AB. Describe the circle through the points A, B, E, and determine points, D, C by stepping round the circle.

The arithmetic and protractor may be dispensed with and the problem solved entirely by a graphic method thus: On the line containing the given side AB describe a semicircle having AB for radius. Divide the semicircle into the same number of equal parts as there are sides in the given polygon—5 in this case—and number them as shown in the diagram. From A draw a line to the second division counting from the end of the diameter remote from B. Then A—2 is a second side of the polygon, which can be completed as explained above.

109. The area of *any* polygon is found by dividing the figure into a number of triangles, finding the area of each triangle separately, and adding all the areas together.

In the case of a *regular* polygon of n sides, we can find the centre of the figure and join to each corner. We have then a number (n) of triangles all equal in area, and the area of the polygon = the area of one triangle $\times$ n.

Ex. To find the area of the regular pentagon ABCDE, *Fig.* 107. Determine (O) the centre of the polygon. Join OB, OA, etc. Then the area of polygon = 5 $\times$ area of $\triangle$ OAB.

The **area of a regular hexagon** can be shown to be equal to $2{\cdot}598 \times (\text{length of side})^2$.

Examples.

(1) *A room is* 15 *ft. long,* 12 *ft. wide, and* 9 *ft. high. It has a door* 6 *ft. high and* 3 *ft.* 6 *in. wide, and a window* 8 *ft. long and* 4 *ft.* 6 *in. high. Find the cost of distempering the walls at* 1*d. per square foot; also cost of papering the walls with paper* 27 *in. wide at* 1*s.* 6*d. per dozen yards. Find also the length of carpet* 2 *ft. wide, which would be necessary to cover the floor.*

The area to be distempered is the area of the four walls less the area of door and window.

Area of four walls $= 2\{(15 \times 9) + (12 \times 9)\} = 486$ sq. ft.
Area of door $= (6 \times 3{\cdot}5) = 21$ sq. ft.
Area of window $= (8 \times 4{\cdot}5) = 36$ sq. ft.
$\therefore$ Area to be distempered $= \{486 - (21 + 36)\} = 429$ sq. ft.
Cost at 1*d.* per sq. foot $= 429$ pence
$= £1\ 15s.\ 9d.$

The area to be papered is, as the area to be distempered, 429 sq. ft.

To find the number of yards of paper 27 in. wide required, we note that 1 yd. of such paper covers an area 3 ft. by $2\frac{1}{4}$ ft. or $6\frac{3}{4}$ sq. ft.

Hence number of yards required is $\frac{429}{6\frac{3}{4}} = \frac{429 \times 4}{27}$

The number of "dozens" of yards is $\frac{429 \times 4}{27 \times 12}$

The cost at 1*s.* 6*d.* per dozen $= \frac{429 \times 4}{27 \times 12} \times \frac{3}{2}$ shillings

$$\frac{143}{18} = 8 \text{ shillings (approximately).}$$

Area of floor to be covered by carpet is $(15 \times 12) = 180$ sq. ft. Carpet is 2 ft. wide.

One yard length of carpet thus covers $2 \times 3 = 6$ sq. ft. of floor.

Hence number of yards required is $\frac{180}{6} = 30$ yds.

(2) *A cistern* 10 *ft. long*, 6 *ft. wide, and* 3 *ft. deep, is to be lined internally with lead which weighs* 5 *lb. per square foot. Find the weight of lead necessary.*

Area of two long sides of cistern $(2 \times 10 \times 3) = 60$ sq. ft.
" " " short " " " $(2 \times 6 \times 3) = 36$ sq. ft.
Area of bottom of cistern $(10 \times 6) = 60$ sq. ft.
Total area $= (60 + 36 + 60) = 156$ sq. ft.
Weight $= 156 \times 5$ lb.
$= 780$ lb.

(3) *A hexagonal plate of* 5*-ft. side has a rectangular hole* 2 *ft. by* 3 *ft. punched from it. Find the area of the remainder of the plate.*

Area of plate = area of hexagon − area of rectangle.
$= (2{\cdot}598 \times 5 \times 5) - (2 \times 3)$
$= 64{\cdot}95 - 6$
$= 58{\cdot}95$ sq. ft.

(4) *A pavement* 1 *mile long and* 8 *ft. wide is to be paved with tiles* 9 *in. by* 4 *in. How many are required?*

Total area of pavement $= (5280 \times 8)$ sq. ft.
$= (5280 \times 8 \times 144)$ sq. in.
Area of each tile $= (9 \times 4) = 36$ sq. in.
Number required $= \frac{5280 \times 8 \times 144}{36} = 168{,}960.$

The Construction of Quadrilaterals and Irregular Polygons.

110. Consider the quadrilateral ACDB, Fig. 108. If we divide this figure into triangles by drawing the diagonal BC, we know that three conditions are in *general* necessary to make the construction of that triangle a definite problem. Consequently we must have three conditions in order to

fix three corners of the quadrilateral in correct relative positions. Having BC in correct position, then two other conditions suffice to fix the corner D. Hence, in *general*,

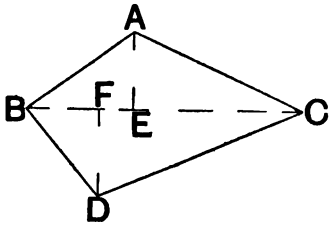


FIG. 108.

five conditions are necessary and sufficient to make the drawing of a quadrilateral a definite problem.

111. Consider now the irregular pentagon ABCDE, Fig. 109. Divide this into triangles as shown, then evidently three conditions suffice to fix the corners A, B, E; two others suffice for C, and two more for D; *i.e.*, 3 + 2 + 2 = 7 in all. Consider now a figure ABCDEFGH. Three conditions suffice to fix the corners A, B, H; two others suffice for C; two more for D; two more for E; two more for F, and two more for G; *i.e.*, 3 + 2 + 2 + 2 + 2 + 2 = 13 in all.

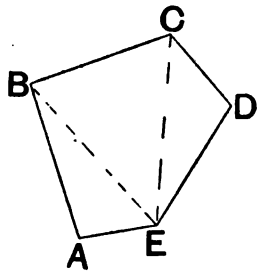


FIG. 109.

These results may be expressed in general terms, thus: *In an irregular polygon of* n *sides we must know* 2n − 3 *independent magnitudes in order to construct the figure.*

Ex. Construct a figure ABCDE *to the following dimensions*:—
AB = 4·25 cm., BC = 4 cm., AE = 5 cm., angle ABC = 110°, angle CDE = 90°, angle AED = 120°, diagonal BE = 6½ cm.

First draw the triangle ABE by Problem 88
Secondly „ „ ABC by Problem 91
Lastly „ „ CDE.

What are the lengths of ED, CD?

112. **Area of any quadrilateral ABCD**, Fig. 108. Divide it into two triangles by drawing the diagonal BC, and draw from the corners A, D the perpendiculars AE, DF. Then

$$\text{Area of triangle ABC} = \tfrac{1}{2}\text{BC} \times \text{AE}$$
$$\text{,, ,, ,, BCD} = \tfrac{1}{2}\text{BC} \times \text{FD}$$
$$\text{Total area of quadrilateral} = (\tfrac{1}{2}\text{BC} \times \text{AE}) + \tfrac{1}{2}(\text{BC} \times \text{FD})$$
$$= \tfrac{1}{2}\text{BC (AE + FD)}$$

$= \frac{1}{2}$ diagonal $\times$ sum of perpendiculars from opposite corners upon the diagonal.

Ex. In a quadrilateral, the diagonal is 12 ft. long, and the lengths of perpendiculars from the opposite corners of the parallelogram upon this diagonal are 5 ft. and 8 ft. respectively. Find the area.

$$\text{Area} = \tfrac{1}{2} \text{ diagonal} \times \text{sum of perpendiculars}$$
$$= \tfrac{1}{2} \times 12 \times (5 + 8)$$
$$= 6 \times 13$$
$$= 78 \text{ sq. ft.}$$

The area of a quadrilateral may be expressed in terms of the diagonals and the angle included between them. Thus if d_1, and d_2 represent the diagonals, and θ the included angle, the area $A = \frac{1}{2} d_1 d_2 \sin. \theta$.

Ex. The diagonals of a field are 17·2 and 24·3 chains, the angle included between them is 40°. What is the area of the field in acres?

From the formula $A = \frac{1}{2} d_1 d_2 \sin \theta$, we have

$$A = \frac{17{\cdot}2 \times 24{\cdot}3}{2} \times \sin. 40°$$
$$= \frac{17{\cdot}2 \times 24{\cdot}3}{2} \times {\cdot}6428$$
$$= \frac{417{\cdot}96}{2} \times {\cdot}6428$$
$$= 134{\cdot}332 \text{ sq. chains}$$
$$= 13{\cdot}43 \text{ acres.}$$

113. **The area of an irregular polygon** is found by dividing the figure into triangles, as shown in Fig. 109, and adding the areas of these triangles together.

An irregular polygon may be reduced to an equivalent triangle by the construction of Art. 159 and the area of the triangle found by measuring its base and height.

Exercises.

NOTE. *The diagrams used in the following exercises should be re-drawn to twice their lineal dimensions. They are on p.* 108.

(1) A rectangular grass plot, the sides of which are as 2 : 3, cost £14 8*s.* for turfing, at the rate of 4*d.* per sq. yd. Find the lengths of its sides.

Answer. 24 and 36 yds.

(2) A room 40 ft. long, 22 ft. wide, and 15 ft. high, has 4 windows in one side, each occupying a space 7 ft. wide and 13 ft. high; 2 doors, each 4 ft. 9 ins. wide and 8 ft. high; and 2 fire-places, each 5 ft. 8 ins. wide and 4 ft. 6 ins. high. Find the number of square yards of painting required for the walls; deducting 1 ft. for the height of the cornice, and 15 ins. for the skirting-board.

Answer. 1,089 sq. yds. 1 sq. ft.

(3) Construct the polygon from the particulars in Q. 3 and carefully measure and write down the distance from D to E. Scale $\frac{1}{10}$ in. = 1 ft. What is the representative fraction of this scale? How many figures can be drawn to satisfy the data?

(4) A trench is dug of the shape and size shown in Q. 4. The scale of the figure being 1 cm. to 1 ft. Find the cross-sectional area of the trench in square feet.

(5) The plans of two halls are given (Q. 5, Q. 7), scale $\frac{1}{2}$ in. to 10 ft.; L is the platform and K the body of the hall. Find the area of K in square feet. Calculate the seating accommodation of K, 30 per cent. of its area being occupied by passages, and allowing one person to every 4 sq. ft. of the remainder. (B.E. 1904.)

The material expands 10 per cent. in bulk, and is piled alongside the trench in an embankment of triangular section with its sides sloping at 45°. Draw this embankment to scale. State its height. (B.E. 1905.)

(6) The diagram Q. 6 gives the dimensions of a plot of level land. Draw this figure to a scale of $\frac{1}{8}$ in. to 1 ft. Measure the lengths of the two diagonals in feet. Determine the area of the plot in square yards. (B.E. 1910.)

(7) Two sides of a triangle are 25 and 21·25 chains, and the included angle 45°. Find the area.

Answer. 18 ac. 3 rd. 5 sq. po.

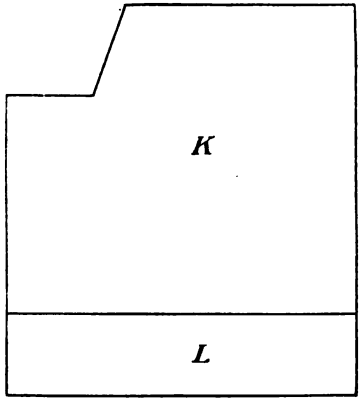
(8) The diagonals of a quadrilateral are 30 and 40 chains, and they intersect at an angle of 45°. Find the area.

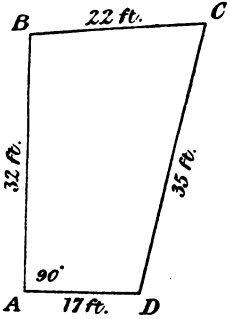
Answer. 44 ac. 2 rd. 14 sq. po.

(9) The parallel sides of a trapezoid are 7·5 and 12·25 chains, and the perpendicular is 15·4 chains. Find the area.

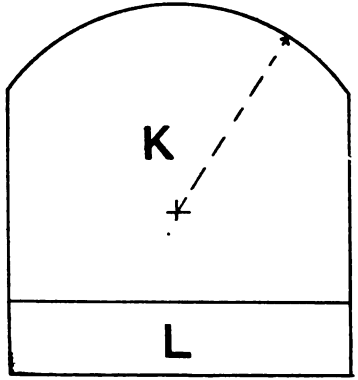
Answer. 15 ac. 33 sq. po.

Q. 7.

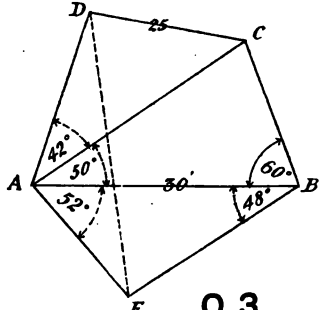

K
L


B
22 ft.
C
32 ft.
35 ft.
90°
A
17 ft.
D

Q. 6.

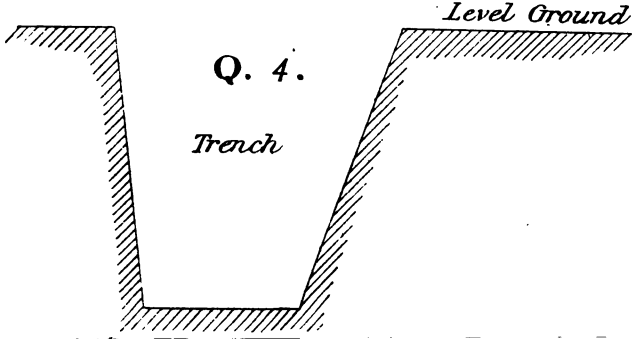

K
L

Q. 5


D
25
C
42°
50°
A
30
60°
B
52°
48°
E

Q. 3.

Q. 4.


Level Ground
Trench

CHAPTER IX.

THE CIRCLE. CHORD AND TANGENT PROPERTIES. CONSTRUCTIONS.

114. In this chapter, our attention will be confined to circles, and problems directly connected with circles.

The *perpendicular bisector vv of any chord* AB *of the circle*, Fig. 110, *is the locus of all points equidistant from* A

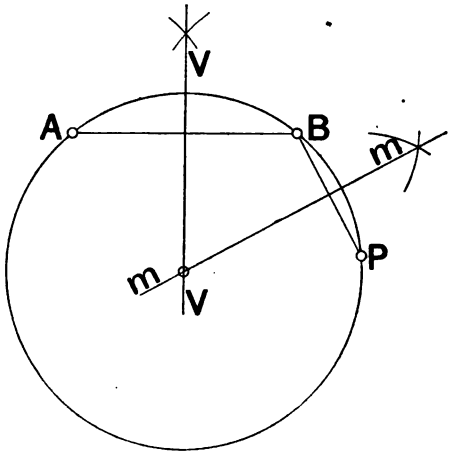


FIG. 110.

and B (Art. 30) ; *and consequently passes through the centre of every circle which contains* AB. This property provides a solution of the following :—

Problem. *Determine the centre of a circle to pass through any three given non-colinear points* A, B, P, *Fig.* 110.

Join AB, BP and draw the perpendicular bisectors *vv*, *mm* of the two lines. Then the line *vv* is the locus of all points equidistant from A and B; and the line *mm* is the locus of points equidistant from B and P. Therefore the point in which the lines *vv* and *mm* intersect, is a point equidistant from the three points A, B, P, and is the centre of the required circle.

The above problem may be set in a slightly different

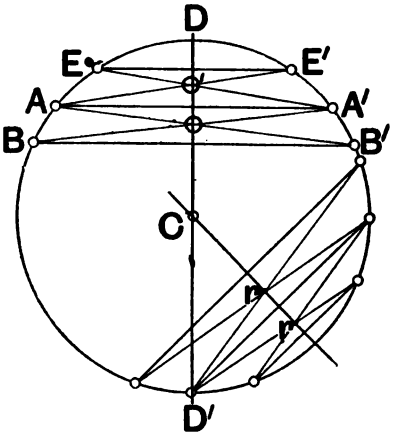


FIG. 111.

form, viz.: *Given an arc of a circle* (Fig. 110), *determine the centre of the circle.*

Select any three points, as A, B, P, on the arc, and find the centre of the circle passing through A, B, P, which is the centre required.

Exercises.

(1) Draw any triangle and draw its circumscribing circle.

(2) Draw a circle to pass through two given points A and B, and to have its centre on a given straight line.

Hint. The centre is on the perpendicular bisector of AB.

(3) Draw a circle to pass through two given points P, Q and to have its centre on a given circle.

Note. If the perpendicular bisector of PQ intersects the circle in two points, either point may be taken as the centre of the required circle. If it falls wholly without the given circle, the problem cannot be solved.

115. Fig. 111. The centre of a given circle may be found with a parallel ruler, the construction being based on a property of parallel chords, which the student can verify as

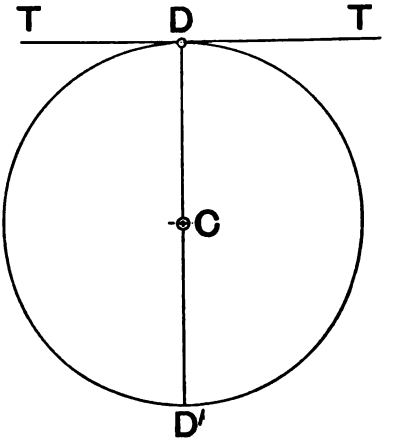


FIG. 112.

follows: Draw any circle, a diameter DD', and any two chords AA', BB', perpendicular to DD'. Draw AB', A'B, and observe that these lines intersect in a point which is on the diameter DD'. Hence, to find the centre of the circle, draw any three parallel chords as AA', BB', EE'. Join AB', A'B intersecting in O. Join A'E, AE' intersecting in O'. Then a line through OO' passes through the centre of the circle. Draw three other parallel chords, and in a similar manner find *rr* intersecting OO' in C. Then C is the required centre. This principle can be used with advantage to find the centre of a circular section of a solid

figure, *e.g.*, the centre of a steel shafting, or the position of the centre on the bed of a circular column of stone or other material.

116. Fig. 112 shows a circle centre C, a diameter DD′, and a line TT which passes through D and is perpendicular to CD. The line TT is said to be **tangent** to the circle at D; D is called the **point of contact**. DC, the perpendicular from the point of contact D, is called a **normal**.

All normals of a circle pass through the centre of the circle. This property is characteristic of the circle, and thus, *the*

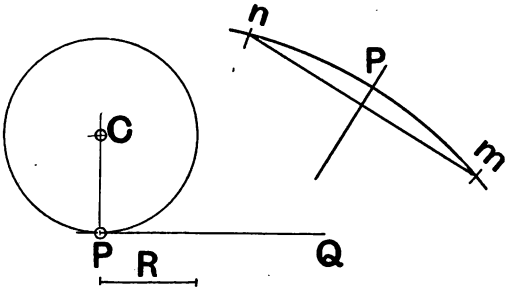


FIG. 113.

tangent at any point D *on the circumference of a circle is perpendicular to the radius* DC.

117. **Problem.** Fig. 113. *To draw the tangent at any point* P *on the circumference of a circle.*

Join P to the centre C, then draw the required tangent perpendicular to CP.

When the centre is inaccessible, as in the case of the circular arc *n*P*m*, the normal at a point P may be found thus: Mark points *m*, *n*, on the curve, one on either side of and equidistant from P; then draw the perpendicular bisector of *m*, *n*, which passes through P and the centre of the circle. See Art. 114.

When a tangent has to be drawn at a point near to an extremity of the arc, apply the construction of Art. 145.

118. **Problem.** Fig. 113. *Draw a circle of given radius* R *to touch a given straight line* PQ *in the point* P.

Draw PC perpendicular to PQ and make PC equal to R (the given radius), then C is the centre required. There are two positions of the circle, one on each side of the line.

119. **Problem.** Fig. 114. *Draw a circle to touch the given straight line m at* P *and to pass through the fixed point* Q.

Draw PO perpendicular to the given line *m*. Then the centres of all circles that touch the line *m* at P are on PO.

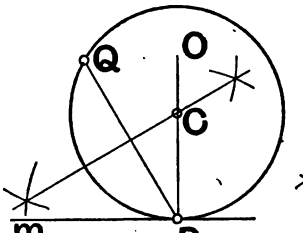


FIG. 114.

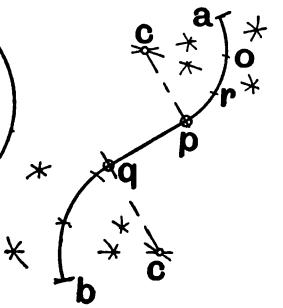


FIG. 115.

Draw the perpendicular bisector of PQ, which crosses PO in the required centre C.

120. **Problem.** Fig. 115. *A line ab is made up of two circular arcs and a straight line joined tangentially. Determine the points of contact of the arcs and the straight line.*

Select three points on the circular arc which contains the point *a*, as *a*, *o*, *r*, and find the centre *c* from which this arc is described (Problem 114). Draw a perpendicular from *c* on to the straight line. This perpendicular meets the straight line in *p*, which is the point of contact of the arc through *a*. Proceed in a similar manner to find the point *q* of contact of the straight line and the arc containing the point *b*.

121. From any given external point O, Fig. 116, two tangents can always be drawn to a circle. When the centre of the given circle is accessible, the points of contact of the

tangents from the given point O are readily found by joining O to the centre C, and describing a circle on OC as diameter

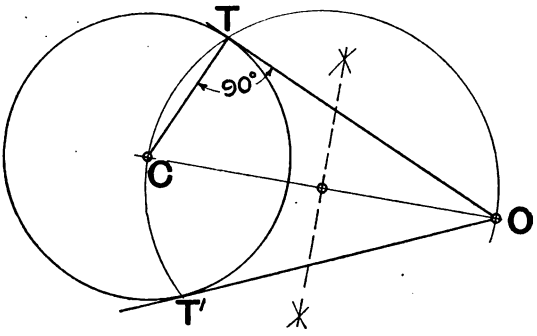


FIG. 116.

this circle intersects the given circle in two points T and T′ which are the points of contact of the respective tangents.

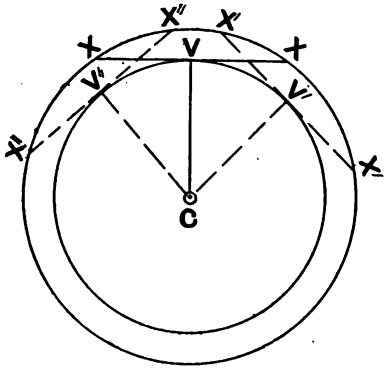


FIG. 117.

For CT is the normal of T, and the angle CTO is a right-angle. See Art. 28.

When the centre of the given circle is inaccessible, the

point of contact of a tangent from O may be found as explained in Art. 150.

In practical drawing, tangents from an external point are drawn without geometrical construction thus: Place the pencil point on the given point O, slide a straight-edge against the pencil, and rotate so as to bring the edge as near as the eye can judge in contact with the circle.

122. **Experiment.** Fig. 117. Describe any circle and draw any chord as *xx*. Draw *cv* a perpendicular on to the

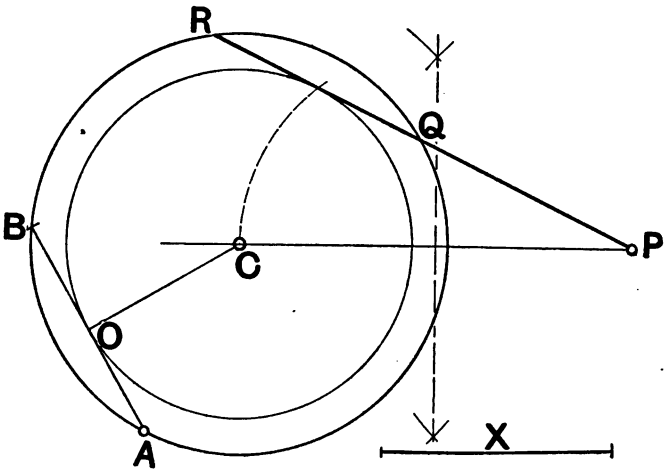


FIG. 118.

chord *xx* from *c*. With *c* as centre and radius *cv* describe a circle. Make a tracing of the chord *xx* and the line *cv*. Insert a pricker at *c* and rotate the tracing about this point. Observe—

(*a*) Both ends of the line *xx* fall always on the circumference of the larger circle, or, in other words, *xx* is always a chord of the large circle.

(*b*) *xx* is always tangent to the smaller circle at the point *v*. Hence—

Theorem. *Equal chords in a circle are equidistant from the centre of the circle.*

Problem. Fig. 118. *From a fixed point* P *draw a line intersecting the given circle, centre* C, *in points* R *and* Q, *such that* RQ *shall equal a given length* X.

With any point A on the circumference of the circle as centre, and radius equal to the given line X, describe an arc intersecting the circle in B. Draw CO perpendicular to AB, and with C as centre and radius CO, describe a circle. Draw PQ a tangent to this circle from the point P, then PQR is the required line.

The student will observe that the given point P may

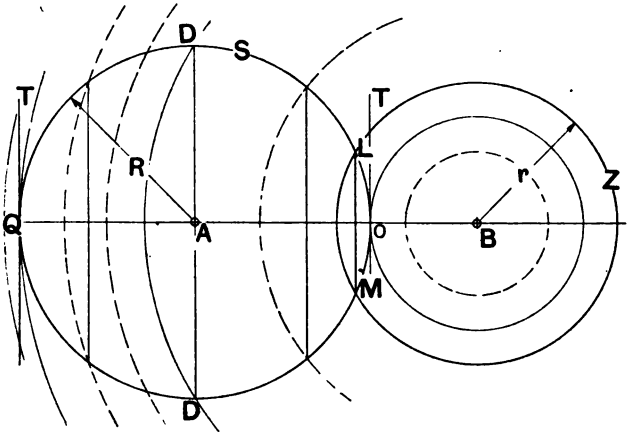


FIG. 119.

be placed inside the circumference, but in that case the minimum distance of the point from the centre of the circle is given by the radius of the smaller circle.

When the distance of P is at a minimum from the centre of the given circle, the point is on the circumference of the small circle, *i.e.*, the circular locus of the mid-point of the given chord, in which case only one line can be drawn to satisfy the conditions of the problem. In all other cases, there are two lines which satisfy the given conditions.

123. On a straight line, Fig. 119, mark points A and B

4 in. apart. With A as centre, and radius R equal to $2\frac{1}{2}$ in., describe the circle S. With B as centre, and radius r equal to 2 in., describe a circle Z. The circles S, Z intersect in *two* points L, M, and *two only*. Join LM; then LM is a chord of both circles and perpendicular to AB, Art. 114. Hence, *the common chord of two circles is perpendicular to the line joining their centres, and is bisected in that line:* also, *if more than two circles have a common chord, all the centres lie on the same straight line.* If the radius r of the circle, centre B, be increased repeatedly, the common chord will get longer and longer until it coincides with the diameter DD of the circle S; but after that position is reached, the chord gets shorter and shorter until it coincides with the point Q. Again, if the original value of r be reduced, the common chord gets shorter and shorter until it coincides with the point o. There is thus a maximum and a minimum value of r between which limits the two circles always intersect in *two finite points*, and *two only*.

$$\text{Maximum } r = AB + R$$
$$\text{Minimum } r = AB - R.$$

In each limiting case, the two circles have a common point represented in the former case by Q; and in the latter by o. These common points lie in the respective tangents QT, oT. Hence, where r is a maximum, QT is tangent to both circles at the point Q. Similarly, where r is a minimum, oT is tangent to both circles at the point o.

If two or more circles touch the same straight line, such line is called a common tangent.

If two or more circles have a common tangent at one and the same point, the circles are said to be tangent at that point.

When two circles which are in contact lie on different sides of the common tangent, the circles are said to have external contact.

When two circles which are in contact lie both on the same side of the common tangent, the larger, or outer, circle is said to have internal contact.

124. **Problem.** *Fig.* 120. *Draw a circle of given radius* R *to touch a given circle centre* C *at a point* P.

Join CP and produce. Make PO = R (the given radius); then O is the required centre.

If R does not exceed the radius of the given circle, centre C, a second circle may be drawn inside the given circle, the centre of which is O′ on the radius CP.

125. **Problem.** *Fig.* 121 *is a cross-section of a sewer, the inner curve being formed with the semicircle* a*D*b*, and three other circular arcs. Determine the centres of these circles, and the points of contact.*

Find *c*, the centre of the semicircle *a*D*b*, and prolong *ab* in both directions.

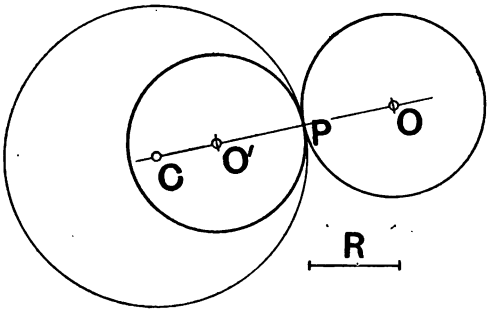


FIG. 120.

Then, because the curve below *a* is tangent to the semicircle *a*D*b* at *a*, the centre from which this curve is described lies on *ab* or *ab* prolonged. Select some point P which is seen to be on the curve from *a*, and draw the perpendicular bisector of *a*P crossing *ab* in c_1, which is the centre for the arc *a*P, etc. Next, select two points *q*, *s*, one on either side of R, and such as to lie on the same arc as R. Find the centre c_2 of the circle containing *q*, R, *s*. Then, because the arcs described from centres c_1, c_2 are tangential, the point of contact lies on c_1c_2 produced. Complete by symmetry.

126. Examination of Figs. 119 and 122 shows that, when two circles intersect, *only* **two** common tangents can be drawn; when the circles are tangent, there are **three** common tangents; when the two circles are separated, **four**

common tangents can be drawn. Observe that in every case the line drawn through the centres of the two circles is an axis of symmetry, not only as regards the circles, but also as regards the common tangents.

Problem. Fig. 122. *To draw a tangent to the given circles, centres a and b. Radii* R *and r.*

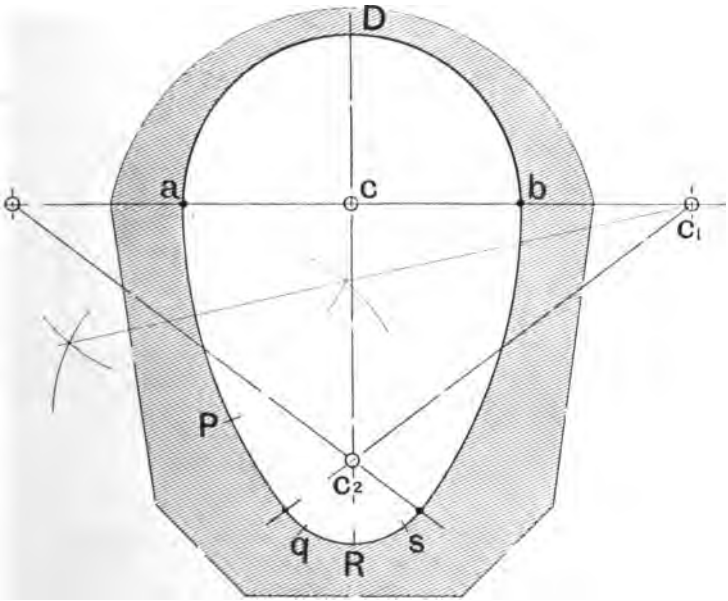


FIG. 121.

In this case, a tangent can be drawn, without geometric construction, by manipulating a straight-edge until it touches the circumference of each circle. A better method is to make *a* the centre of the larger circle, the centre of circles of radii equal to R $-$ *r*, and R $+$ *r*, and draw from *b* the centre of the smaller circle tangents to these subsidiary circles, as shown at *bt*, *bt'*. These lines are parallel to the common tangents TT, MM of the two circles.

The points of contact are determined by perpendiculars from a and b on to the tangents.

There are two other tangents Pp, Qq, which are omitted to avoid overloading of the diagram.

Problem. *Fig.* 123. *Two vulleys of radii* R *and* r, *respectively* 10 *in. and* 4 *in., are fixed on parallel shaftings and connected by a crossed belt; the centres of the pulleys are*

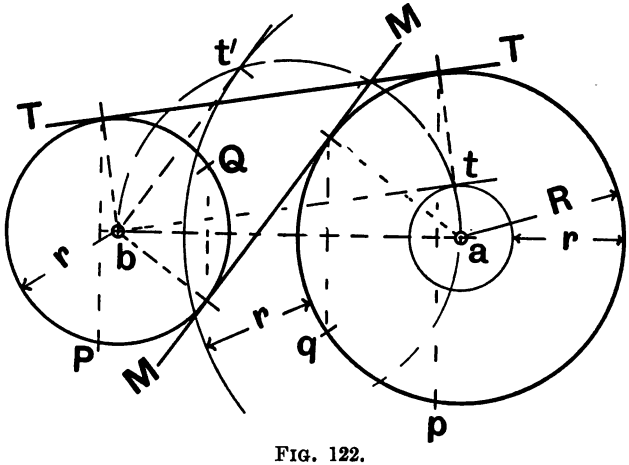


FIG. 122.

30 *in. apart.* (*a*) *Make a diagram showing the arcs of contact of the belt.* (*b*) *Measure the length of the belt. Use a scale of* $\frac{1}{4}$.

Draw circles of given radii at the given distance apart, then draw the cross tangents to the two circles and find the points of contact Tt, T't'. The arcs of contact are TMT', tmt'. Measure the length of the belt with tracing-paper as explained in Art. 20.

127. **Concave and convex curves.** A curve is said to be *concave* with regard to a given point, when the curve and the point are on the *same side* of a tangent; and *convex* when the curve and the point are on *opposite* sides of the tangent. Thus, the curve aa', Fig. 124, is concave with regard to the point S, but convex with regard to R.

In Fig. 125, a curve EE bends in different directions, and the tangent at Q is seen to cut the curve on the left at V, whilst a tangent at Q′ cuts the curve on the right at W.

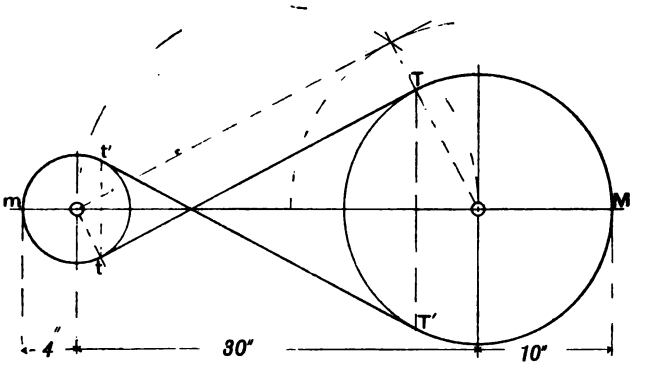


FIG. 123.

There is thus a portion of this curve *convex* and a portion *concave*, with regard to the point S.

There is somewhere a point P where EE passes from the

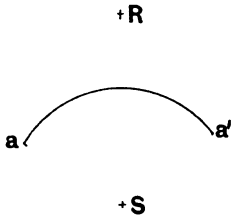


FIG. 124.

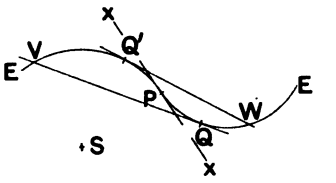


FIG. 125.

concave to the convex with regard to any external point S. This point P is called the **point of inflection**.

The tangent at a point of inflexion crosses the curve at the point of contact.

Beginners often have difficulty in understanding a tangent of this type, as it seems directly contrary to the preconceived

notion of tangency which implies contact, whereas XX appears to cut the curve only, at P.

The following simple experiment may help to clear away the difficulty.

Draw a straight line XX, Fig. 126, and place a set-square with an edge perpendicular to XX; place a penny piece against the set-square and touching the line. The line XX is then tangent to the curved edge of the penny at a point P. Now place a second penny in contact with the first and the set-square, then XX is tangent to this second penny also

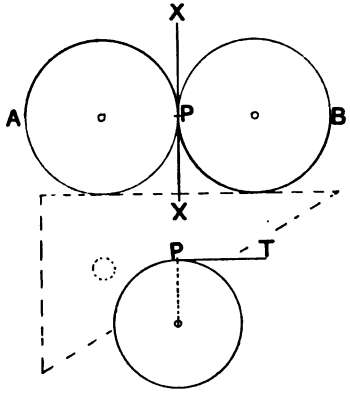


FIG. 126.

at the point P. Consider now a segment of each circumference, as AP, BP, forming a sinuous line APB. The position and curvature of the two segments remain unchanged, and XX, the common tangent of the two circles at P, is also tangent to the compound curve APB at the same point.

All sinuous curves, whether of constant or varying curvature, although forming a continuous and flowing line, nevertheless undergo a fundamental change in direction at a point of inflexion, and each segment, by virtue of the change in direction, may be considered a separate curve.

128. **Parallel circles.** Parallel lines have been defined (Art. 6) as lines which have the same direction.

A circle, or indeed any curve, may be defined as the locus of a point which *continually changes its direction.*

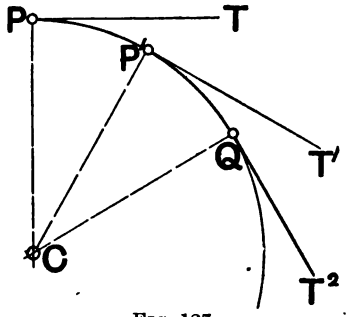


FIG. 127.

Let a point P, Fig. 127, turn about a fixed point C as centre into a new position Q, generating the circular arc PQ. The moving point P is constrained to remain at a fixed distance from the fixed point C, and therefore at any instant it can move in one direction only, viz., at right-angles to the radius. Thus, when the generant is at P its direction is along the tangent PT; when at P^1, its direction is along P^1T^1; and when at Q, along QT^2.

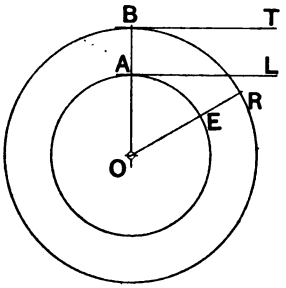


FIG. 128.

(It will be understood that the moving point P occupies any particular position in its path, as P^1, for a period of time infinitely small, and therefore the tangent P^1T^1 indicates the direction of the moving point only for an infinitely small period at the particular instant when the point occupies the position P^1.)

Consider two concentric circles AE, BR, centre O, Fig. 128. Draw OB which is normal to both curves.

Imagine the circles as generated by points A and B rotating about the fixed point O; then the corresponding points A and B have the same direction, viz., along the parallel tangents AL, BT. Now, let OR be drawn from the common centre O, terminating at R on the larger circle, and crossing the smaller circle in E. Then OR is normal to the two circles at the respective points R and E, and consequently the two circles have the same direction at those points, from which we conclude that circles which have a common centre, have (a) always the same direction, and are parallel curves; (b) either circle is the locus of points in the plane of the circles at a fixed distance from the other; inside or outside as the case may be.

Exercises.

(1) Draw the pattern shown (including the dotted lines). The arcs are all to be of $\frac{1}{2}$-in. radius. (B.E. 1895.)

(2) Divide a circle of $1\frac{1}{2}$-in. radius similarly to the given diagram, the radii of the smaller circles (r_1, r_2, and r_3) being $\frac{3}{8}$ in., $\frac{1}{4}$ in., and $1\frac{1}{8}$ in. respectively. (B.E. 1896.)

(3) Draw a line AB 3 in. long, and describe two circles, each of $\frac{3}{4}$-in. radius, touching AB at A and B respectively. Then describe a third circle of 1-in. radius, touching both the other circles. (B.E. 1895.)

(4) Set out the figure to the given dimensions $\frac{1}{4}$ linear size. (B.E. 1910.)

(5) Draw the figure from the given dimensions. (B.E. 1895.)

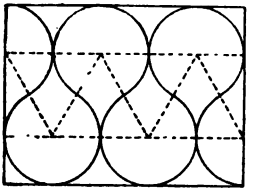
(6) The given figure is to be drawn according to the following data:—

The point P is to be 1 in. from the line CD. Each of the two small circles is to be of 0·75-in. radius. The large arc is to be of 1·75-in. radius. (B.E. 1902.)

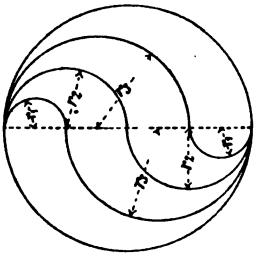
(7) The diagram, a rampant arch ring, 18 in. deep, made up of eleven voussoirs or arch stones S, S, S, ..., having equal lengths of arc *aa*, *aa*, ... along the soffit ACB, the joints *ab*, *ab*, ... being perpendicular to the curve of the soffit. The soffit is made up of two circular arcs AC, CB, each of 90 degrees, and of 6-ft. and 3-ft. radius. Draw the figure to a scale of $\frac{1}{2}$ in. to 1 ft. (B.E. 1908.)

(8) Two pulleys whose axes are 10 ft. apart are respectively 3-ft. and 2-ft. diameter. They are joined by a crossed belt. Draw the arrangement to scale geometrically.

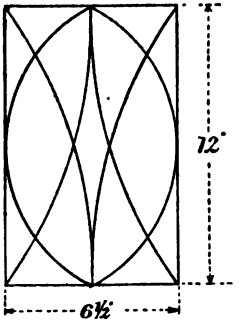
(9) Three shafts A, B, and C are situated at the three corners of an equilateral triangle whose side is 12 ft.; a pulley 3-ft. diameter at the corner A is connected by an open belt to a pulley 4-ft. diameter at B. A second pulley 2-ft. diameter at B is connected by a crossed belt to a pulley 3-ft. diameter at corner C. Draw the arrangement geometrically. Scale $\frac{1}{2}$ in. to 1 ft.



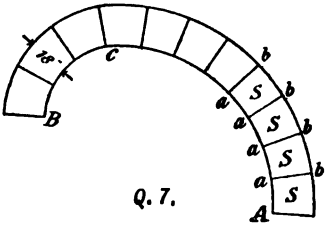
Q. 1.



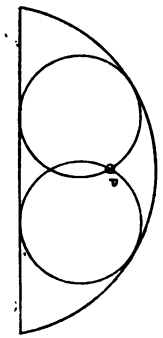
Q. 2.

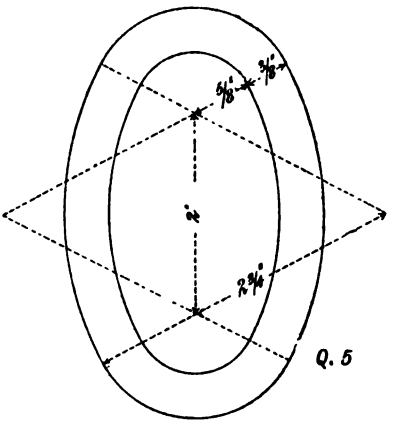


Q. 4.



Q. 7.





Q. 5

CHAPTER X.

CONSTRUCTIONS AND AREAS OF CIRCLES.

129. If the student carefully reviews the work he has already done in connection with the circle, he will learn from his experiments and the constructions employed that *a circle can in general be drawn to satisfy three geometrical conditions.* Thus, from among the examples already considered we see that a circle can (1) be described through three given points ; (2) contain two given points and have its centre on a given line—straight or curved ; (3) contain a given point and touch a given line at a given point.

This last case, at first sight, may seem to include only *two* conditions, but in reality there are three, because a tangent with its point of contact gives a point in the curve and also fixes the position of the line on which the centre must lie. It should be noted that fixing the centre of a circle is always equivalent to two conditions.

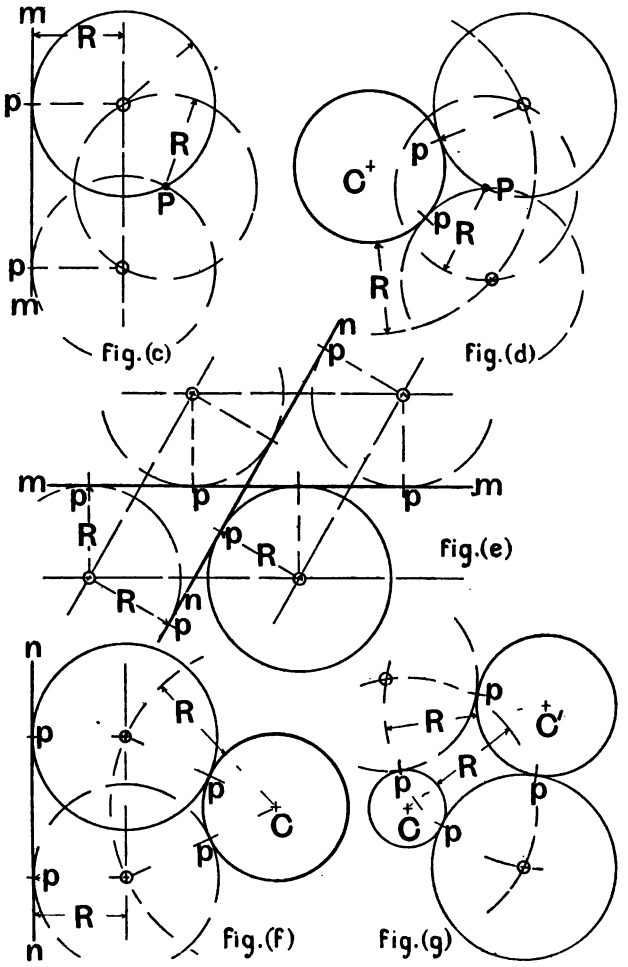
The term *in general,* as used above, is necessary because the relative positions of the lines or points given in the conditions may be such as to render the problem impossible.

130. **A circle of given radius R located by certain given conditions:—**

The student should draw a circle of any fixed radius R on a piece of tracing-paper, and aided by this tracing he should satisfy himself that a circle of given radius can be drawn to satisfy the different sets of conditions given below. He should also pay careful attention to the limitation of the data, and the different possible positions of the circle.

Case (*a*). To touch a given straight line in a given point ; two circles can be drawn. See Fig. 113, Art. 118.

Case (*b*). To touch a given circle in a given point. If the radius of the given circle is greater than R, two circles can be drawn. See Fig. 120, Art. 123.


m
R
p
R
P
p
m
Fig.(c)
C+
p
P
p
R
R
Fig.(d)
n
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p
R
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R
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Fig.(e)
n
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R
p
C
p
p
R
Fig.(f)
p
R
C'
R
p
C
p
p
Fig.(g)

Case (*c*). To touch a given straight line *mm* and pass through a given point P. See Fig. (*c*).

The distance of the given point P from the given line must not exceed 2R. If this distance is less than 2R, two circles can be drawn.

Case (*d*). To touch a given circle centre C, and to pass through a given point P. See Fig. (*d*).

The same limitations are applicable to this case as to (*c*). If the point P is inside the given circle the problem may be impossible, or, there may be one or two circles depending on the position of the point P.

Case (*e*). To touch two given non-parallel straight lines *mm*, *nn*, Fig. (*e*).

Four circles may be drawn.

Case (*f*). To touch a given straight line *nn* and a given circle centre C, Fig. (*f*).

The distance of the given straight line from the given circle must not exceed 2R. If this distance is less than 2R, two circles can be drawn.

Case (*g*). To touch two given circles centres C, C'.

The distance apart of the given circles, measured along the line of their centres, must not exceed 2R, nor must the circles be concentric.

In general, two circles can be drawn. See Fig. (*g*).

Investigate Cases (*f*) and (*g*) and find under what conditions, three, four, five, six, seven, and eight, circles can be drawn.

In Figs. (*c*) to (*f*) above, different positions of the centre of the circle are indicated by small circles; points of contact are marked by the letter *p*. The constructions are based on a property previously referred to, viz., *the locus of a point which is always the same distance from a given straight line is a parallel straight line; and the locus of a point which is always the same distance from a given circle is a concentric circle.*

131. **Problem.** *Fig.* 129. *The section of a hand-rail is shown. Draw this figure to the given dimensions, which are in millimetres.* (B.E.)

In all problems of this type, make a careful examination of the diagram and discover which lines and dimensions must be located first, and in what order the other conditions

can best be brought into the figure. If the figure is seen to be symmetrical about one or more lines, always commence by drawing such lines.

The rail is symmetrical about the dotted line through O. Draw first this centre line, and describe the circle of radius

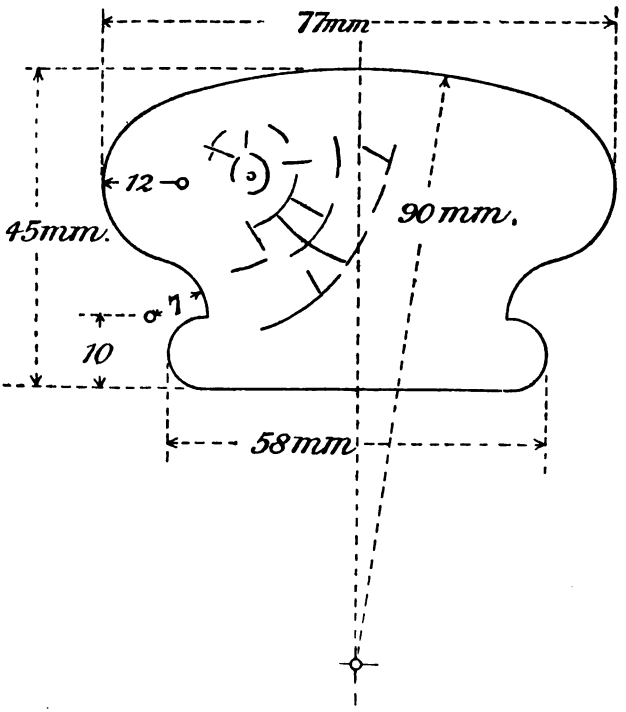


FIG. 129.

90 mm. Next, draw the base line perpendicular to the centre line and 45 mm. below the highest point of the section. Then draw a parallel to the centre line at a distance of 38·5 mm. and describe a circle of given radius 12 mm. to touch this line and the circle which forms the

top of the rail. Draw a parallel to the centre line at a distance of 29 mm., and describe a circle of given diameter 10 mm. to touch this line and the base line. Draw a

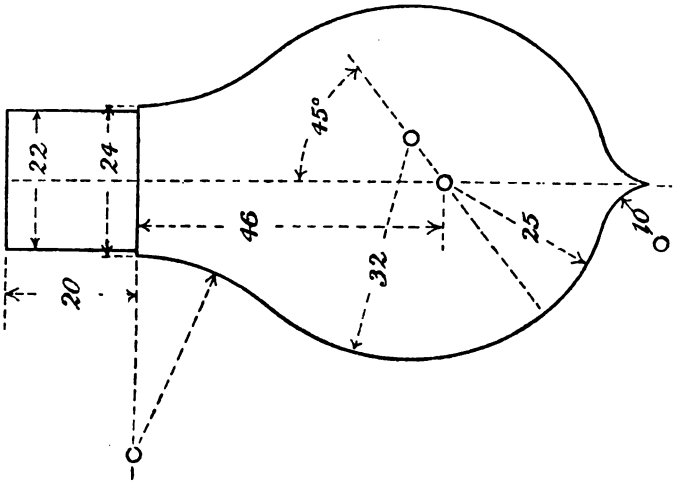


FIG. 130.

parallel to the base line at a distance of 10 mm. above same, and describe a circle of given radius 7 mm. having its centre on this line, and to touch the previously drawn circle of radius 12 mm. Complete by symmetry.

132. Describe any circle, and draw any straight line. How many circles can you draw to touch both the straight line and the fixed circle? Answer: *Any number.* Experiment and satisfy yourself that this answer is correct. Also, verify the following statement: *A circle of any given radius can be drawn to touch two straight lines which intersect; and the centre of such circle lies always on the bisector of the angle between the given lines.*

133. **Problem.** *Draw a circle which shall touch two given lines m and n, and pass through a given point P.*

First case. Fig. 131. When the point P is on one of the given lines.

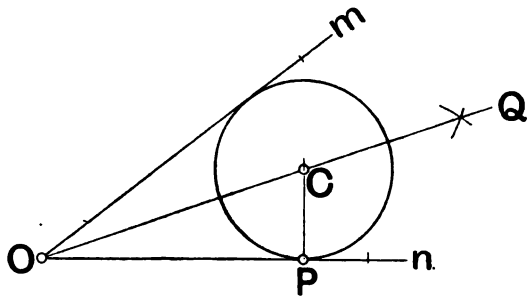


FIG. 131.

Draw OQ the bisector of the angle mOn, and draw PC perpendicular to On, intersecting OQ in C, which is the required centre.

Refer to Expt. Art. 38.

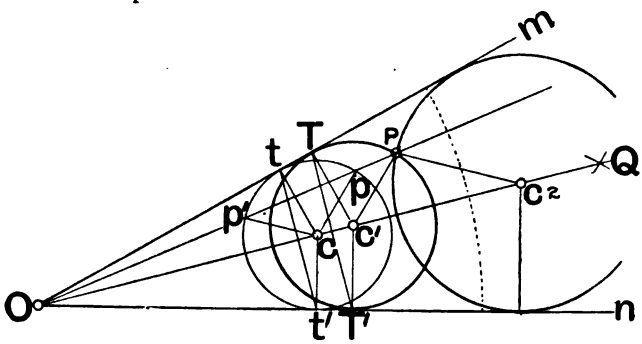


FIG. 132.

Second case. Fig. 132. When the point P is not on either of the given lines.

Draw OQ the bisector of the angle between the lines. Describe any circle centre c touching both lines. Draw OP intersecting the circle in p, p^1. Draw Pc'^1 parallel to pc; or Pc^2 parallel to p^1c, then c^1, c^2 are centres of circles, both of which satisfy the problem. Observe that the triangles tct^1, Tc^1T^1, are similar.

134. **Problem.** Fig. 133. Describe a circle to touch a given circle centre O, and to touch a given straight line *mm* at a point P.

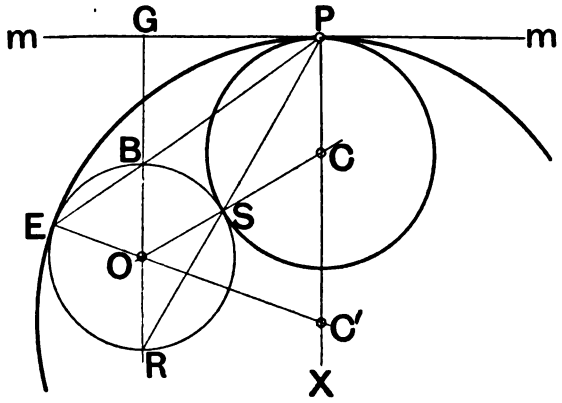


FIG. 133.

Construction. Draw PX perpendicular to the line *mm*, then the required centre lies on PX.

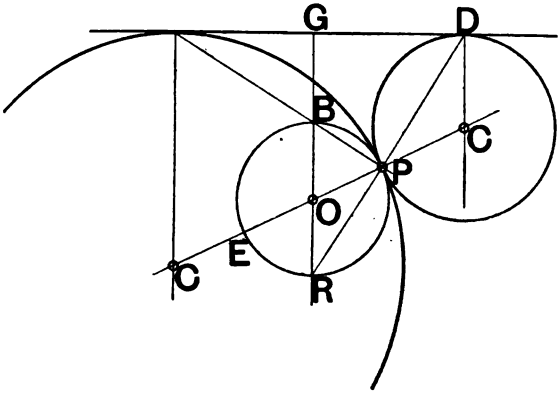


FIG. 134.

Through the centre O draw a perpendicular to *mm* at G intersecting the given circle in R, B.

Join PR intersecting the circle in S, and draw PB intersecting the circle in E. Draw OS, EO intersecting PX in C, C'. Then these are the centres of two circles each of which touches the given circle, and also the given line at P.

Observe that the triangles C'PE, OBE are similar and isosceles triangles; also, that the triangles CPS, ORS are similar and isosceles triangles.

Fig. 135 shows the above construction applied to the setting-out of the soffit curve of an *ogee* arch. AB is half the width of the opening; AD is the rise of the arch; and the radius of the bottom portion of the soffit is given by R = OB. Then the problem is to determine the other centre C and the point P.

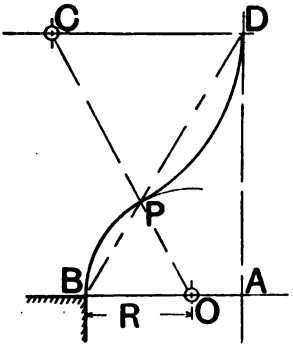


FIG. 135.

Problem. Fig. 134. Describe a circle to touch a given straight line GD and touch a given circle centre O at a given point P.

The construction is based on the same principle as problem 134.

Exercises.

(1) Fig. 130 shows an electric lamp. Draw this to the given dimensions, which are in millimetres. (B. E. 1906.)

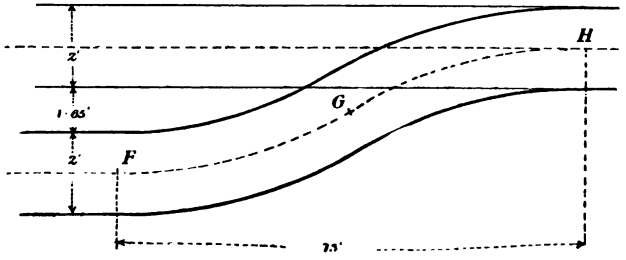


FIG. 136.

(2) Fig. 136 shows a junction of rails for small wagons, the lines drawn dotted being midway between the rails. Set out the figure to a scale of 1 cm. to 1 ft. (B.E. 1904.)

135. We have seen in Art. 70 that an arc of a circle whose length is $\frac{1}{n}$th part of the whole circumference $= \frac{1}{n} \times 2\pi r$ (where r = radius).

Now, such an arc will subtend at the centre of the circle an angle $= \frac{1}{n} \times 360°$.

Hence, *the length of a circular arc is proportional to the angle subtended by that arc at the centre of the circle.*

Ex. 2. *In a circle* 4 *ft. diameter, find the length of the arc which subtends an angle of* 135°.

The circumference of the circle $= 4\pi$ ft.

The angle subtended by the whole circumference $= 360°$.

Hence $\frac{\text{Length of arc}}{\text{Length of circumference}} = \frac{135°}{360°}$.

Let the length of the arc be denoted by a ft.

Then $$\frac{a}{4\pi} = \frac{135}{360}.$$

$$a = \frac{4 \times \pi \times 135}{360} = 4{\cdot}7124 \text{ ft.}$$

136. The length of an arc AB may be found graphically.

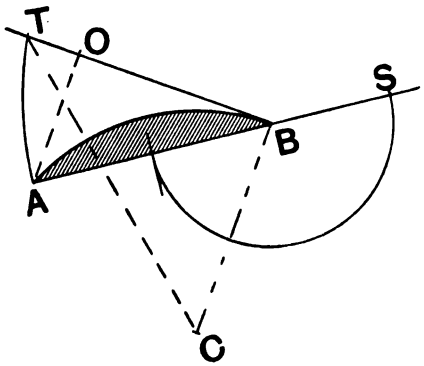


FIG. 137.

At an extremity B, Fig. 137, draw BT tangent to the arc; draw the chord AB and produce to S such that BS = ½AB. With S as centre, and radius = SA, describe an arc meeting the tangent from B in T. Then BT is very nearly equal to the length of the arc AB.

When the given arc is greater than a quadrant, first find πr, *i.e.*, the length of the semi-circumference; then find the length of a segment, which together with the given arc equals a semicircle, and subtract this length from πr.

Another method. Fig. 138. Take a point E on the arc such that arc BE $= \frac{1}{4}$ arc BA. Through the centre of the circle draw CE and produce to meet the tangent from B in P. Join PA. Then BP + PA is slightly longer than the arc AB.

This last construction can be used to mark off a given length along a given circular arc.

Ex. Fig. 138. Mark off on the circumference of the circle BA, centre C, an arc of given length Bq. Set off the given length Bq along the tangent at B. Take Bp on this tangent equal $\frac{1}{4}$ Bq. With centre p and radius pq describe a circle meeting BA in Q. Then arc BQ is very nearly equal to Bq.

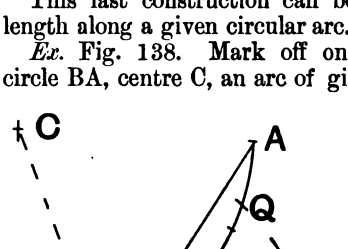


FIG. 138.

137. **Area of circle expressed in terms of its radius.** In Fig. 139, the circle centre O is divided into 12 equal sectors.

In Fig. 140, these sectors are arranged in a form somewhat resembling a parallelogram. Now, suppose the circle

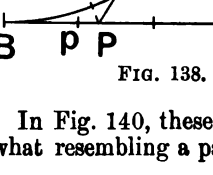


FIG. 139.

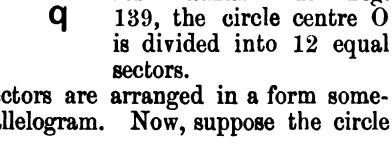


FIG. 140.

divided into 24 equal parts, separated as before described and arranged as in Fig. 140. The various circular arcs, being now more numerous, consequently approach more nearly to a straight line. If this process of division is continued until the number of sectors becomes greater than any assignable number, the arc of each sector will not differ from a straight line by an assignable magnitude, and the sectors, if arranged as in Fig. 140, would form a rectangle having for length, DB = semi-circumference of circle; and for breadth, r = radius of circle. Hence

$$\text{Area of circle} = r \times \pi r.$$
$$= \pi r^2.$$

Ex. What would be the cost of paving a circular courtyard the diameter of which is 33 *yds., at* 5*s. per square yard?*

$$\text{Area of yard} = \pi r^2 \text{ sq. yds.}$$
$$= 3{\cdot}1416 \times (16{\cdot}5)^2$$
$$= 3{\cdot}1416 \times 272{\cdot}25 = 855{\cdot}3 \text{ sq. yds.}$$
$$\therefore \text{Cost} = £\frac{855{\cdot}3}{4} = £213\ 14s.\ 6d.$$

138. **Area of a sector of a circle.** If a circle be divided into two or more sectors, the areas of the sectors will be proportional to the lengths of their arcs; and, since the angles at the centre of a circle are proportional to the arcs on which they stand, it follows that *the area of a sector of a circle is proportional to the angle of the sector, whether expressed in degrees or radians.*

Graphic construction for the area of a sector. Fig. 137. On the tangent at B set off BT = the arc AB (prob. 136). Then $\frac{1}{2}$ BT . BC = the area of sector ABC.

139. **The area of a segment** AB, Fig. 137, **of a circle** is equal to the area of the sector ABC − the area of triangle ABC.

Graphic construction for the area of a segment. Fig. 137. Determine BT = arc BA (prob. 136). Draw AO ∥ CB meeting BT in O. Then triangles CAB, COB are equal in area, and triangle CTO = CTB − COB = CTB − CAB.

$\therefore$ $\frac{1}{2}$ OT . BC = area of segment AB.

140. **To find the area of an annulus,** or the space included between two concentric circles. (Fig. 141.)

Let R = radius of the outer circle ABC.
r = „ „ inner „ DEF.
A = the area of the annulus.

Then, $A = \pi R^2 - \pi r^2$ (or area of larger circle − area of smaller circle).

$= \pi(R^2 - r^2)$
$= \pi(R + r)(R - r)$
$= \pi \times$ (sum of radii) $\times$ (difference of radii).

Ex. Surrounding the circular base of a monument 24 *ft. in diameter is a path* 12 *ft. wide. What would be the cost of constructing this path if each yard super. cost* 12*s.* 6*d.?*

Making use of above formula

$A = \pi(R + r)(R - r)$
$= 3{\cdot}1416 \times (24 + 12) \times (24 - 12)$ sq. ft.
$= \dfrac{3{\cdot}1416 \times 36 \times 12}{9}$ sq. yds. $= 150{\cdot}8$ sq. yds.

As each yard super. costs 12*s.* 6*d.*, or £$\frac{5}{8}$, the cost of path
$=$ £$(150{\cdot}8 \times \frac{5}{8}) =$ £94 5*s.* 0*d.*

141. **The sectorial area of the annulus** lying between the parallel arcs BC, EF may be shown to be equal to

$\dfrac{(C + c)h}{2}$, where C = length of outer arc BC, c = length of inner arc EF, and h = distance DA between them.

Graphic construction. Fig. 141. Rectify the arc CB

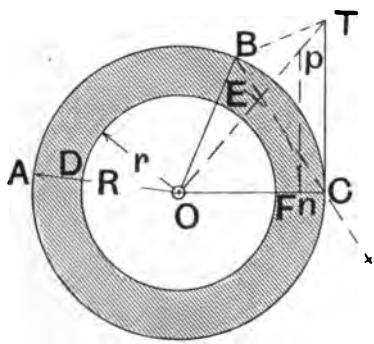


FIG. 141.

along the tangent CT (prob. 136). Join TO ; bisect FC in n, and draw $np \parallel$ CT. Then FC . np = area of FEBC.

Exercises.*

(1) The figure represents a continuous outline composed of portions of three circles and of their common tangents. Draw the figure *to the indicated dimensions.* (B.E. 1904.)

(2) Draw the figure to the given dimensions.

(3) Construct a figure made up of a trefoil of tangential arcs with the circumscribing circle, like the one shown, but to the dimensions given. Show all the construction lines clearly. (B.E. 1901.)

(4) A pinion whose pitch circle is 6 in. diameter gears with a wheel 2 ft. diameter, which in turn gears with a rack 18 in. from the axis of the pinion. Draw the arrangement geometrically to a scale of $1\frac{1}{2}$ in. to a foot, representing wheels and rack by their pitch circles and pitch line respectively.

(5) Two spur wheels are 1 ft. and 8 in. diameter respectively, and are placed on shafts 18 in. apart. A third wheel gears internally with the two wheels and is 3 ft. 6 in. diameter. Draw the arrangement geometrically representing each wheel by its pitch circle.

(6) The curve representing the curtail of a stone step is constructed of circular arcs, each of 90°. Set out the figure to the given dimensions, which are in centimetres. Scale $\frac{1}{4}$. (B.E. (2) 1910.)

[N.B.—Do not copy this figure.]

(7) A straight line is ·75 in. from the centre of a $2\frac{1}{2}$-in. circle. Describe a circle tangent to both line and circle. The point of contact on given circle to be $1\frac{1}{2}$ in. from the line.

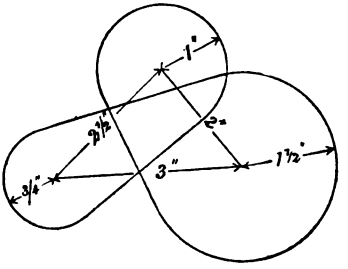
(8) Describe a circle equal in area to the cross-hatched part of the given figure.

(9) A steam engine piston is 20 in. diameter, and the rod is 2 in. diameter. The steam pressure on this side of the piston is 200 lb. per square inch. Find the total pressure on the piston. *Ans.* 62,203 lb.

(10) The area of a circle is 100 sq. in. What is its diameter? (B.E. 1909.) *Ans.* 11·3 in.

(11) A circular path has an outer diameter of 50 ft. and an inner diameter of 40 ft. Find the cost of laying it at one shilling per square foot. Find also the cost of fencing it on both sides at five shillings per yard. *Ans.* Cost of laying the path, £35 6*s.* 10*d.* Cost of fencing, £23 11*s.* 3*d.*

* The diagrams for these exercises are on p. 139.



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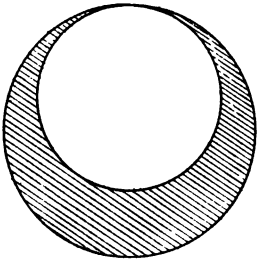
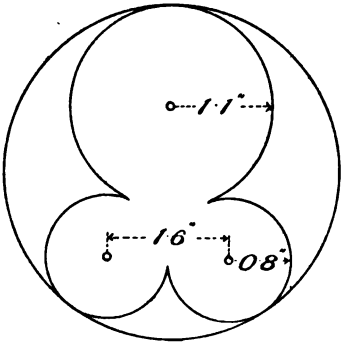
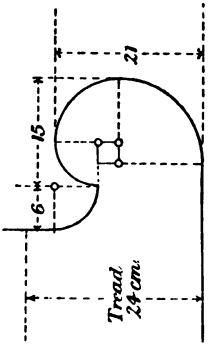


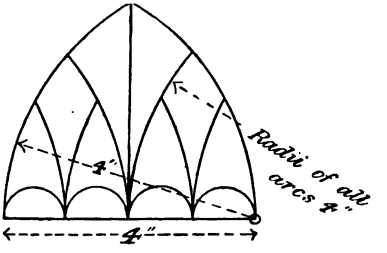
Q. 8.



Q. 3.



Q. 6.



Q. 2.

CHAPTER XI.

ANGLE PROPERTIES OF CIRCLES.

IN this chapter, are considered some angle properties of the circle, and some constructions based on those properties.

142. **Experiment.** Fig. 142. Draw any circle centre C and any chord PQ. Take any point B on the smaller arc and join to the extremities of the chord PQ. Let PB, QB respectively extend to

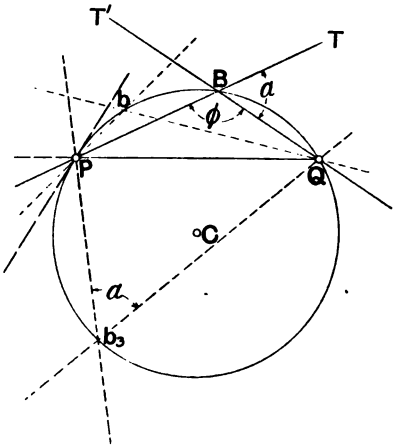


FIG. 142.

T, T'. Denote the angle PBQ by ϕ, and the adjacent angle by a. Now make a tracing of the chord PQ and the lines PBT, QBT'; move the tracing and place B in a new position b on the arc PQ; insert a pricker at the point b to secure the tracing at that point, and rotate the tracing so as to make the line T'BQ pass through the fixed point Q on the drawing; then observe, that TBP *on the tracing passes through the fixed point* P. Repeat this operation, taking different positions on

the arc PQ, and observe that no matter what point on the arc be taken, the angle formed by lines drawn from the point to the extremities of the fixed chord PQ is always the same and equal to ϕ.

Suppose the tracing to move with the line T'BQ always passing through Q, and TBP always passing through P. As B moves from its initial position towards the fixed point P, it traces out the circular arc BP. When the moving point arrives at P, the line BQ on the tracing falls along the fixed chord PQ.

Let B continue moving beyond P; then the segment BT of the moving line PBT will be passing always through the fixed point P, and at the instant when the moving point is at b_3, the angle Pb_3Q is equal to the angle TPQ, *i.e.*, $\alpha = 180 - \phi$. Apply the tracing in several positions, placing the point B on the tracing always on the larger circular arc PQ, the line T'BQ always passing through Q: and observe that BT passes always through the fixed point P. You have thus verified the following important theorems.

Theorem 1. *The angles in the same segment of a circle are constant.*

Theorem 2. *The opposite angles of any quadrilateral inscribed in a circle are supplementary.*

It will be observed that the smaller segment contains the larger angle, and therefore if the two segments are equal the included angles are equal, each being a right-angle. Now, when the chord divides a circle into two equal segments, that chord is a diameter of the circle. Hence,

Theorem 3. *The angle in a semicircle is a right-angle.*

Theorem 4. *The angle in a segment greater than a semicircle is less than a right-angle.*

Theorem 5. *The angle in a segment less than a semicircle is greater than a right-angle.*

143. When the tracing in Fig. 142 occupies its initial position with the point B over the point B on the drawing, PB, BQ are chords of the circle, PB being the longer of the two. As the point B moves towards P, the chord BQ increases, and the chord BP decreases continually, until when B arrives at P, BQ lies along PQ; and the line PBT on the tracing is tangent at P. This experiment verifies an important **Theorem.** *If at an extremity of a chord of a circle a tangent of the circle be drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments.* Thus, the small angle α is the angle in

the larger segment Pb_3Q, and the larger angle ϕ is the angle in the smaller segment PBQ.

144. **Theorem.** *The angle which an arc of a circle subtends at the centre is double the angle it subtends at any point on the circumference.* For, let β and ϕ, Fig. 143, be the angles subtended by the arc PBQ at the centre and circumference respectively. Then the triangle CPQ is isosceles and the angles marked $\underline{\alpha}$ are equal.

Let the perpendicular from C on the chord PQ meet PQ in O, and the tangent from P in T. Then the right-angled triangles CPO, CTP, PTO are similar, and the angles which are similarly marked in the diagram are equal.

The angle TPQ $= \phi$ Art. 143

$\therefore$ The angle PCO $= \phi$

but „ „ PCO $= \frac{1}{2}\beta$

$\therefore$ „ „ $\beta = 2\phi$.

The facts enunciated in Arts. 142,—3,—4, supply solutions to many problems. Some examples will now be given.

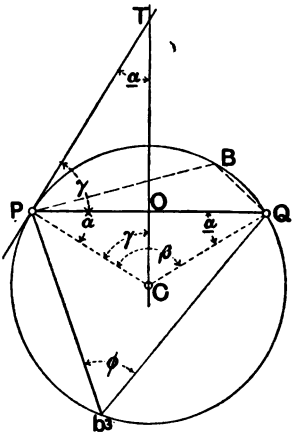


FIG. 143.

145. (*a*) **Problem.** Fig. 142. *To draw a circular arc through three given points* PBQ *the centre being inaccessible.*

Place a piece of tracing-paper over the points and draw thereon straight lines PB, QB, passing through the points. Move the tracing into different positions, taking care that the line BQ always passes through the fixed point Q, and PB always passes through the point P, and mark with a pricker the different positions taken up by B. A fair line through these points is the required circular arc.

(*b*) Now place the point B on the tracing to the point P on the drawing, and adjust BQ on the tracing to fall along PQ. Prick through a second point anywhere in PB on tracing, and join to the fixed point P. You have now solved the following problem: *to draw a tangent at the end of a circular arc, the centre of the circle being inaccessible.*

146. Fig. 144 shows a mechanical device used to draw a circular arc by continuous motion. PQ is the clear width between the supports of a segmental arch; B is the highest

point in the curve measured above the chord line PQ ; and P, Q are points in the curve. The point B is equidistant from P and Q. We desire to draw the circular arc through the three points P, B, Q.

Let a light wooden triangle be made having two straight edges, and shaped to the angle PBQ. Round nails, or other suitable guiding-pins, are driven into the drawing-board at points P, Q. The triangle is placed in position against these pins and given a sliding motion, the sides of the triangle being kept constantly against the pins ; a

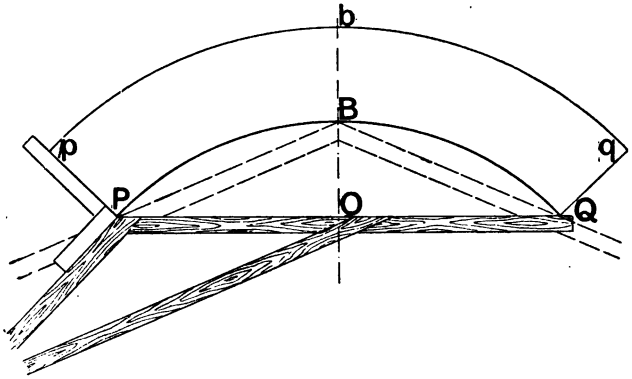


FIG. 144.

pencil is held close to the apex of the moving tool, and thus traces out the required circular arc through P B Q.

An important line in arch problems is the *normal* at the extremity of the arc ; this is obtained by placing a square on the triangle when it occupies the limiting position shown in the diagram.

The curve we have just dealt with forms the soffit line of the arch, and in general another curve parallel to this one has to be drawn to form the outer curve or extrados of the arch.

The triangle already used will serve to draw the outer curve, or, indeed, any number of arcs parallel to the first.

For, consider the parallel arc *pbq*; then the sectors CPQ, *Cpq* are similar, and the angles PBQ, *pbq* are equal.*

The distance between the supports indicated in Fig. 144 by PQ is called the **span of the arch**; the height of the mid-point B above the chord line PQ is called the **rise.**

147. In actual work, the span and rise of the arch are nearly always given by the designer. The draughtsman, when preparing a full-size drawing, prefers to

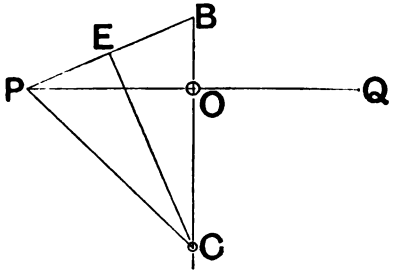


FIG. 145.

calculate the length of the radius from these known values. A formula for this purpose may be deduced as follows :—

Fig. 145. Let S = PO = ½ the span = 1 ft. 9 in.
„ V = OB = the rise = 9 in.
and R = BC = the radius.

Let the perpendicular from C on to PB meet same in E. Then the triangles POB, CEB are similar and right-angled at O and E respectively.

Whence $OB : BP :: EB : BC$

$$BC = \frac{BP \cdot EB}{OB.}$$

But $EB = \frac{1}{2}PB$;

hence, $$BC = \frac{PB^2}{2OB} = \frac{OP^2 + OB^2}{2OB}$$

$$\therefore \quad R = \frac{S^2 + V^2}{2V}.$$

By substitution, $$R = \frac{21^2 + 9^2}{2 \times 9} = 2 \text{ ft. } 5 \text{ in.}$$

* The centre C is not shown in the diagram.

148. **Problem.** Fig. 146. From a given point P on the circumference of a given circle cut off a segment which shall contain a

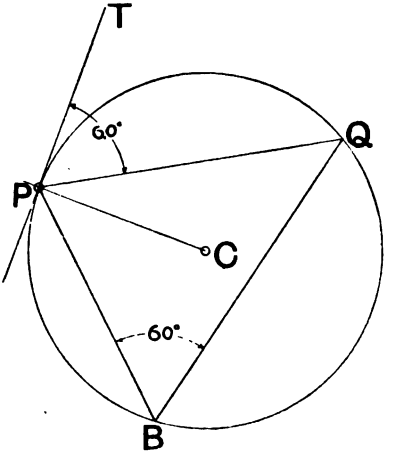


FIG. 146.

given angle, say 60°. Draw the radial PC and the tangent PT. Draw PQ making the given angle 60° with PT. Then the segment PBQ contains the given angle.

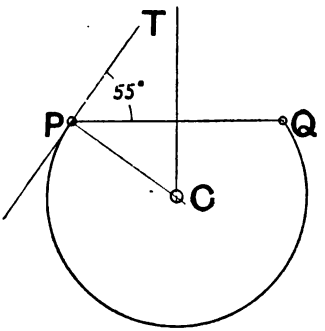


FIG. 147.

149. **Problem.** Fig. 147. On a given line PQ as chord, construct a segment of a circle which shall contain a given angle, say 55°.

At one end of the chord as P, draw PT making the given angle 55°

with QP. Then PT touches the required circle at P, and its centre lies on the perpendicular from P. The centre also lies on the perpendicular bisector of PQ (Art. 30). It is therefore at C, where the two loci intersect.

Exercise.

Given two points P and Q and a fixed straight line m, determine a point S on the line m such that the angle PSQ shall be a given angle, say 70°. When is this problem impossible? When is there more than one solution? Can there be more than two solutions?

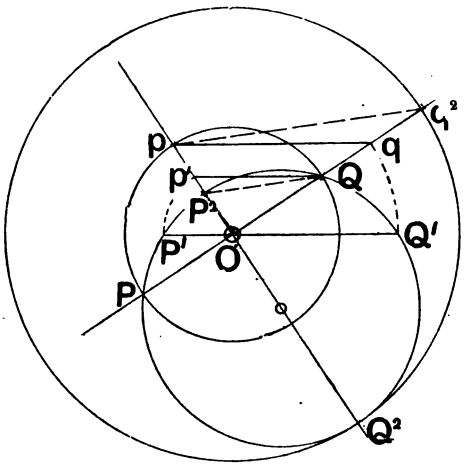


FIG. 148.

150. **Experiment.** Fig. 148. Draw any circle, and through any fixed point O inside the circle draw a diameter Q^2OP^2 and a perpendicular chord POQ. Then PO = OQ (Art. 30). With O as centre and radius OP describe a circle intersecting Q^2P^2 prolonged in p, and from the same centre but with radius OQ^2 describe a circle intersecting PQ prolonged in q^2. Join pq^2, P^2Q, and observe that these two lines are parallel and therefore the triangles P^2OQ, pOq^2 are similar.

Hence $OP^2 . Oq^2 = Op . OQ$, *i.e.*, $OP^2 . OQ^2 = OP . OQ = OP . OP$.

Now draw any chord through the fixed point O, as P^1Q^1. Make $Op^1 = OP^1$; $Oq = OQ^1$, and observe that p^1Q, pq are parallel, and the triangles p^1OQ, pOq are similar.

Hence $Op^1 . Oq = Op . OQ$, *i.e.*, $OP^1 . OQ^1 = OP . OQ = OP . OP$.

Draw other chords through the point O, and proceeding as above verify that the product of the segments is equal to OP . OP = constant.

Now take a fixed point O outside a circle (Fig. 149) and from O draw a tangent, and any second line cutting the circle. Let P^2 be the point of contact of the tangent, and P, Q the two points in which the second line intersects the circle. Draw Op^2 perpendicular to the tangent and make $Op^2 = OP^2$; make Op on $Op^2 = OP$; make Oq on the tangent equal OQ, then observe that the lines P^2p, qp^2 are parallel, and the triangles P^2pO, qp^2O are similar. Hence $Op \cdot Oq = Op^2 \cdot OP^2$, *i.e.*, $OP \cdot OQ = OP \cdot Op^2 = OP_2 \cdot OP_2$.

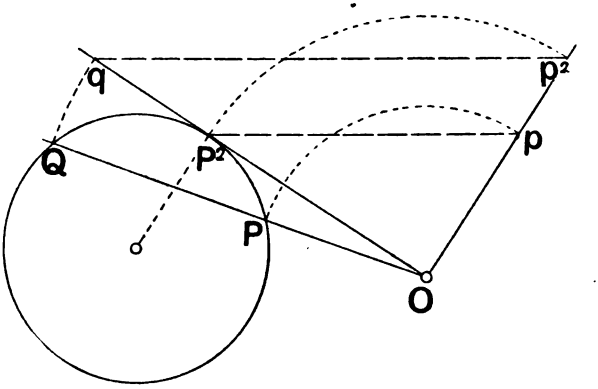


FIG. 149.

Draw other lines from O cutting the circle and, proceeding as above, verify that the product of the segments of all such lines $= OP_2 \cdot OP_2$ = constant.

The facts just verified are expressed in the following

Theorem. *If from any fixed point two straight lines be drawn intersecting the circle in two points, the product of the segments of one line shall equal the product of the segments of the other, and equal the square on the tangent from the point when the point is without the circle.*

151. **Problem.** No figure. Given a circular arc whose centre is inaccessible and a fixed point O, determine the length of the tangent from O.

From the given point O draw any straight line intersecting the arc in P and Q. Then $OP \cdot OQ$ is equal to the square of the tangent from O.

Hence determine OT the geometric mean of OP, OQ (Art. 329), which is the length of the tangent from the given point O.

152. **Problem.** Fig. 150. Describe a circle to contain two given points P and Q and touch a given straight line *mm*.

Join QP and produce to intersect the given line *mm* in O. Then OP . OQ = the square on the tangent from O.

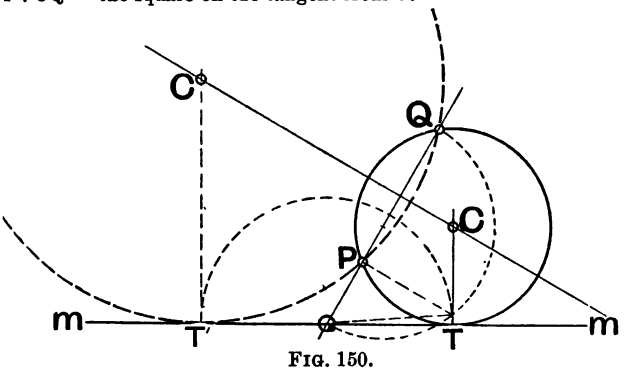


FIG. 150.

Determine the length of the tangent by the last problem and set off this length on either side of O at T, T'. Having found T the problem is reduced to that of Art. 114.

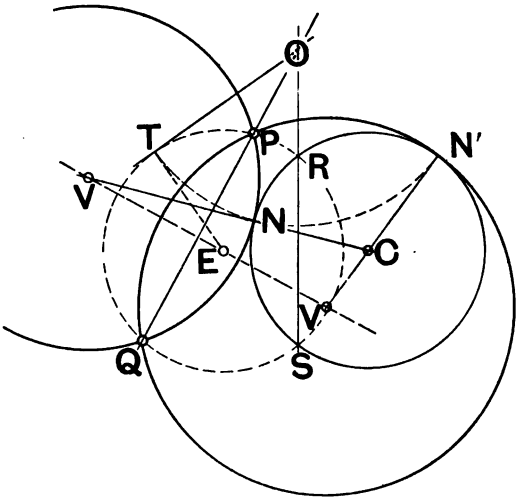


FIG. 151.

There are two circles which satisfy the conditions of the problem; their centres are C, C.

The student will observe that when PQ is parallel to the given tangent *mm* the above construction fails; but in that case the point of contact T is on the perpendicular bisector of PQ, and only one circle can be drawn.

153. **Problem.** Fig. 151. Describe a circle to pass through two given points P and Q, and touch a given circle, centre C.

Draw any circle, centre E, through the two given points, and intersecting the given circle in R, S. From O, the meeting point of chords QP, SR produced, draw OT tangent to the circle centre E. (Then the length OT is constant for all circles that contain the given points P and Q.)

With centre O, and radius OT, describe an arc intersecting the given circle in points N, N′. Then the circle which can be drawn to contain the three points P, Q, N has external contact with the given circle at the point N. The circle that can be drawn to contain the three points P, Q, N′ has internal contact with the given circle at the point N′. Either of these circles satisfy the conditions of the problem.

Exercises.*

(1) It is required to make a railway curve pass through the points *b* and *a*, and run into line *xy*. Find the geometrical centre of the required curve.

(2) Draw a circle to pass through P, having its centre on the line *b*P and to touch the smaller segment of the circle centre C. Measure and write down the length of the radius of the smaller circle, also the length of the curve P*b*.

(3) The perimeter of a triangle is 6 in., the angle at the apex is 40°, and the height is 1½ in. Construct the triangle.

(4) A man is 100 ft. above the earth, which is assumed to be a sphere of 8,000 miles diameter; what is his distance in miles from the furthest point he can see on the surface? Do not give more than three figures in the answer. (B.E. (2) 1906.)

(5) Draw a circle of 2·25-in. radius. In this circle inscribe a quadrilateral ABCD, having given—

Sides, AB = 2·87 in.; DC = 2·5 in.

Angle BCD = 76·5°.

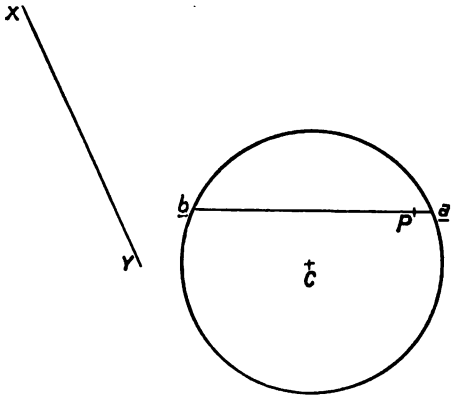
Measure in degrees the angle BAD.

Draw the tangent to the circle at A. Join AC, and measure the angles which AC makes with the tangent. Also measure the angles ABC and ADC. (B.E. 1901.)

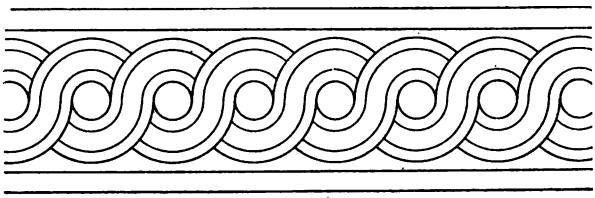
(6) Draw the patterns (*a*) and (*b*), making the diameter of the smallest circle ½ in.

(7) ABCD is a rectangle, AB = 1½ in., BC = 2½ in. Find two points (P and Q) on the line BC, such that the angles APD, AQD, shall be each 70°. (B.E. 1896.)

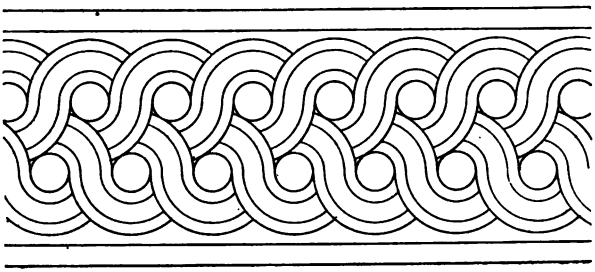
* The diagrams are on page 150.



1 and 2.



6(a)



6(b)

CHAPTER XII.

AREAS OF CURVILINEAR FIGURES.

154. **To find approximately the area of a plane figure bounded by any curve.**

As an example, take Fig. 152, which is a reproduction of an actual indicator diagram for a steam engine.

Draw two parallel lines HN, HM touching the curve, and draw the line HH perpendicular to HN and HM. Divide HH into a convenient number of equal parts (in the diagram 10 divisions are shown). Through these points of division draw lines parallel to HN, thus dividing the area into 10 strips each of width $h = \frac{1}{10}$HH. (The intercepts of these parallels are conveniently referred to as *ordinates* of the figure.)

Now the area of a strip of the figure between two consecutive ordinates is approximately equal to the distance between the ordinates multiplied by the length of an ordinate placed midway between them; thus, the area of the strip between the ordinates 8 and 9 is approximately $h \times y_9$. Hence, if $y_1, y_2, y_3, \ldots$ are the mid-ordinates of the various strips, the total area is equal to $\{(y_1 \times h) + (y_2 \times h) + (y_3 \times h) + \text{etc.}\}$. And since, in this example, $h = \frac{1}{10}$HH, we have, total area $= \frac{1}{10}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10})$HH. The area is thus equal to a rectangle on base HH and having height equal to the average height of the mid-ordinates of all the strips.

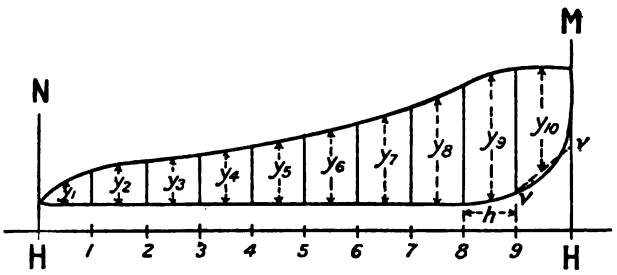


FIG. 152.

Where the curvature is sharp, as in the strips at the ends of Fig. 152 we can replace the curves by a straight line which will, to the best of our judgment, cut from, and add to, the strip portions equal in area. Such a line is shown in *vv*; it is called an **equalising line.** The area of the trapezoid is then taken as equivalent to the actual strip, and the mid-ordinate terminates on the line *vv* as shown.

155. **Simpson's Rule for finding Area of an Irregular Figure.** Divide the area into an **even** number of strips; then, referring to Fig. 153,

$$\text{Area} = \frac{h}{3}\left\{ y_1 + y_7 + 4\,(y_2 + y_4 + y_6) + 2\,(y_3 + y_5) \right\}$$

or in words: Add together the first and last ordinates, four times the sum of the even ordinates, and twice the sum of the odd ordinates (omitting the first and last). Multiply the sum by one-third of the distance between consecutive ordinates, and the product gives the area.

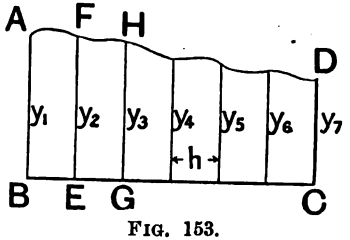


FIG. 153.

When the first and last ordinates are zero, the area is

$$\frac{h}{3}\left\{ 4\,(y_2 + y_4 + y_6) + 2\,(y_3 + y_5) \right\}.$$

156. (*a*) The weights of two or more sheets of cardboard, zinc, or other material of uniform thickness, are, provided the composition of the material is also uniform throughout, proportional to the size of the sheets. This fact may be utilized to determine approximately the area of any plane figure. Cut the figure out of cardboard, sheet-zinc or other available material, and compare its weight with the weight of a known unit of area of the same material.

(*b*) Another method is to draw the irregular figure upon squared paper to scale, and count the number of squares enclosed. The area of each square depends upon the scales adopted. Part of a square may be neglected, if less than half the area of a square, and counted as a whole square if it exceeds the half.

157. Graphic method of finding the area of an irregular curved figure.

Let AEO, Fig. 154, be the given figure in which AE $\perp^r$ OQ. Draw OM $\perp^r$ OQ. Divide the figure into a number of strips equal in width by ordinates parallel to OM. Draw the mid-ordinates of these strips (shown dotted). Project these ordinates on to MM′ at 1′, 2′, 3′. Take any point P on QO produced and draw radials to the points 1′, 2′, 3′. Draw OB ∥ P1′ meeting the ordinate through I in B. Similarly draw BF ∥ P2′; FH ∥ P3′. Then the area of the strip on OI = IB . PO.

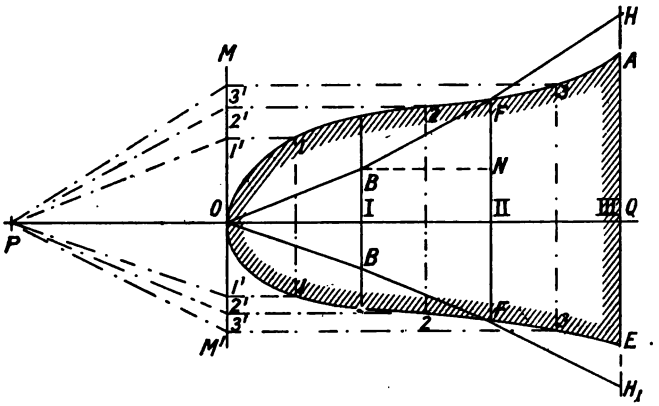


FIG. 154.

For the triangles PO1′, OIB are similar, hence $\frac{IB}{OI} = \frac{O1'}{PO}$ and IB × PO = OI × O1′ = base × average height.

Also the area of the figure OIIF = PO . FII, for, let BN ∥ OQ meet IIF in N. Then the triangles BNF, PO2′ are similar, hence NF . PO = O2′ . BN = O2′ . I II

and since IIF = IB + NF

∴ IIF . PO = IB . PO + NF . PO,

i.e., the area of the first strip and the area of the second. Similarly the area of the whole figure above OQ is QH . PO.

In a similar manner, we find the area of figure below OQ to be QH_I . PO.

The line OBFH is called a **sum curve.**

158. Surveyors and others who have frequently to measure irregular areas on drawings make use of **planimeters.**

A planimeter is any instrument that may be used to determine mechanically, from a drawing, distances and areas.

In Fig. 155, we illustrate a simple planimeter which the student can make out of sheet metal, or cardboard, or mark out on stout tracing paper. The curves ET, FT′ are arcs of a *rectangular hyperbola** which has OX, OY for a symptotes.

For any point (P) on the arc ET, we have xy = ON . NE = constant.

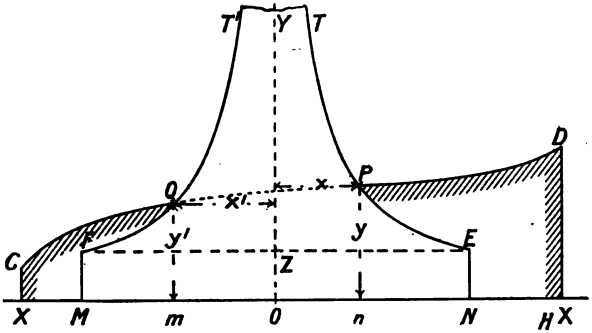


FIG. 155.

Similarly, for any point (Q) on the arc FT′, we have $x'y'$ = OM . MF = constant.

And since OM = ON, and MF = NE = OZ ∴ $x'y'$ = xy = ON . OZ = constant.

And $x'y' + xy$ = 2ON . OZ = constant.

Let the figure XCDX have a straight side along XX and an irregular side CD passing through Q, P. Let ON = 2 in. (centimetres or other convenient linear unit) and OZ = ·5 in. Then the area of the strip *mn*PQ, lying between the ordinates y, y' of P and Q is approximately †

$$xy + x'y' = 2\ ON . OZ = 2 \times (2 \times \cdot 5) = 2 \text{ sq. in.}$$

* See Art. 285.

† It will be observed that the product xy is *greater* than the area of the strip between OY and P*n*, whereas $x'y'$ is *less* than the area of the strip between OY and Q*m*, thus an increase on one side operates against a decrease on the other. If the curve between Q and P approximate a straight line, the "triangles of error"

Ex. Fig. 156. To measure the irregular figure BACD, having AB, AC, BD straight and CD curved, place the tool with the line OX along AB and so that FT′ passes through C. Mark the point I on the curve CD, where the arc ET crosses the boundary. Then the area of the strip ACI1 = 2 sq. units. Next slide the tool along the line AB until FT′ passes through I, and mark the point II where the arc ET crosses the boundary. The area ACII2 = 2 × 2 = 4 sq. units. Continue in this way until, in n moves, the whole area has been traversed. If in the last position the arc ET passes through D the area of the figure $= n$. MN . OZ $= 2n$ sq. units. Since MN . OZ = 2 sq. units.

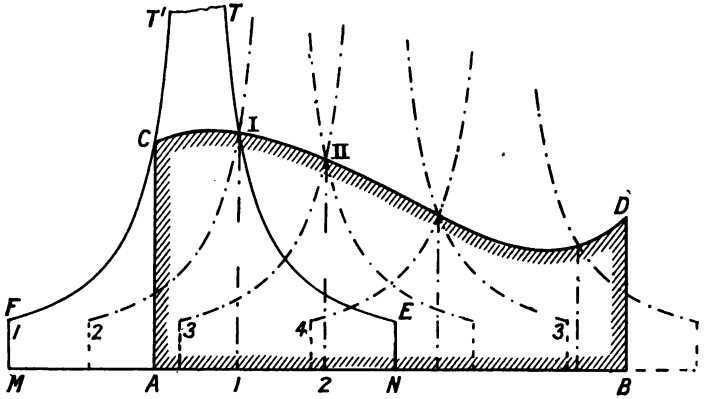


FIG. 156.

Should the tool not exactly measure the figure in a definite number of moves, we must multiply the constant (2 in this case) by the number of complete steps, and add to this product an estimated measure of the fraction remaining. After a little experience, this magnitude may be estimated very nearly ; *e.g.*, suppose in measuring the figure we have 4 complete strips and a piece remaining, the width of which

will not be equal except when PQ is ∥OY ; the steeper the slope of QP the greater the deviation from equality in the triangles. This variation in the case of a curved line QP will be greater or less according to the form of the curve, but in many cases the totality of + and − quantities is negligible.

we estimate at ·3 of a strip. Then if our constant MN . OZ = 2 in., area of figure = 4·3 × 2 = 8·6 sq. in.

If the boundary of the figure is entirely curved, as in Fig. 157, select some fixed point O in or near the centre, place the point O of the tool to this fixed point and mark the points 1, 2 where the arcs FT′ and ET cross the boundary line, then rotate the tool about O until FT¹ passes through 2, and mark the new point 3 where the arc ET crosses the boundary, and continue until the whole figure has been traversed. If the points 1, 2, 3, etc., be joined to the fixed point O, the figure will be divided into a number of triangles all equal in area, and $= \frac{MN \cdot OZ}{2} = 1$ sq. unit.

If the whole figure be exactly traversed in n moves, total area = n square units.

If the tool does not measure the area in a definite number of moves, the area of the piece between the initial and last lines may be estimated as in the previous example.

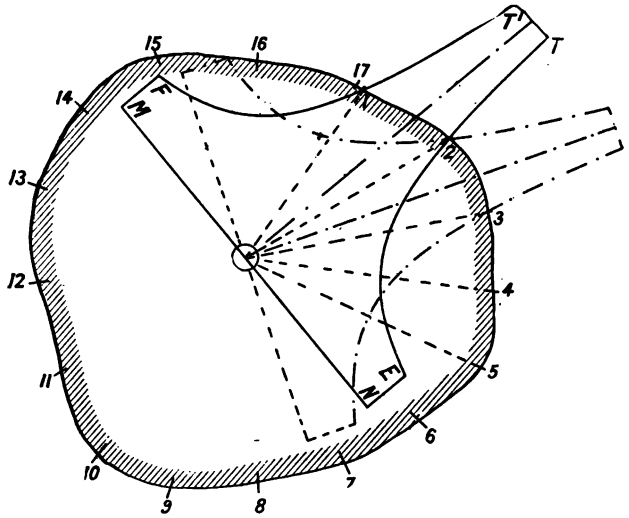


FIG. 157.

Exercises.

(1) Map out a piece of tracing-paper in squares of 1 cm. side, place this tracing over Fig. 157, and by the method of Art. 156 (*b*) determine the area of the figure. Move the tracing a little and repeat the operation. Find the *mean* of the *two answers*. Now experiment by placing the tracing in numerous positions and satisfy yourself that the *mean of two readings* is very nearly constant.

(2) Employ the method of 156 (*b*) to determine the area of (a) the trefoil in Q. 3, p. 139; (b) the lamp, Fig. 130; (c) Fig. 129; (d) Fig. 121; (e) Figs. 3, 5, p. 165.

(3) A field is bounded on three sides by straight lines OD, OA, DB, and on the fourth side AB by a stream. In order to show this field on a plan the side OD was measured and found to be 215 yds. long, OA and DB were measured and found to be respectively 33 yds. and 41 yds. long. OA and DB were found to be perpendicular to OD. To determine AB, equidistant ordinates were measured from side OD, with the following result, going from O toward D: 42, 48, 50, 59, 66, 60, 57, 57, 58, 60, 51, 43, 40. Draw this field to a scale of 0·1 in. = a yard. Determine its area by each of the methods of the preceding chapter. (Simpson's Rule (Art. 155) gives Area = 11,182·08 sq. ft. We also find that the mid-ordinate rule (Art. 154) gives a result very near to this.)

(4) The equidistant ordinates of a curvilinear area are 0, 40, 57, 60, 60, 56, 53, 45, 40, 32, 23, 15, 0; measured from a base line 240 ft. in length. Plot this area to scale, and determine its magnitude by the methods of Arts. 155, 154. (Simpson's Rule gives Area = 9,720 sq. ft., which is considerably more than the result obtained by the mid-ordinate rule.)

Note. From these two examples we learn that the methods of Arts. 154 and 155 will, in some examples, give results very much alike, and in others give results which differ considerably. If we compare the figures in Exercises 1 and 2 above we find the curve in Exercise 2 is wholly concave to the base line; and we see at once that the sum of the trapezoids must be less than the true area. In Exercise 1 the curve is of an undulating nature, and whilst the mid-ordinate in some of the strips is in excess of the mean ordinate, it is less in others, and the errors nearly balance each other.

CHAPTER XIII.

EQUIVALENT AREAS.

159. The area of a parallelogram is equal to the base multiplied by the height. If, therefore, two or more parallelograms have the same, or equal, bases, and the same, or equal, height, they are equal in area. Hence, **Theorem.**

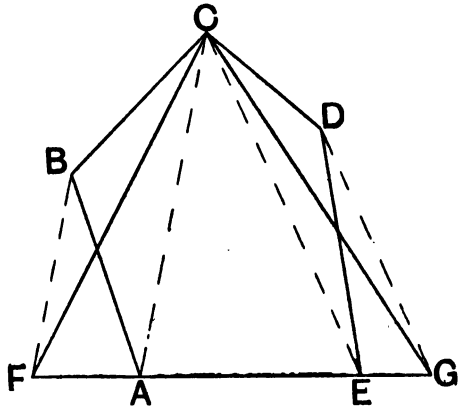


FIG. 158.

Parallelograms on the same, or equal, bases, and between the same parallels, are equal in area.

Again, since the area of a triangle is equal to half the area of the parallelogram on the same base and having the same height, the theorem above can be extended to triangles, thus—

Theorem. *Triangles on the same, or equal, bases, and having the same height, are equal in area.*

Now, as the area of a parallelogram is given by bh, and the area of a triangle by $\frac{1}{2}$ bh, where b = base, and h = height, *the ratio of the areas of parallelograms*, or *triangles on equal bases, is equal to the ratio of their heights*; or if the heights are *equal* and their bases *unequal*, then *the ratio of the areas is equal to the ratio of the bases.*

160. *To reduce an irregular polygon* ABCDE *to an equivalent triangle*, Fig. 158.

Join CA, and draw BF parallel to CA to meet EA pro-

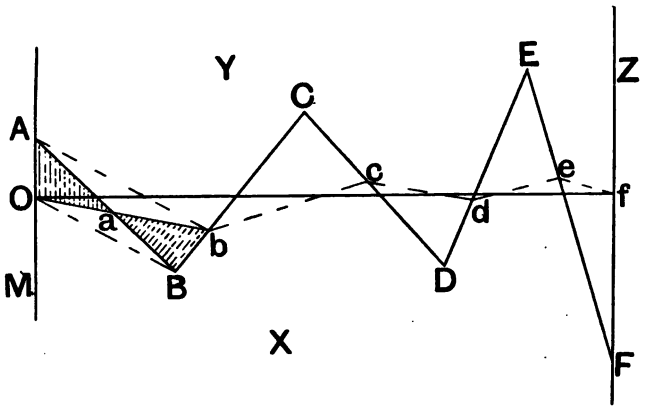


FIG. 159.

duced in F. Join CF. Similarly join CE, draw DG parallel thereto, and join CG.

Now, the polygon ABCDE = triangles ACE + ABC + CDE. But triangles ABC, AFC are on the same base AC, and between the same parallels AC, BF; hence they are of equal area. Similarly, triangles CDE, ECG are of equal area. Hence polygon ABCDE = triangles ACE + ACF + ECG = triangle FCG.

161. **To rectify a crooked line.** *In Fig.* 159 *a crooked line* ABCDEF *separates two pieces of land owned respectively by* X *and* Y. *It is required to substitute a straight line boundary which it is agreed shall pass through* O *in the frontage line* MA. *Determine this line.*

Join O to the *second* corner (B) in the crooked line, counting from O. From the first corner A draw A*b* parallel to OB meeting BC (produced if necessary) in *b*. Join O to C (the next turning point) and draw *bc* parallel to OC meeting CD in *c*. Join OD, and draw *cd* parallel to OD meeting DE in *d*. Join OE, and draw *de* parallel to OE meeting EF in *e*. Join OF, and draw *ef* parallel to OF meeting FZ in *f*. Draw *f*O, which is the required rectifying line.

To verify the construction: Let O*b* be drawn meeting

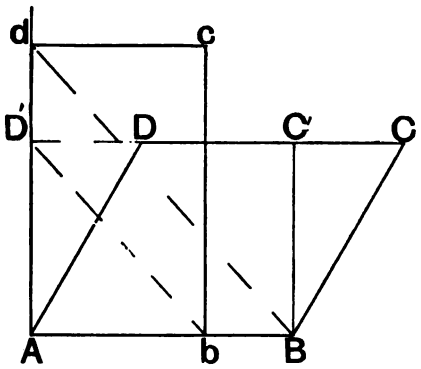


FIG. 160.

AB in *a*. Then the triangles A*b*O, A*b*B are equal in area (since they are on the same base A*b*, and between the same parallels A*b*, OB). Subtracting the triangle A*ba* we have the triangles *b*B*a*, O*a*A also equal in area, *i.e.*, equal areas are cut from the lands of X and Y respectively. Proceed to verify each segment in this way.

162. If a parallelogram having base b and height h is equal in area to another parallelogram which has base b' and height h', then we have $h \times b = h' \times b'$. Which can be written as a proportion $\frac{h}{b'} = \frac{h'}{b}$. And provided any three of these terms are known we can determine the other both *arithmetically* and *graphically*.

Ex. Convert the given parallelogram ABCD, Fig. 160, *into an equivalent rectangle which shall have one side* $\frac{2}{3}$ *AB.*

At A erect a perpendicular to AB. Divide AB at *b* such that $Ab = \frac{2}{3}$ AB. Draw BC′ perpendicular to AB. Then ABC′D′, ABCD are equal areas. Join *b*D′, and draw a parallel B*d* meeting the perpendicular from A in *d*. Draw *dc*, *bc* parallel to AB, and A*d* respectively meeting in *c*.

Then $\frac{Ab}{AD'} = \frac{AB}{Ad}$ or $Ab \,.\, Ad = AB \,.\, AD'$. Hence A*bcd* is the required rectangle.

Ex. Convert the given triangle ABC, Fig. 161, *into an equivalent square.*

Drop a perpendicular from C on to the base line. Bisect CO in N.

Then AB × ON is a rectangle equal to the triangle. We now require to find the side of a square whose area is equal to the

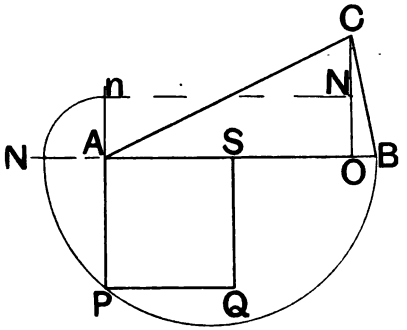


FIG. 161.

rectangle, having sides AB, ON. This will be a mean proportional between AB, ON (see Art. 84). The construction is shown in the figure. AP is the side determined, and APQS the required square.

163. **Areas of similar figures: Theorem.** *The areas of similar figures, whether rectilinear or curved, are proportional to the squares of their linear dimensions.*

Thus, the areas of *similar triangles* are to one another as the squares of corresponding lines.

Circles of unequal diameters D, and *d*, have areas A, *a*, in the ratio of $\frac{D^2}{d^2}$.

The theorem of Art. 329 therefore applies not only to triangles but to all similar figures, and may be written thus :—

The area of any figure described on the hypotenuse of a right-angled triangle is equal to the similar and similarly described figures on the sides about the right-angle.

Fig. 162 indicates how a circle having an area equal to the sum or the difference of two other circles may be drawn.

The circle on diameter AB is equal to the circles on AC,

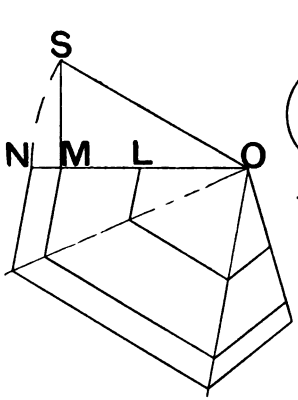


FIG. 163.

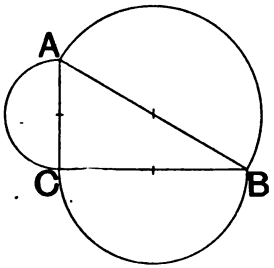


FIG. 162.

BC added together. And, either of the smaller circles is equal to the circle on AB minus the other.

A polygon equal to the sum or the difference of two similar polygons may be drawn by making corresponding sides of the figures form the three sides of a right-angled triangle. The drawing of such polygons is simplified by arranging the figures so as to have a common angle, and one side of each figure along the same line. See Fig. 163, where ON = OS, OL = MS. The polygon on ON = that on OM + that on OL.

164. **Problem.** Fig. 164. *Draw a rectilinear figure similar to a given figure, the area of the new figure being to that of the given figure* (1) *as* 2 : 3. (2) 3 : 2 *say.*

Let ABCD be the given figure.

Case I. On BA produced lay off A2 = 2 units (any unit) and A3 = 3 units. On B2 as diameter describe a circle meeting the $\perp^r$ from A in r. On B3 as diameter describe a

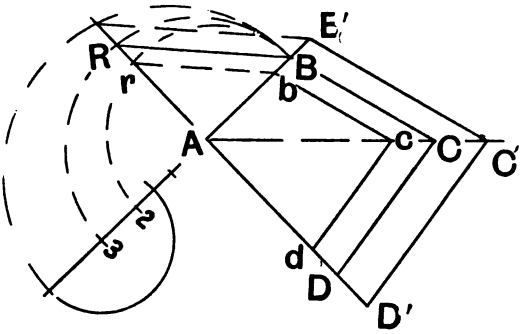


FIG. 164

circle meeting the $\perp^r$ from A in R. Join BR and draw $rb \parallel$ RB. Then Ab is the side of required figure corresponding to AB.

Observe that $Ab : AB :: Ar : AR$

and $\overline{Ar}^2 = AB \cdot A2$, $\overline{AR}^2 = AB \cdot A3$ (Expt. 150)

hence $\dfrac{\text{area } Abcd}{\text{area } ABCD} = \dfrac{(Ab)^2}{(AB)^2} = \dfrac{(Ar)^2}{(AR)^2} = \dfrac{AB \cdot A2}{AB \cdot A3} = \dfrac{2}{3}$.

Case II. The figure is shown at AB'C'D'. The construction for finding the side AB' is clearly shown in the diagram.

165. **Problem.** Fig. 165. *To construct a figure of given species and to have a given area.*

As an *example*, let it be required to *construct a rectangle when the adjacent sides are in the ratio of* 5 : 2, *and its area is* 2 *sq. in.*

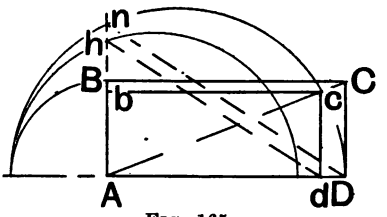


FIG. 165.

Draw any rectangle ABCD having sides AD, AB in the given ratio. Determine An on AB the geometric mean of AD and AB (prob. 329). On AB (produced if necessary) set

off Ah = b, the side of a square of given area, *i.e.*, Ah = $\sqrt{2 \times 1}$. Join nD and draw $hd \parallel n$D. Then Ad is the side corresponding to AD, and Adcb is the required figure.

166. **Problem.** Fig. 166. *Determine a solid round*

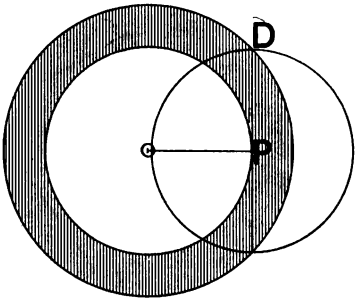


FIG. 166

column to have the same sectional area as the hollow column.

Draw OP the radius of the interior, and draw PD perpendicular to OP. Then PD is the radius of required column.

For, $PD^2 = OD^2 - OP^2$.

Hence circle of radius PD = the annulus lying between P and D.

Exercises.

(1) The building plot is drawn to a scale of $\frac{1}{2}$ in. to 10 yds. Find the area of the plot in square yards.

A rectangular plot of land of the same area has the frontage (or side AA). Find the depth (that is the other side) of the plot in feet, and draw the rectangle to scale. (B.E. 1907.)

(2) Find geometrically the side of a square which shall have an area equal to that of a rectangle whose sides are respectively 5 in. and 3 in.

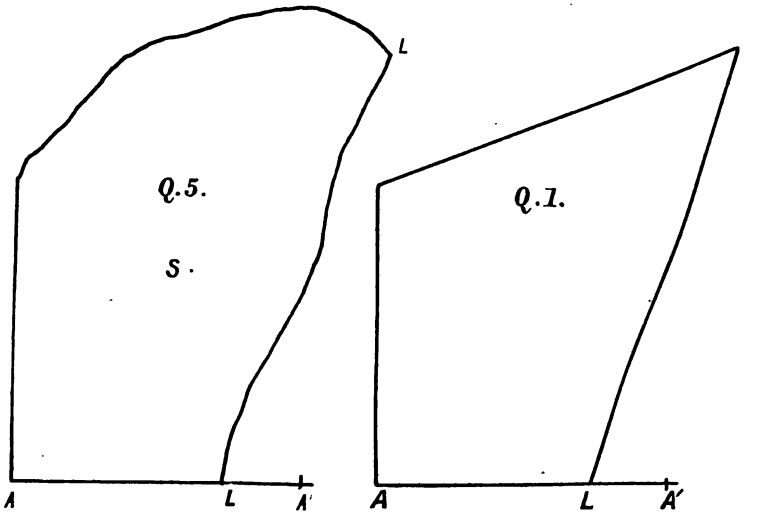
(3) Having pricked off the given polygon, enlarge it to double size, Ef becoming EF. Then draw a rectangle with an area equal to that of the enlarged polygon, and having EF for one side. (B.E. 1901.)

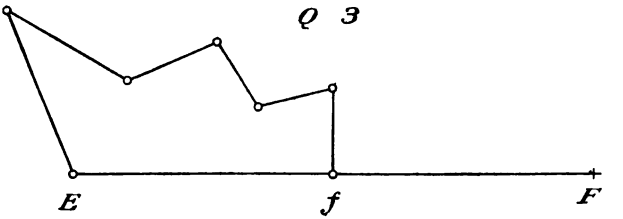
(4) Construct an isosceles triangle, base 2 in., height 3 in. Measure the three angles in degrees. Draw also a rectangle with one side 2 in. long, and equal in area to the triangle.

(5) The building plot S is drawn to a scale of $\frac{1}{2}$ in. to 10 yds. Find the area of the plot in square yards. The portion L L of the boundary may be replaced by a well-judged "equalising line."

A rectangular plot of the same area has the frontage (or side) AA′. Find the depth (*i.e.*, the other side) of the plot and draw the rectangle to scale. (B.E. 1907.)

(6) Reduce Fig. 157 to (1) a square, (2) a circle, of equal area.


L
Q.5.
S.
A
L
A'
Q.1.
A
L
A'


Q 3
E
f
F

CHAPTER XIV.

TANGENTIAL ARCS OF CIRCLES.

167. Numerous examples of curves built up of arcs of circles arranged tangentially are dealt with in Chapters IX., X., and XI. In the present chapter, we give some practical applications.

For convenience, two essential facts are recapitulated :—

(i.) *If a circle touches a straight line, the point of contact is at the foot of the perpendicular drawn on to the line from the centre of the circle* (Art. 116).

(ii.) *If a circle touches another circle, the point of contact is on the line joining their centres* (Art. 123).

168. Fig. 167. Draw any square—or what is better—select a square on a piece of squared paper, and letter the corners O_1, O_2, O_3, C. Insert pins at the corners of the square and fasten a thread C*m* to the pin at C. Wind this thread outside the square in an anti-clockwise direction bringing it against the pins O_3, O_2, O_1 and C. Attach a pencil to the free end of the thread at C and unwind. The pencil will trace a sort of spiral C*gknm*, which is the *involute* of the square.

The conditions controlling the movement of the pencil show clearly that the locus C*gknm* is made up of a series of quadrants of circles ; C*g* is a quadrant of the circle centre O_1, radius O_1C ; *gk* is a quadrant of the circle centre O_2, radius O_2g ; *kn* is a quadrant of the circle centre O_3, radius O_3k ; *nm* a quadrant of the circle centre C radius C*n*. Thus, the whole line is made up of four quadrants of circles.

The final direction of the fourth radius C*m* is the same as the initial direction of the first radius O_1C, and the pencil in moving from C to *m* causes the thread O_1C to turn through four right angles or 360°. Hence the curve

C*m* is called *one convolution;* the arc C*n* is $\frac{3}{4}$, C*k* $\frac{1}{2}$, and C*g* $\frac{1}{4}$ of a convolution. If x represent the length of side of the square $O_1O_2O_3C$, and r_1, r_2, r_3, r_4 respectively represent the radii of the arcs described from the centres O_1, O_2, O_3, C, we have—

$$r_1 = x,\ r_2 = r_1 + x = 2x,\ r_3 = r_2 + x = 3x,\ r_4 = r_3 + x = 4x.$$

Hence, each radius is a multiple of a side of the square $O_1O_2O_3C$, and if we call $x = 1$, the radii r_1, r_2, r_3, r_4 form an arithmetic series 1, 2, 3, 4. *All the radii are therefore*

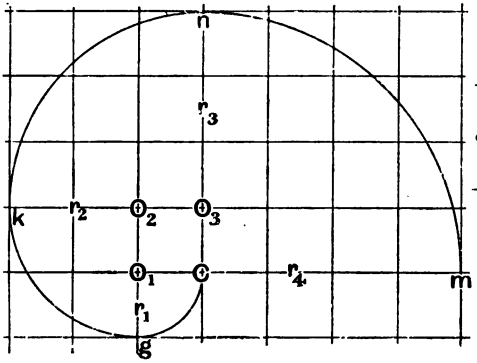


FIG. 167.

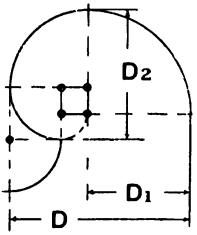


FIG. 168 A.

interdependent, and when any one radius is given, the others can be derived from that one.

Ex. *Given* C*m, the length of the largest radius, determine the centres of the other arcs.* Divide C*m* by 4 (the number of quadrants in the curve) and construct a square having sides equal to one of these parts. The corners of this square will be the 4 centres of the required line.

Similarly, if the radius of the 3rd quadrant be given, *one-third* of this radius will be the smallest radius and, hence, side of square. If the 2nd radius be given, *one-half* of this will be the smallest radius.

Fig. 168A shows a curve of this type applied to form the end of a *curtail step.*

Suppose D is given. Then—

$\because$ D = the sum of the two largest radii,

$\therefore$ $\frac{1}{7}$ D = the smallest radius and side of square

If D_1 be given, then $\frac{1}{4}$ D_1 = side of square.

If D_2 be given, how would you determine r_1, or side of square ?

Ans. Divide D_2 by 5. (Since $D_2 = r_2 + r_3 = 5\,r_1$).

If a second convolution is required, the additional centres

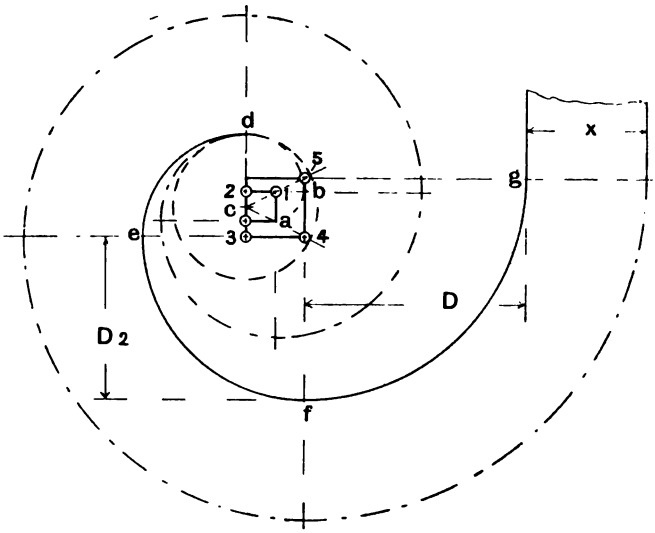


FIG. 168B.

may be taken at the corners of a second square which is twice the size of the first, and placed with one side along a side of the first square, and with the mid points of these two sides in the same point *c*. Fig. 168B.

The curve *abdefg* consists of five quadrants of circles described from centres 1, 2, 3, 4, 5. If we denote the radii of these quadrants by r_1, r_2, r_3, r_4, r_5 respectively, and call the side of the smaller square unity ; we have, $r_1 = 1$; $r_2 = 2$; $r_3 = 3{\cdot}5$; $r_4 = 5{\cdot}5$; $r_5 = 7{\cdot}5$. These values

enable us to employ the construction to draw a curve when the radius of either quadrant (or equivalent data) is given.

Ex. 1. *If* D *be given as* 11 *in. determine the lengths of the sides of the two squares.*

Here we have D $= r_5$; and $r_5 = 7{\cdot}5\ r_1$.

Hence $\frac{2}{15}$ths of D = 1·47 in. = side of small square, and $\frac{4}{15}$ths of D or 2·94 in. = the side of the large square. If D be given as a line not figured, the length of the sides of squares are easily found graphically by dividing D into fifteen equal parts, and taking two and four of these parts.

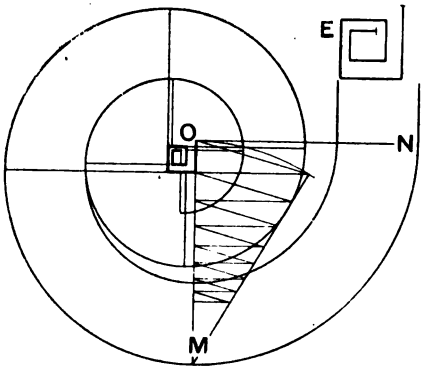


FIG. 169.

Ex. 2. If D_2, the radius of the fourth quadrant be given, r_1 is equal to $\frac{2}{11}$ths of D_2.

The same centres may be used to describe a curve parallel to the first. One such curve is shown by the chain dotted line.

This figure is the plan of a scroll terminal to a hand rail. The distance x represents the width of the rail: this is usually equal to the periphery of the small square.

169. Fig. 169 shows a construction in which the radii of successive quadrants are continued proportionals. If given two successive radii, the others can be found graphically as in the figure, or by calculation. The centres of curvature lie at the angular points of a *fret-like* figure, shown enlarged at (E). Observe that the length of a side of this figure is equal to the difference in the radii of consecutive arcs described from its extremities.

Any number of arcs can be used in this construction, and the resulting curve has a better appearance than that obtained by the construction of Fig. 168B.

The diagram shows the construction applied to a scroll terminal of a hand rail. The ratio of the two largest quadrants ON, OM being 6 : 5, which makes ON = $\frac{6}{11}$ths of the extreme width of the scroll.

170. Fig. 170. On squared paper select a square *abdc*. Draw cO_1 || to the diagonal *da*, meeting *ba* prolonged in O_1. Draw O_1O_2, $\perp^r$ cO_1, meeting *db* prolonged in O_2. Draw O_2O_3 || O_1c meeting *cd* prolonged in O_3. Draw O_3O_4 || O_2O_1 meeting *ac* prolonged in O_4.

Every one of these points coincide with some corner of a square on the drawing paper, and $O_1O_2 = 2\ O_1c$; $O_2O_3 = 3\ O_1c$; $O_3O_4 = 4\ O_1c$. Now draw O_4M perpendicular to O_4O_3, and, starting from O_4, mark off along O_4M the lengths O_4O_3, O_3O_2, O_2O_1, O_1c at III, II, I, and M. Insert pins at points O_1, O_2, O_3, O_4. Fasten a thread to the pin O_4; make it taut along O_4M, and attach a pencil to it at the point M. If the thread be wound in an anti-clockwise direction the pencil will move in a circle centre O_4, radius O_4M, until the thread comes in contact with the pin O_3; it then moves in a circle centre O_3, radius III M, until the thread touches the pin O_2, then in a circle centre O_2, radius II M, until the thread touches the pin O_1, and thence in a circle centre O_1, until it arrives at the point *c*. The path of the pencil is thus a sort of spiral consisting of 4 quadrants of circles. If we call O_1c the shortest radius = 1, and denote the four radii having centres O_1, O_2, O_3, O_4 by r_1, r_2, r_3, r_4, we have $r_1 = O_1c = 1$, $r_2 = O_2O_1 + r_1 = 2 + 1 = 3$, $r_3 = O_3O_2 + r_2 = 3 + 3 = 6$, $r_4 = O_4O_3 + r_3 = 4 + 6 = 10$, *i.e.*, the radii are interdependent and form a summation series 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4. Similarly, if the curve were extended beyond one convolution, there would be one additional centre for each quadrant as illustrated at O_5, O_6, obtained by constructing other triangles. The lengths of the additional radii are obtained by extending the series; thus $r_5 = 1 + 2 + 3 + 4 + 5$, $r_6 = 1 + 2 + 3 + 4 + 5 + 6$, and generally, where n is any number, $r_n = \frac{n}{2}(n + 1)$. Hence, in any given example, if we know the length of any one radius and know the quadrant it describes relative to the origin of

the curve, or equivalent data, we can determine the other radii.

Ex. 1. *Given* D = 6 *cms., determine corresponding values of other radii, and the side of the root square.*

The drawing shows D = radius of 4th quadrant, and according to our notation = $r_4 = 10r_1$.

Hence, if we divide a straight line 6 cms. long (the given length) into 10 equal parts, one of these parts = r_1, 3 parts = r_2, 6 parts = r_3, 15 parts = r_5, and $\frac{n}{2}(n+1)$ parts = r_n.

The side of the square is found by constructing a square on $r_1 = \frac{1}{10}$ D as a diagonal.

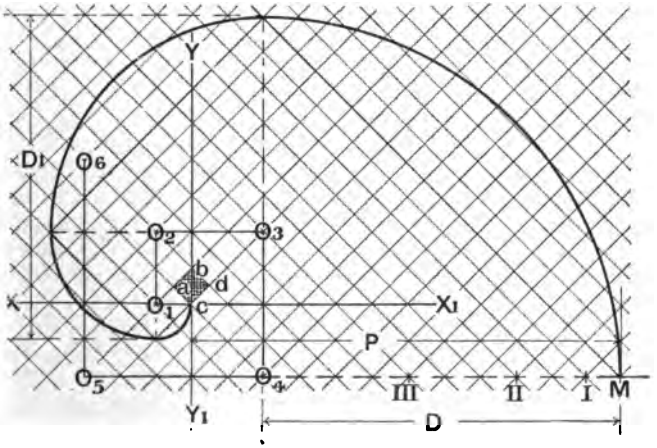


FIG. 170.

Ex. 2. *Let* D_1 *be given* 7 *cms. say.* Then since D_1 = radius of 3rd quadrant + radius of 2nd quadrant,

We have $$D_1 = r_3 + r_2.$$

$$= \frac{3}{2}(3+1) + \frac{2}{2}(2+1)$$

$$= 6 + 3 = 9.$$

$$\therefore \quad r_1 = \frac{D_1}{9} = \frac{7}{9} \text{ cms.}$$

171. Fig. 170. Take the point c as origin, and rectangular axes of reference XX_1, YY_1, the former along O_1c. The co-ordinates of the different centres, in terms of r_1, are found by counting squares on the paper to be—

x	y		x	y		x	y
O_1 ... -1,	0	O_3 ...	2,	2	O_5 ...	-3,	-2
O_2 ... -1,	2	O_4 ...	2,	-2	O_6 ...	-3,	4

Hence, the perpendicular distance of any centre from the origin c measured parallel to the final direction of the radius of that centre, is a multiple of the smallest radius. This enables us to find all the radii when given data which includes the distance of a known centre from the origin of the curve.

Ex. *Let* P *be given as* 8 *cms.*

The given length includes the radius of 4th quadrant r_4. The abscissa of O_4 is known to be $2\,r_1$, whilst

$$r_4 = \frac{4}{2}(4 + 1)\,r_1 = 10\,r.$$

$$\therefore \mathrm{P} = 2\,r_1 + 10\,r_1 = 12\,r_1.$$

hence $\frac{1}{12}\,\mathrm{P} = r_1$ the smallest radius.

172. In Fig. 171, the centres O_1, O_2, etc., and the dotted spiral line *cgkn*M are obtained as in Fig. 170. Fasten a thread at O_4 and make taut along O_4X. Attach a pencil to the thread at a point P, *which is farther from* O_4 *than is point* M, and wind the thread against the pins, as previously explained. The pencil will describe a curve, PQ . . . p, parallel to its original locus *cgkn*M at a distance therefrom = MP. After arriving at p in one convolution, the pencil describes a complete circle, centre c, radius cp = MP. The complete locus is thus a whole circle and four quadrants.

If we apply to this diagram a similar analysis to that used in the preceding Art., and denote the distance MP—*i.e.*, the radius of the complete circle—by R, and the radii of the quadrants in ascending order by r_1, r_2, r_3, r_4 . . ., and call $r_1 = 1$, we find

$$r_2 = \mathrm{R} + 1 + 2, \qquad r_3 = \mathrm{R} + 1 + 2 + 3,$$

$$r_4 = \mathrm{R} + 1 + 2 + 3 + 4, \qquad r_n = \mathrm{R} + \frac{n}{2}(n + 1).$$

And, as before, if we know any one radius and the quadrant it describes relative to the origin and also the radius of the

inner circle, *or equivalent data,* the other radii are easily determined by finding the corresponding radii of the parallel curve which would result if MP were reduced to zero, and adding to these values the constant MP.

Ex. *Let the abscissa of* P = 6 *cms. and* MP = 1·3 *cms. be given,* where *c* is the origin, and axis of abscissæ is parallel to O_4M.

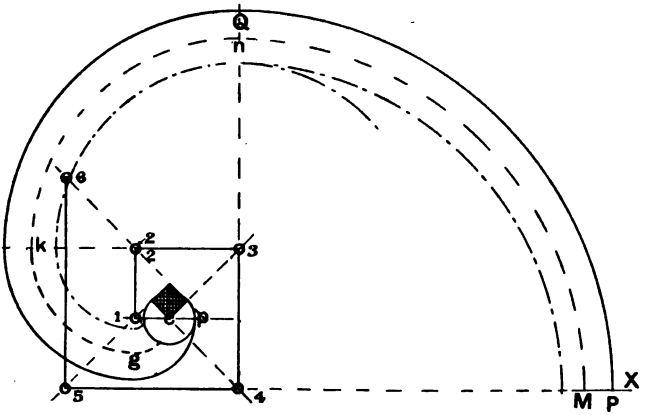


FIG. 171.

Eliminate MP to get a curve parallel to the one required and at a distance MP inside. Then (making $r_1 = 1$)

$$D = r_4 + r_3 - (r_2 + r_1),$$
$$= \frac{4}{2}(4+1) + \frac{3}{2}(3+1) - \left(\frac{2}{2}(2+1) + 1\right),$$
$$= 10 + 6 - 3 - 1,$$
$$= 12.$$

Hence, divide a line 6 cms. long into 12 equal parts, then

1	of these parts	+ R	$= r_1$ — *where* R = MP.
3	,, ,,	+ R	$= r_2$.
6	,, ,,	+ R	$= r_3$.
10	,, ,,	+ R	$= r_4$.
$\frac{n}{2}(n+1)$	,, ,,	+ R	$= r_n$.

Suppose that MP in Fig. 171 is greater than $O_1 c$; then O_1 lies inside the central eye, or the initial circle, but this does not affect the construction at all. The eye of the scroll is sometimes so large as to include O_1, O_2, O_3, or even more centres.

If we have a curve as PQp, (Fig. 171), having a central eye, the centres of the arcs which form the curve may be used to draw another curve at any given distance *inside* the first one. One such curve is shown in the diagram by the chain dotted line.

Exercise 1. Fig. 172. *The given volute of one convolution is constructed of circular arcs each of* 90°. *Set out the curve, when the dimensions are as figured in centimetres.* (B. E., 1910.)

Hint.—If the eye shrink to zero, the distance 6 cms. represents r_4 + abscissa of O_4 for a curve parallel to the arc required.

Exercise 2. Fig. 173. *The figure shows a construction for describing a volute of one convolution. Set out a similar curve of one and a half convolutions when the dimensions are as figured in centimetres.* (B. E., 1910.)

Hint.—In one and a half convolutions there are six quadrants, hence the given length 8 cms. = r_6 + abscissa of centre O_6, both measured in the same direction.

hence $$8 \text{ cms.} = \frac{6}{2}(6 + 1)\, r_1 + 4\, r_1,$$

$$r_1 = \frac{8}{25} \text{ cms.}$$

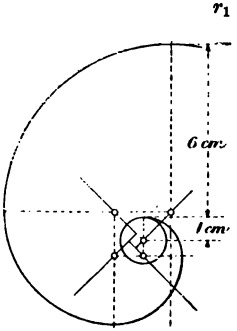


FIG. 172.

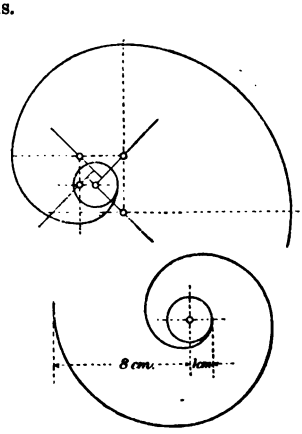


FIG. 173.

173. A curve can be built up of circular arcs closely approximating to any plane curve.

Ex. Fig. 174. *Given a curve xvy; determine three centres from which a curve closely resembling the given curve can be described.*

Select any three points x, u, s in the curve, and not too far apart. Find the centre o of the circle that can be drawn through the points x, u, s. Select some point v in the curve and draw the $\perp^r$ bisector of sv meeting so prolonged in o'. Draw the $\perp^r$ bisector of vy meeting vo' prolonged in o''. A line formed of arc xs described from o, arc sv described from o', and arc vy described from o'', will lie close to the given curve xvy.

The greater the number of centres employed, the nearer will the new line lie to the original.

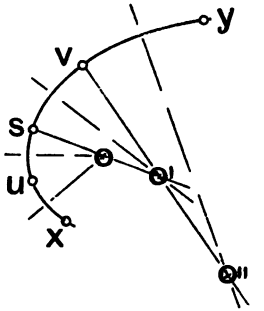
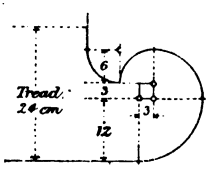


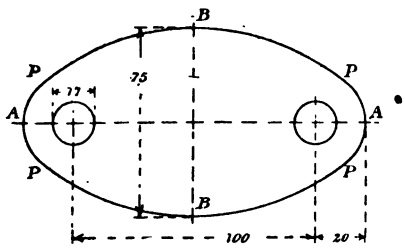
FIG. 174.

Exercises.

(1) The curve representing the curtail of a stone step is constructed of circular arcs, each of 90°. Set out the figure to the given dimensions, which are in centimetres. Scale $\frac{1}{4}$. (B.E. 1910.)



(2) The given Figure is symmetrical about the axes AA, BB. The circular arcs PAP, PBP join tangentially at P. Draw the Figure to the given dimensions, which are in millimetres, determining by construction the centres of the two arcs PBP, and the four points P. (B.E. 1907.)



174. **Mouldings.** In Fig. 175 (a to g), certain mouldings used in classic architecture are shown in outline. The geometrical construction of each figure is indicated by the marked centres or construction lines. The mouldings are named as follows :—

a. Ovolo or Echinus.
b. Cavetto or Hollow.
c. Fillet or Listel.
d. Cyma, Cyma Recta.
e. Inverted Cyma, Ogee.
f. Scotia, or Trochilos.
g. Torus or Tore.

Figs. 176 to 180 are typical examples of the use of tangential arcs in Gothic architecture.

Fig. 176 shows an elaborate moulding based on equilateral triangles. Copy this moulding, making the sides of the triangles = 1 cm. Isometric Paper will be found useful.

Fig. 177 is based on equilateral and right-angled triangles. Set out this moulding, working to the given dimensions.

Fig. 178 is the plan of a pier-shaft and base. Set out this when the side of the exterior octagon is $1\frac{3}{4}$ ins. and the diameter of the reeds and fillets is $\frac{3}{16}$ in.

Fig. 180 is an outline of window *tracery* in stone. Copy this, making the diameter of the circle containing the centres (marked by small circles) $5\frac{1}{2}$ ins.

Fig. 179A is a cross section of a moulded surface ; draw this to the given dimensions.

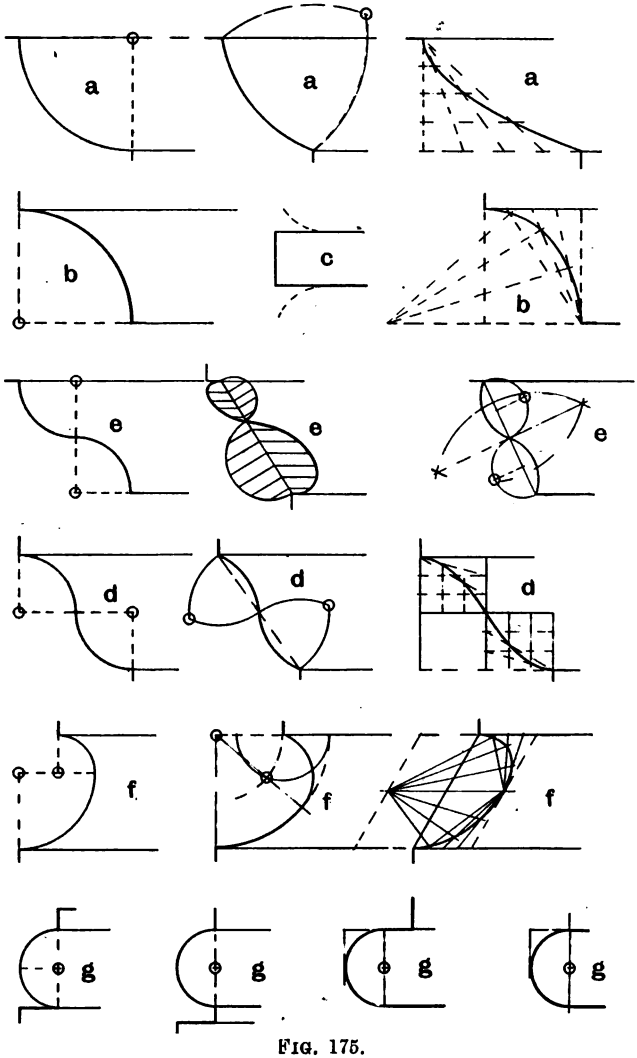


FIG. 175.

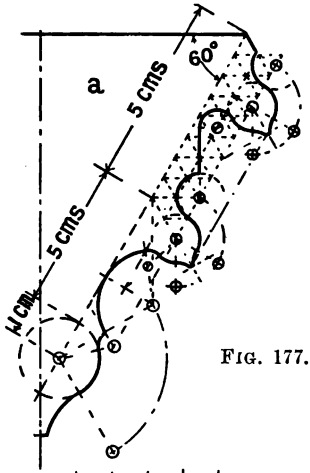


FIG. 177.

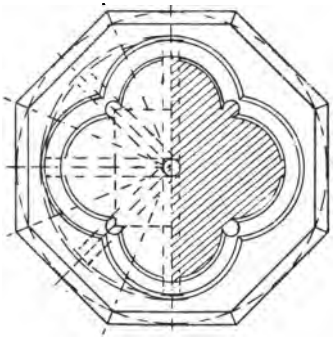


FIG. 178.

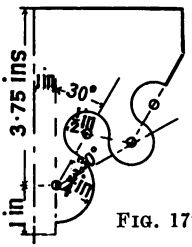


FIG. 179.

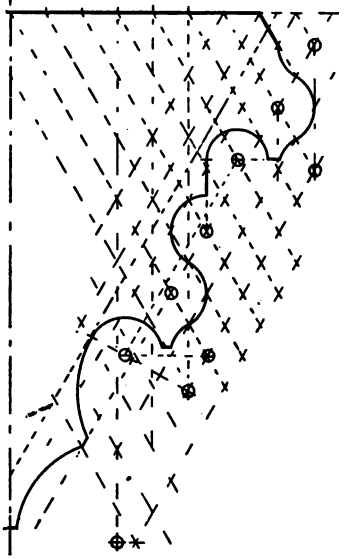


FIG. 176.

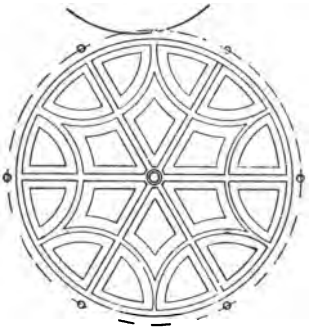


FIG. 180.

CHAPTER XV.

MISCELLANEOUS PROBLEMS AND CONSTRUCTIONS.

In this chapter, a few problems of a representative character are solved by constructions which will in general suffice to solve problems of the same class.

175. **Polar, central, or radial projection.** Let any polygon ABCDE, Fig. 181, be given. From any point *o* draw lines through each corner of the polygon. Draw any line parallel to AB cutting the radials *o*A, *o*B in *a*, *b*.

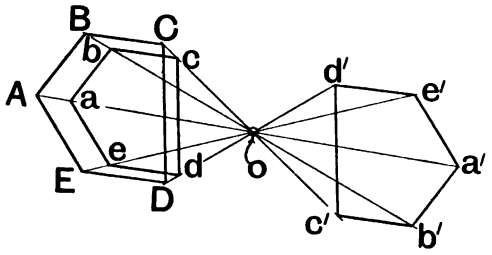


Fig. 181.

Draw *bc* parallel to BC cutting *o*C in *c*. Draw *cd* parallel to CD, cutting *o*D in *d*. Draw *de* parallel to DE, and join *ea*. Then AB is said to be **radially projected** at *ab*, and *ab* is called the *projection* of AB. Similarly *bc* is a projection of BC, and so on for the other lines. Using the term "projection" in the same sense, the polygon *abcde* is called a radial projection of the figure ABCDE. The fixed point *o* is called the **centre of projection**; it is sometimes called the **pole.** Hence the names *Central* and *Polar*, by which *radial projection* is sometimes called.

The radials from o are called *projective rays*, or simply *projectors*.

176. *The radial projection of any figure is a figure similar to the original.*

This may be shown as follows :—

In the triangles oAB, oab, we have :—

(i.) The angle at o the same in both triangles. (Since oA falls along oa, and oB falls along ob.)

(ii.) Angle $BAo = bao$; and angle $ABo = abo$. (Since ab is parallel to AB.)

(iii.) From i. and ii. the ratio $\frac{ab}{AB} = \frac{ob}{oB} = \frac{oa}{oA} =$ constant.

If each pair of triangles, as oBC, obc ; oED, oed, etc., be treated in this way, we find $\frac{bc}{BC} = \frac{cd}{CD} = \frac{de}{DE} = \frac{ea}{EA}$;

$\frac{ob}{oB} = \frac{oc}{oC} = \frac{od}{oD} = \frac{oe}{oE} = \frac{oa}{oA} =$ constant.

Again, because ab, bc are respectively parallel to AB, BC, the angles at b and B are equal. In a similar manner we can show that angles $c = C$; $d = D$; $e = E$; $a = A$. And also, $\frac{bd}{BD} = \frac{ce}{CE} = \frac{ad}{AD} = \ldots = \frac{oa}{oA} =$ constant.

Hence, *corresponding angles in the two figures are equal, and corresponding lines are in a constant ratio ; therefore the figures are similar.*

177. Similar figures are said to be *similarly placed* when straight lines drawn through corresponding points converge to one and the same point. Similarly placed similar figures are called **homothetic** figures.

The centre of projection, or pole, is called the **centre of similitude.**

In $a'b'c'd'e'$ we have a projection of the polygon on the opposite side of the pole o. This figure is similar to the original, since lines in the projection are parallel to corresponding lines in the original.

In making use of radial projection to solve problems, it must be thoroughly understood that the *pole* can be placed in any position as most convenient.

In Fig. 182, two projections of a square ABCD are shown at A′B′C′D′, *abcd*, the centre of projection *o* being inside the figure.

Sometimes the construction is simplified by placing the pole at a corner of the figure (see Fig. 183). After a little experience, the student will be able to select the most convenient position for the pole.

178. *Enlarging and reducing plane figures.* Two similar

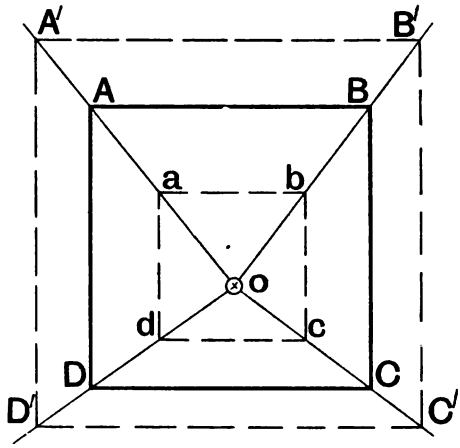


FIG. 182.

figures, one larger than the other, may be conceived as drawings of the same object drawn to different scales.

Properties of *homothetic* figures can be used with advantage in problems on enlarging and reducing figures.

Problem. Fig. 183. *Draw a figure similar to the given quadrilateral ABCD, but smaller, the ratio of AB to the corresponding side being* $\frac{3}{2}$.

Let the corner A serve as pole. Divide AD at *d* such that $\frac{Ad}{AD} = \frac{2}{3}$ (the given ratio) by problem 236. Join AC and draw *dc* parallel to DC meeting AC in *c*. Draw *cb* parallel

to CB meeting AB in *b*. Then the figures are homothetic, and A*bcd* is the required figure.

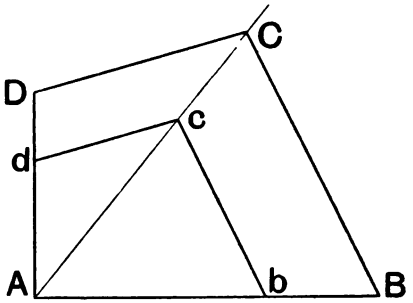


FIG. 183.

Problem. Fig. 184. *The profile of a moulding is shown at ABCDEFH.*

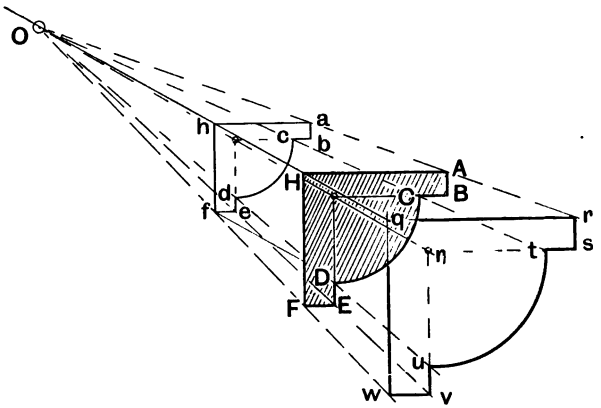


FIG. 184.

(*a*) *Enlarge this moulding, making the linear dimensions* $\frac{1}{8}$ *longer than in the figure.* (*b*) *Reduce it by* $\frac{1}{3}$ (*linear dimensions*).

(*a*) Draw a ray through H inclined both to AH and FH. Take a pole *o* anywhere on this ray and divide HO in *h*, such that H*h* = $\frac{1}{3}$HO. Make *q* on OH produced equal H*h*, *i.e.*, $\frac{1}{3}$OH.

From O draw rays through A, B, C, D, E, F. Draw *q*W parallel to HF meeting OF in W. Draw *qr* parallel to HA meeting OA in *r*. Proceeding in this way, determine points *s*, *t*, *u*, *v*, and join them up in proper order.

The curve CD is a quadrant of a circle, the centre of which projects on to r_1, the meeting point of *st* and *vu*; $r't = ru$ is the radius of the arc, which can be drawn with compasses.

(*b*) Make H*h* = $\frac{1}{3}$HO. Draw *hf* parallel to HF; *ha* parallel to HA; *ab* parallel to AB; etc. The centre of the circular arc is at the meeting point of *bc* and *ed*.

179. Mouldings have sometimes to be enlarged or reduced more in one direction than another, in which case the new figure is *not similar to the initial figure*. The radial method can, however, be used to determine the new figure.

Problem. Fig. 185. Make an enlargement of the given moulding *abcdefh*, increasing the depth by $\frac{3}{5}$*fh*, and the projection by $\frac{1}{3}$*ah*. In this case, two poles are necessary, one from which to project distances measured parallel to *hf*, and another from which to project distances measured parallel to *ah*. In the figure, these poles O, O′ are placed in any convenient position on *ah*, and *fh* produced. Make *f*F equal $\frac{3}{5}$*hf*, the given ratio of increased depth; join FO, and draw *fg* parallel to *ah* meeting FO in *g*. Draw *gk* parallel to *fh*. From *c* and *d* draw parallels to *ah* meeting *kg* in *s* and *t*. From *o* draw rays through *s*, *t* meeting *hf* in *u* and *r*. Then *hu* : *ur* : *r*F = *ks* : *st* : *tg*. Make *a*A on *ha*, produced equal to $\frac{1}{3}$*ah*, the given ratio of increased width. Join AO′ and draw *ag*′ parallel to *h*O′ meeting AO′ in *g*′. Draw *g*′*k*′ parallel to A*h*. From *c* and *d* draw parallels to *fh* meeting *k*′*g*′ in *s*′, *t*′. From O′ draw rays through *s*′, *t*′ meeting A*h* in *u*′, *r*′. Then *hr*′ : *r*′*u*′ : *u*′A = *k*′*t*′ : *t*′*s*′ : *s*′*g*′. Draw *u*B parallel to *h*A meeting verticals from A and *u*′ in B and C. Draw *r*D parallel to *h*A meeting a vertical from *r*′ in D. Draw FE parallel to *h*A meeting *r*′D in E.

Points in the projection of the curve can be found as follows—point P, for example :—Draw *pv*, *pv'*, respectively

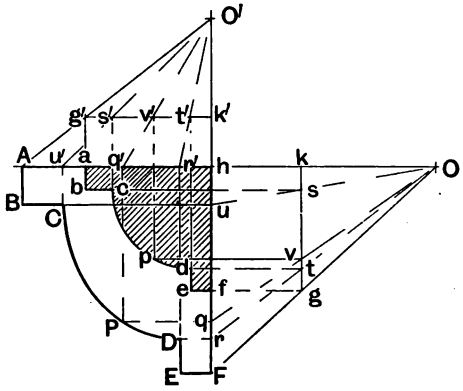


FIG. 185.

parallel to *ah*, *hf* meeting *kg*, *k'g'* in *v* and *v'*. Project *v'* from O' on to *ah* at *q'*. Project *v* from O on to *hf* at *q*. From *q* and *q'* draw parallels *ha*, *hf* meeting in P. Then P is the projection of a point corresponding to *p*. In a

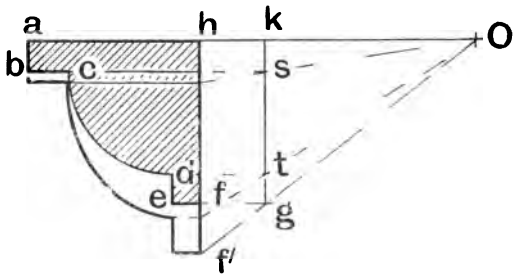


FIG. 186.

similar manner determine other points, and draw a fair curve by freehand through the points.

Fig. 186 shows how a moulding is increased in depth whilst the width *ah* remains the same.

Inscribed and Circumscribed Figures.

180. A figure A is said to be *inscribed* in a figure B, and B is said to *circumscribe* A, when no part of A falls outside

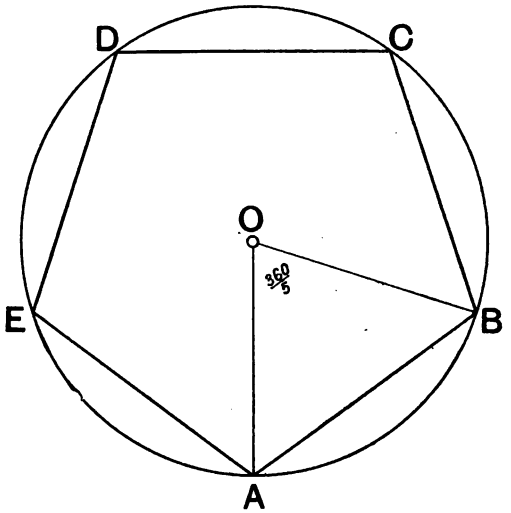


FIG. 187.

B, and every corner on the boundary of A lies on the boundary of B, and *every side* of B contains, at least, one point in the boundary of A.

181. *In a given circle, any regular rectilinear figure can be inscribed.*

Problem. Fig. 187. To inscribe a regular *polygon* in a given circle centre O. Let the given *polygon* be a *pentagon* (5 sides). Draw any radius OA. Then divide the circumference into the same number of equal parts as there are sides in the polygon, viz., 5. This is most conveniently done, either (1) by setting out the angle AOB subtended at the centre O by a side of the figure; this angle AOB $= \frac{360^\circ}{5} = 72^\circ$.

The angle 72° may be plotted direct from a protractor, and in this way a side AB of the figure is determined. The other angular points C, D, E are found by stepping round. (2) The circumference may be divided by trial with dividers.

If the given figure is a *hexagon*, each side is equal to the radius of the circle.

182. In the 17th century, Renaldinus discovered a

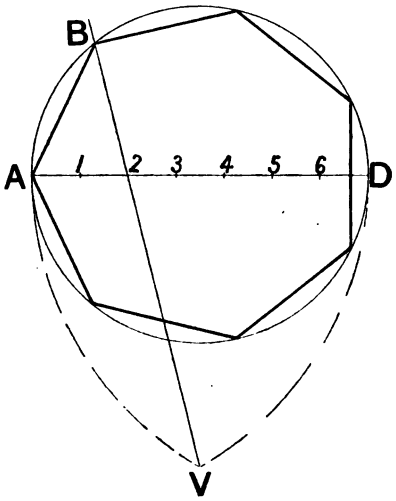


FIG. 188.

graphic method of inscribing any *polygon* in a given circle To illustrate the construction, let it be required to inscribe a *heptagon* in the given circle, Fig. 188. On a diameter AD describe an equilateral triangle AVD. Divide AD into n equal parts, where n = the number of sides in the required figure (7 in this case). From V draw a line through 2, the second division from A, to intersect the circle on the side of AD remote from V, in B. Join AB, and this line is one side of the inscribed figure.

In all cases, divide the diameter into a number of equal parts equal to the number of sides of the required figure

and draw the straight line from V through the second division from A.

The construction gives accurate results when $n = 3, 4,$ and 6, *i.e.*, for an equilateral triangle, a square, and a hexagon. For other numbers the result is only approximate, but the error is less than 1 per cent. for numbers below 10.

When making large drawings, it is best to employ the principle of Art. 107 to determine the angle subtended at the centre; then compute the length of a side of the *polygon*. Thus, take from a table the numerical value of

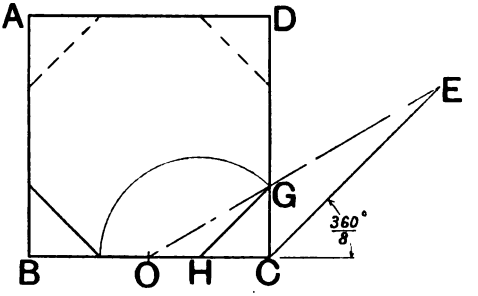


FIG. 189.

the chord of the predetermined angle, and multiply by the length of radius of circle. The product is the length of a side of the inscribed figure.

183. In a regular rectilinear figure of n sides, we can inscribe a regular figure of $2n$ sides, so that each side of the circumscribing figure shall contain a *side* of the inscribed figure.

Ex. Fig. 189. In a given square ABCD, inscribe a regular *octagon*.

Let us suppose that BC is one side of a regular octagon. Draw (CE) the adjacent side of such figure. Now bisect BC in O. Join EO intersecting CD in G. Draw GH parallel to EC meeting BC in H. Then GH is a side of a regular octagon homothetic with that on BC, O being the pole. The remaining sides can be marked off with compasses.

184. **Refraction of Light.** It is shown in works on Light that a luminous ray is propagated in a straight line through the ether, or through any homogeneous medium, such as glass.

The velocity of light depends on the medium through which it is passing ; its velocity through good quality glass is about two-thirds of its velocity through the air.

FIG. 190.

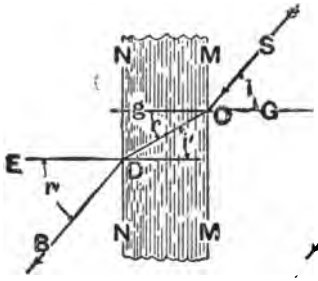
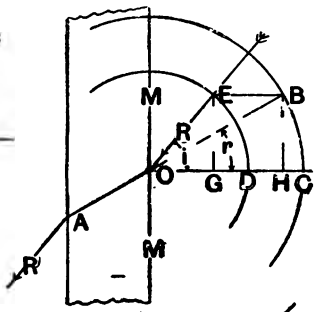


FIG. 191.



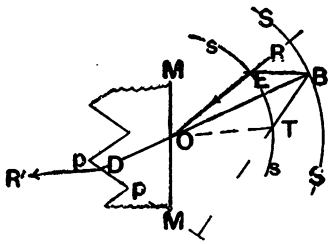


FIG. 192.

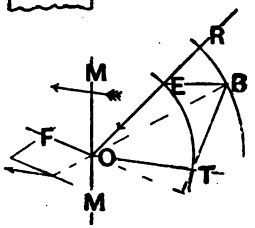


FIG. 193.

Light changes its direction when it passes obliquely from one medium to another ; this change in direction is called **refraction.**

If SO, Fig. 190, indicates a ray of light falling on a plane sheet of glass at O, and OD is the path of the ray through the glass ; DB the direction of the ray after it emerges from the glass ; and OG, DE respectively are normal to the

surfaces MM and NN ; then SO is called the *incident* ray ; OD is called the *refracted* ray ; DB is called the *emergent* ray ; the angle SOG (i) is called the angle of *incidence ;* the angle DOg (r) is called the angle of *refraction;* the angle BDE (r') is called the angle of *emergence.* It can be shown that

$$\frac{\sin. i}{\sin. r} = \frac{\sin. r'}{\sin. i'} = \frac{\text{velocity of light in air}}{\text{velocity of light in glass}}$$

The above ratio is called the **index of refraction, or refractive index.**

Problem. Fig. 191. *A luminous ray* R *falls at* O *on the surface* MM *of a plate of glass, with parallel faces. Trace the path of the ray through the glass and show its direction after it emerges from the glass, having given the angle of incidence* (i) = 50°, *refractive index* = 1·55.

With centre O and any radius OD describe a circle : with centre O and radius OC = 1·55 OD describe a second circle. From E, the point in which the incident ray intersects the smaller circle, draw EB ⊥^{r} to MM. Draw BOA, which is the path of the ray through the glass.

For $\sin. i = \frac{EG}{EO}$ $\quad \sin. r = \frac{BH}{BO}$

but $EG = BH.$ $\quad \therefore \frac{BO}{EO} = \frac{\sin. i}{\sin. r}$

To find the direction which the ray takes on leaving the glass, draw AR' ∥ RO.

Problem. Fig. 192 *is a section through a piece of glass, the outside face of which is plane, whilst prisms are cut on the inside.* RO *indicates a ray of light falling on the glass at* O. *Determine the path of the ray after it leaves the glass ; having given the refractive index of the glass* = 1·53.

With O as centre describe two circles *ss*, SS the ratio of their radii being equal to the refractive index. Draw EB ⊥^{r} MM meeting the arc SS in B. Draw BO and produce to meet *pp* in D. This is the path of the ray in the glass. Draw BT perpendicular to the face *pp* of the prism, meeting the smaller circle in T. Draw DR' ∥ TO. Then DR' is the path of the emergent ray.

Problem. Fig. 193. *Having given a surface* MM *upon which an incident ray* RO *falls, determine the angle between this surface and a surface which will refract the ray in a direction parallel to that of the given arrow, having given the refractive index* = 1·53.

From centre O describe two circles the ratio of whose radii is equal to the given index 1·53. Draw EB ⊥^{r} MM. Draw OT ∥ to the

direction indicated by the arrow, meeting the smaller circle in T. Join BT and draw OF ⊥r BT. OF is parallel to the refracting surface required.

185. **Reflection of Light.** When a ray of light meets a highly polished surface, part of the light is *reflected* according to the following laws :—

1. *The angle of reflection is equal to the angle of incidence.*
2. *The incident and the reflected rays are in one plane, which is perpendicular to the reflecting surface.*

If an object is held in front of a plane mirror, it appears behind the mirror at a distance equal to the distance of the object in front of the reflecting surface ; this reflection of the object is called an **image.**

When an object is placed between two plane mirrors, it is reflected in both mirrors. The image of an object in a plane mirror is determined geometrically by aid of the fore-

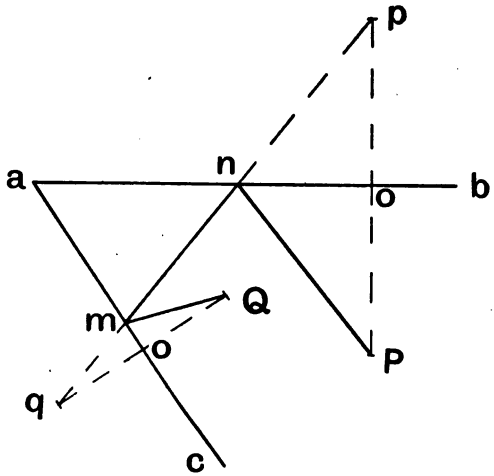


FIG. 194.

going principle which applied to a point becomes —. *The image of a point is on the perpendicular from the point on to*

the mirror and at a distance behind the mirror equal to that of the point in front.

Problem. Fig. 194. *A ray of light from* P *is reflected to* Q *after impinging successively on ab and ac.* Show *the path of the ray.* (B. E. 1892.)

Draw $Pp \perp^r ab$ and make $op = oP$. Draw $Qq \perp^r ac$ and make $oq = oQ$. Join p, q meeting ab in n and ac in m. Join Pn, nm, mQ, which is the path of the ray.

186. **Graphic Determination of Mass Centres.**

If a piece of card, sheet metal, or other material of uniform thickness and density, be placed flat against a vertical plane and fixed thereto by means of a pin inserted in a point O taken at random, the card will rotate on the pin and ultimately come to rest. Mark a point Q near to the edge of the card and vertically under the pin. If the pin be passed through another point O′ in the card and the test repeated, we can find a different point (P) on the card vertically under the pin. If we again change the position of the pin, and pass it through any point *in the straight line* OQ, we shall find that the card will come to rest with the line OQ vertical, *i.e.*, Q will lie either *below* or *above* the pin. Similarly, if the pin be inserted *in any point in* O′P, the card will come to rest with O′P vertical, and P will lie either *below* or *above* O′. Suppose the lines OQ, O′P intersect in C, and let the card be hung on a pin inserted at C, we then find the card has no tendency to turn on the pin and that it will remain stationary in any position. The point C is called the **Centre of Mass** of the card; it is often referred to as the **Centre of Gravity** or **Centre of Area.**

187. The centre of mass of any plane figure may be determined accurately or approximately by graphic construction. A general method is given on p. 355. We give here a few examples which may be solved by easy constructions. In all cases in which the figure has an axis of symmetry, the centre of mass lies in such axis.

Ex. 1. Triangle. — In any triangle, a straight line joining a vertex to the mid-point of the opposite side is an axis of symmetry; hence the centre of mass of a triangle is found by drawing two such lines in the figure.

The following principle is sometimes useful.

Fig. 195. *If from the ends of the base* AB *of a triangle* ABC *we take equal distances* AE, BF, either within or without the triangle, it can be shown that the centre of mass

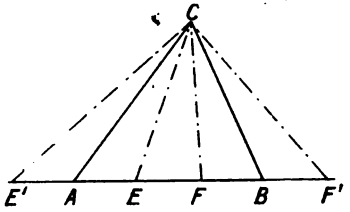


FIG. 195.

of the triangle CBA is also the centre of the triangles CFE, CF'E'.

Ex. 2. A trapezium ABCD, Fig. 196. Bisect AB, DC in E, F. Then EF is an axis of symmetry. Prolong CD to H such that DH = BA; prolong AB to M such that

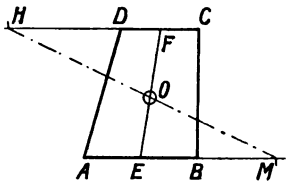


FIG. 196.

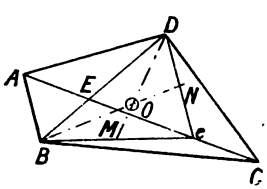


FIG. 197.

BM = DC, and join HM, which determines by its intersection with EF the required point in O.

Ex. 3. A **quadrilateral** ABCD, Fig. 197. Draw diagonals BD, AC, meeting in E. Make C*e* = AE. Join D to M the mid-point of B*e*, and B to N the mid-point of D*e*. Then O is the required centre of mass of Δ B *e* D, and also of Quadrilateral ABCD. Refer to Ex. 1.

Ex. 4. *A piece of wire in the form of a* **circular arc** AB. Fig. 198. Draw the chord AB; Draw PT

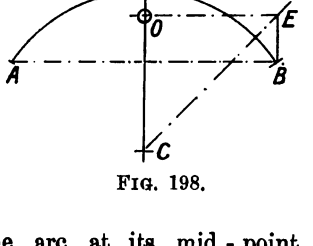


FIG. 198.

tangent to the arc at its mid-point P and make PT = arc PB. Draw BE ⊥ AB meeting CT in E; draw EO ‖ BA. Then O is the required centre of mass.

Ex. 5. A Sector of a Circle APBC. Fig. 199.

If we imagine the sector as built up of an infinite

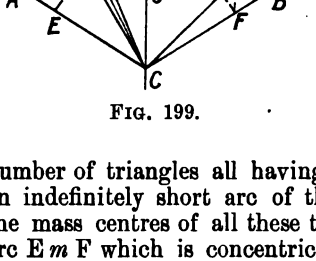


FIG. 199.

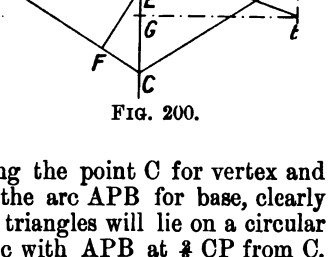


FIG. 200.

number of triangles all having the point C for vertex and an indefinitely short arc of the arc APB for base, clearly the mass centres of all these triangles will lie on a circular arc E *m* F which is concentric with APB at $\frac{2}{3}$ CP from C. Therefore the centre for the sector APBC will be the same as that for the arc E *m* F, and is found as in Ex. 4. It is shown at O.

Ex. 6. A Circular Segment APB. Fig. 200.

The segment APB is the difference between the sector

APBC and the triangle ABC. Determine first G the mass centre of triangle ABC; then G′ the mass centre of the sector. From point E in which CP intersects AB, draw EF $\perp^r$ AC meeting same in F. Draw Gt, G′f respectively equal to (or proportional to) PT, EF (where PT = arc PB). Join tf and produce to meet CP in O, which is centre of mass of the sequent APB.

CHAPTER XVI.

LOCI.

188. The student is already familiar with the idea of a line being traced by a point moving subject to some conditions which compel the point to keep always in that particular course. Certain examples will readily come to mind thus, (1) *a straight line is generated by a point moving always in the same direction;* (2) *a circle is the path of a point which moves in a plane at a constant distance from a fixed point.* The straight line is an example of *simple translation;* the circle is one of *simple rotation* about the fixed centre.

In the constructive arts, we have constantly to deal with curves traced by a point the movement of which is the result of two or more conditions acting simultaneously on the point. In particular, various parts of machines have certain paths of motion, the form and position of which it is often necessary to know.

Many of these loci, even when the conditions which govern the motion are complex, may be drawn without much difficulty by locating a number of positions in the path of the moving object.

A few easy examples will now be given, and these the student should carefully draw to a large scale and satisfy himself on the points discussed. The use of tracing-paper in some of the examples tends to accuracy and considerably reduces the labour. In nearly all the examples, the student will be able to make a model* of a mechanism which would serve to draw the locus mechanically; and although the model which the student is able to make may be of no use in actually drawing the locus, it may serve to determine

* Narrow strips of thin celluloid serve admirably to make models of link-work mechanisms; failing this, stout tracing-paper can often be used. Small drawing-pins placed point upwards will serve as pin-joint connections.

points; and in any case it will prove of great value in showing the kind of movement that takes place in an actual mechanism.

The student must determine the positions of points as accurately as he can, and at commencement he will find much care necessary.

189. **Watt's parallel motion.** In Fig. 201 is shown an arrangement of three links devised by James Watt to

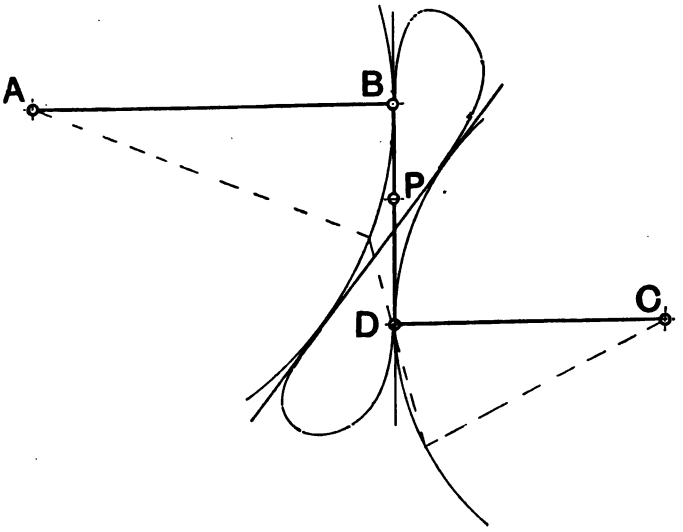


FIG. 201.

constrain piston-rods and other parts of steam engines to move in straight lines. It consists of two links AB, CD, having fixed centres A, C, and a connecting-link BD. If BD be moved into every possible position, the extremities B, D will describe circular arcs about the respective centres A and C, whilst the path of any point as P in the link is a sort of "figure eight." If the position of P be such that the segments PD, PB of the link BD are inversely as the length of the adjacent arms DC, AB, the locus of P

within certain limits of the motion will be approximately a straight line. This important principle the student should verify, either by aid of a simple model or as follows: From fixed centres A and C, with radius AB and CD respectively, describe arcs of circles representing the paths of B and D. Draw a straight line equal in length to BD on a piece of tracing-paper and divide it at P such that $\frac{BP}{PD} = \frac{CD}{AB}$. Place the tracing with the end D of the line on the circular locus of D, and the end B on the locus of B, and prick through the point P. Do this for several positions of the moving line.

190. In Fig. 202 are shown two links CP, OP connected by a pin-joint at P. The link CP rotates on a fixed centre

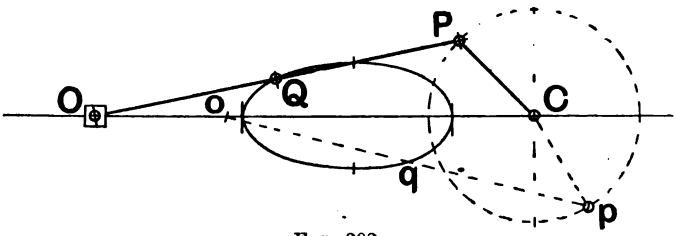


FIG. 202.

C, whilst the end O of link OP is constrained to travel along the straight line OC. Find the locus of Q having given OQ = QP = 2 in., and PC = 1 in.

(CP, OP and O respectively correspond with the crank, the connecting-rod, and the cross-head of a direct-acting engine.)

Construction. With any point *p* on the circular locus of the pin P as centre, and radius PO, describe a circle cutting OC in *o*. Join *op* and make *oq* = OQ. In a similar manner determine other points in the locus and draw a fair curve through the points.

Tracing-paper may be used with advantage. The point Q is one of the points in a mechanism known as Joy's

valve gear, extensively used in locomotive and other steam engines to control the motion of the slide-valve.

191. In Fig. 203, two rods BA, QP are connected by a pin joint at A, the mid-point of QP. The rod BA = PA, and turns on the fixed centre B, the end P of QP slides along the fixed straight line nBn. Find the locus of the point Q.

With centre B and radius BA describe a circle, which is

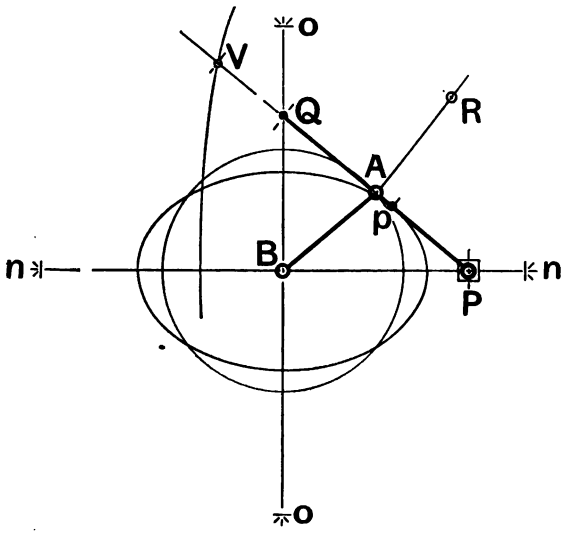


FIG. 203.

the path of A. Mark the points P, A, Q in a straight line on a piece of tracing-paper; and move this tracing-paper over the drawing, keeping A always in the circular path of A, and P on the line nn. Different positions of the point Q may be marked by pricking through the tracing. Observe that the path of Q is a straight line perpendicular to nBn.

Exercise. Reproduce the lines nn, oo, QP, AR, in Fig. 203, and find the locus of each of the points, V, p, R, and A, when P moves along the line nn, and Q moves along oo. (Use tracing-paper.)

The locus of p is an ellipse. The locus of V is also an ellipse; that of R is a straight line.

Exercise. Two straight lines l and m intersect in a point O and include an angle of 60°. A line AB 3 in. long moves with the end A always in l, and the end B in m. (1) Find the path of a point P in AB and 1 in. from A. (2) Find the path of a point Q which forms the apex of a triangle having AB for base and sides AQ = BQ = 2¼ in.

192. In Fig. 204 is an example of the *four-link chain.* The four links are connected by pin joints. The link MN is fixed; NR rotates about the fixed centre N, during which time the link MD vibrates through the angle aMd. Determine the locus of the point Q in the link DR.

Describe the circle centre N, radius NR, in which the

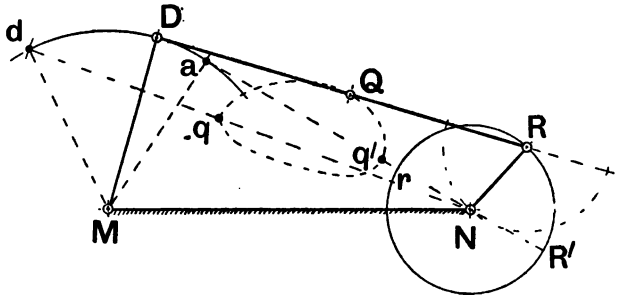


FIG. 204.

point R moves. Describe an arc centre M, radius MD, in which the point D moves. With any point as R′ in the locus of R for centre, and radius equal to RD, describe an arc cutting the locus of D in a. Join aR′ and make R′q' on this line equal RQ. Then q' marks the position of Q when the link DR occupies the position aR′. Other points in the path of Q are found in a similar manner.

Important points in the locus are those which correspond with the limiting positions of D. These may be found as follows. With centre N, and radius equal to DR − NR, and DR + NR describe circles which intersect the arc through D in a and d. The points a, d are the limiting positions of D. Draw dN, intersecting the locus of R in r. Make rq = RQ. Draw aN and produce to cut the circle through R on

the side remote from D in R′. Make R′q' = RQ. Then q and q' are the points required.

193. Fig. 205 shows an arrangement of linkwork forming a parallelogram with two adjacent sides extended: the members of the frame are connected at the corners ABCD by pin joints.

Problem. *If the point* P *in the arm* AD *is made to trace the profile of the moulding* Y *whilst the linkwork rotates about the fixed point* O *as a pivot, determine the locus of a point* N *in the arm* CD *and in the straight line* PO.

Since the points B, A are at a fixed distance from the fixed point O, the paths of these points are circles struck from O. Let P move into a new position p. With this point as centre and radius PA describe an arc intersecting the circle through OA in a. Join aO, ap and make pd = PD. Then b, a, d are new positions of points B, A, and D corresponding to p. With centre d and radius DC describe an arc, with centre b and radius BC describe an arc intersecting the arc drawn from d in c. Join cb, cd and make cn = CN. Then d and n are positions corresponding to p, and n is a point in the required locus. If sufficient points be taken in the path of P a fair line drawn through the corresponding positions of N will be the Fig. y. *The locus of* N *is a curve similar to the curve traced by* P. This may be shown as follows. Let the line diagram (Fig. 205A) represent the framework, and let a straight line PO intersect the four parts of the frame in P, N, R, O.

From similar triangles OBR, OEN, OAP,

$$\frac{OR}{OP} = \frac{OB}{OA'} \quad \frac{OR}{ON} = \frac{OB}{OE}.$$

Let the system rotate about the point O, P moving into a new position p. The position of the parallelogram corresponding to p is at $abcd$, and the triangles Obr, Oen, Oap are similar, hence the points $pnro$ are co-linear.

Also
$$\frac{On}{Op} = \frac{Oe}{Oa} = \frac{OE}{OA}.$$

Join Pp, Nn. Then the triangles Pop, Non are similar. For corresponding sides OP, ON, Op, On are proportional, and the angle at O is common, hence Pp ∥ Nn, and

$\frac{Pp}{Nn} = \frac{OP}{ON} = \frac{OA}{OE}$, that is, the points P and N have parallel paths, and the ratio of the lengths of these paths is equal to $\frac{OA}{OE}$. Consequently if P trace any geometrical figure, N will trace a similar figure.

In a similar manner, we can show that if P, N, R, O be any range of co-linear points one on each side of a parallelogram (or side produced), and if either point remain fixed

FIG. 205A.

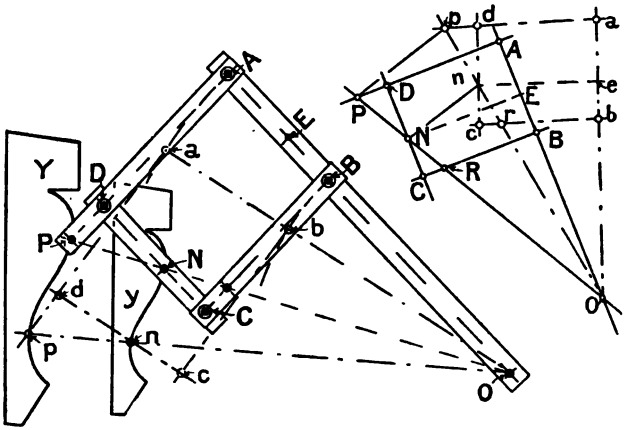


FIG. 205.

whilst one other traces a given figure, the remaining points will describe similar figures.

The profile traced by N in Fig. 205 is a copy of the profile Y to a scale of $\frac{OE}{OA}$.

The linkwork illustrates a device called **a pantograph**, which is extensively employed in machinery, and is also used to make copies of drawings to any scale. By properly selecting the fixed centre O, the position of the tracing point P, and the position of the pencil N, a curve can be copied to any given scale.

194. In Fig. 206 (a), a line PP′ always passes through the fixed point S, and a fixed point A in PP′ moves in the straight line XX. The locus of any fixed point in PP′ is a curve called the **conchoid of Nicomedes.**

Any point, as P′, on the side of S remote from the directing-line XX forms a loop; points as P, p, V′ on the other side, form waves which have points of contrary flexure at V′, V″, V‴.

When given the perpendicular distance of the point S from the directing-line XX and the position of a definite point P in the moving line, the locus of the point is readily found by drawing different positions of the moving line and marking off the constant AP, at ap, $a'q$, etc., and drawing a fair line through the points thus found.

195. The conchoid is often used in architectural columns to give to the column a swell or **entasis.**

Problem. *To apply the conchoid to the entasis of a column.* Fig. 206 (b). *Let* XX′ *represent the axis of the column,* PX the semi-diameter at the base, and AX′ the semi-diameter at the top.

Draw AE ∥ X′X. With centre A and radius PX describe an arc meeting XX′ in V. Divide XV and EA each into the same number of equal parts (four in the diagram) at 1, 2, 3; I, II, III. Draw lines through corresponding points 1, I; 2, II; etc., and make $1P_1$ $2P_2$, etc., = XP. Then P_1, P_2, etc., are points in the curve which can be drawn with the aid of a flexible ruler.

196. Many problems connected with mechanisms have reference to the rolling of some curve on a stationary line or track which may be either straight or curved; and the draughtsman frequently requires to know the path traced by some point attached to the rolling object.

Problem. *The circle centre* C, Fig. 207, *rolls without sliding along the straight line* LM. *Find the path of the point* P *on the circumference.*

Note. In the solution of this problem, we employ a

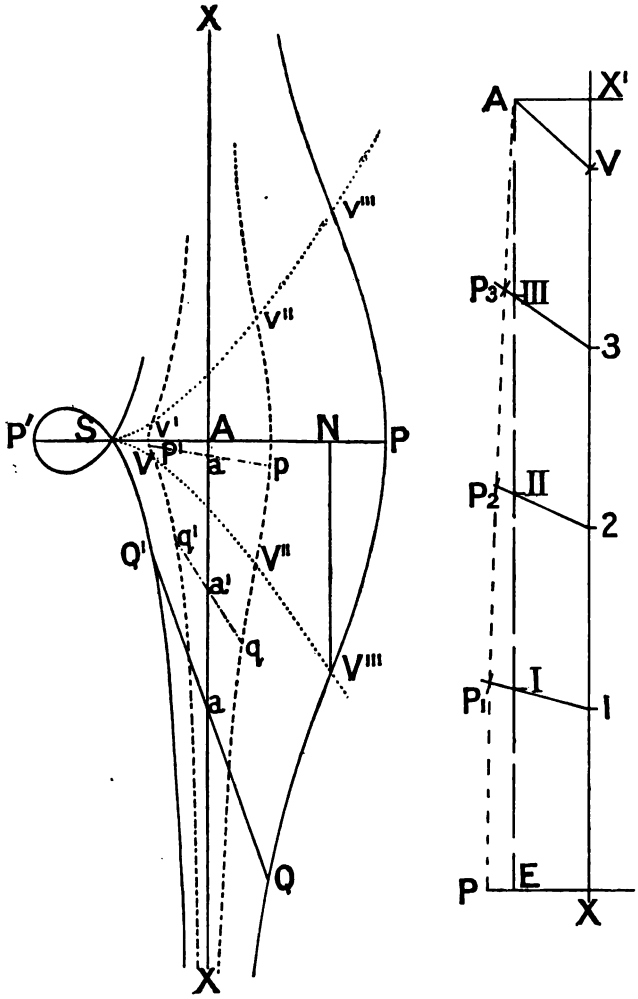


FIG. 206.

mechanical method which does not satisfy the conditions of *perfect rolling contact*; the method, however, is general in its application, and when used with care gives satisfactory results.

Copy the circle on a piece of stout tracing-paper and mark the point P thereon. Place the tracing over the drawing with P in correct position, and insert a pricker at the point of contact O. Rotate the tracing a little on the pricker, bringing a short length of the circumference near O into close agreement with the line LM. This rotation will bring P into a new position P′, which can be pricked through on to the drawing. Hold the tracing still and transfer the pricker to a second point O′, where the circle and line LM are in agreement; repeat the rotation and prick through the new position of P. Continue in this way until the circle has measured its whole circumference along the line LM.

It will be observed that the circle moves forward with a hobbling motion, like a cart-wheel with the felloe off running on the ends of the spokes. The shorter the steps taken, the more nearly the circumference of the rolling object is measured along the track LM.

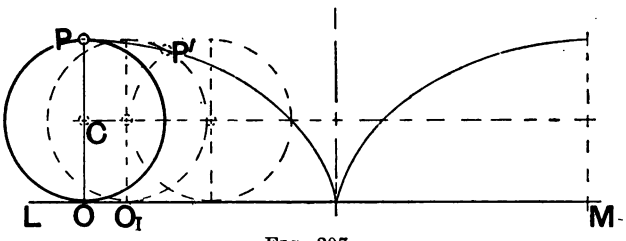


FIG. 207.

The path of the point P is a curve called the **cycloid.** It has a *cusp* on the line LM.

Exercise 1. A circle of radius 1½ in. rolls along a straight line. Find the locus of points Q and Q′ which lie in the plane of the circle, Q being 1 in., and Q′ 2 in. from the centre of the circle.

197. **Cams.** A cam is a member of a machine so

shaped that its motion (which may be either one of rotation, translation, or oscillation) imparts a definite motion to another member upon which it acts by direct contact, or through an auxiliary roll, or block.

Figs. 208 and 209 show two forms. In Fig. 208, the cam plate S rotates on the fixed centre C and gives to the slider P a reciprocating motion in a vertical direction. The plate S which controls the movement of P is called the *driver*, the slider P is called the *follower*. The curved edge of the plate in Fig. 208 is in contact with the end of the slider and is called the *pitch line*. In Fig. 210, a small roller is attached to the follower. The motion of the slider is not affected by the roller. The function of the latter is to obviate friction between the two pieces. The pitch line in this case is *not* the curved edge of the driver, but the dotted line which represents the curve traced by the centre of the roller on the rotating plane of the driver; this line is parallel to the edge of the cam plate.

In the designing of cams, we have geometrical problems of two types :—(1.) To determine positions of the follower corresponding to definite positions of a driver of given shape.

(2.) To determine the shape of a driver to impart a definite motion to the follower.

Ex. Fig. 210. *The cam* S *rotates counter-clockwise with a uniform angular velocity; determine positions of the centre of the roller on the slider* P *at intervals of* $\frac{1}{8}$*th of a revolution.*

Draw CO, CI, CII, etc., including angles of 60°, and intersecting the pitch line in O, I, II, III, IV, V, E. With centre C and radii CI, CII, etc., describe arcs meeting the line of travel of the slider in 1, 2, 3, 4, 5. These points are the positions occupied by the centre of the roller at periods corresponding to the given angular movements of S.

For, suppose the plate S to rotate through the angle a_1, the slider will be forced upwards through a distance equal to CI – CO, and the position of the centre of roller will be at 1. After further rotation through the angle a_2 the position of the roller centre will be at 2, such that C2 = CII, and so on, the slider continuing to rise so long as the distance of the point of contact from the cam axis increases, that is,

until $\frac{5}{6}$ths of a revolution has taken place, when the point V occupies position 5. It will be observed that the portion of the pitch line from V to E is straight and lies along the radial VC, whilst EO is an arc of a circle struck from C. Corresponding to the straight line VE, the slider drops suddenly from 5 to O; and corresponding

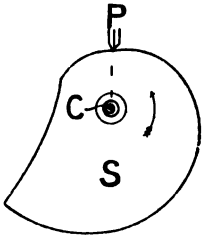


FIG. 208.

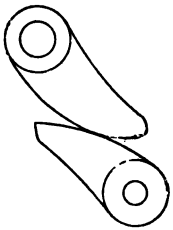


FIG. 209.

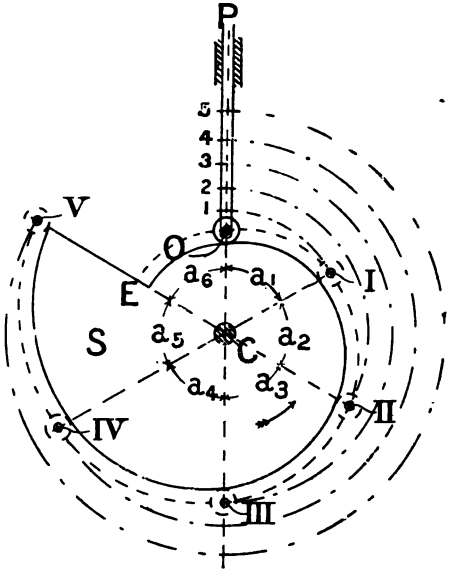


FIG. 210.

to the circular arc EO, the slider remains stationary during the rotation of the driver through the angle a_6.

Ex. 2. Fig. 211 shows a case in which the line of stroke of the reciprocating piece P does not, if produced, pass through the axis of the cam, but passes at a distance aC from the axis.

In this case, if CO be joined we have a right-angled triangle OaC, and if the cam rotates, the locus of point a will be a circle centre C; and for any given angular movement of the cam each side of the triangle will turn through the same angle.

Let 1, 2, 3, . . . be positions of the roller centre on the stroke line of the slider corresponding to positions I, II, III . . . on the cam plate. Let tangents to the circle of radius C*a* from points I, II, III . . . have contact with the circle at a_1, a_2, a_3. Then if C1 and CI be drawn, the triangles C*a*1, Ca_1I are equal in all respects, and the angle θ through which C*a* has rotated is equal to θ, the angle between O*a* and Ia_1. Similarly, triangles C*a*2, Ca_2II are equal, and angle $aCa_2 = \phi$, the angle between IIa_2 and the stroke line. This fact enables us to plot corresponding positions of the roller on the cam plane, and the stroke line of P, *e.g.*, *to show the position of the roller centre on the stroke line, and cam plate, corresponding to a rotation of* 90°. Draw C*a* ⊥r to PO meeting same in *a*. Describe a circle centre C and radius C*a*. Draw Ca_3, making 90° (the given angle), with *a*C meeting the circle in a_3. Draw a tangent to the circle at a_3, crossing the pitch line in III. Then III is the position of the roller in the cam corresponding to a rotation of 90°. With centre C and radius CIII describe an arc crossing the stroke line PO in 3. This point is the position of the roller centre corresponding to the given rotation.

As an illustration of the second type, take the following quantitative example :—Design a cam to operate a sliding piece according to the following conditions. *The centre of the cam-shaft to be* 2¼ *ins. from its edge at the nearest point, and* 2 *ins. from the line of stroke of the slider; the diameter of roller to be* ⅝ *in.; the slider to rise uniformly through* 3 *ins. as the cam-shaft turns through* ½ *a revolution; to remain stationary during turning of the shaft through the next* 60°, *and then descend uniformly to its initial position.*

This cam is shown in Fig. 211. First plot the line of stroke PO, and the cam axis C. Describe the circle of radius C*a*; from O on OP, set off the given rise 3 ins. at O6 and divide O6 into a number of equal parts as at 1, 2, 3, 4, 5, 6. Divide the lower half of circle aa_6 into 6 (the same number) equal parts by radials Ca_1, Ca_2, etc. Draw tangents of the circle at the points a_1, a_2, . . . and with centre C, and radius C1 describe an arc intersecting the tangent from a_1 in I. Similarly, with the same centre C,

describe arcs through 2, 3, 4, 5, 6, intersecting the tangents from a_2, a_3 . . . in II, III, . . . VI. These points are positions of the roller on the pitch line corresponding to points 1, 2, 3, . . 6, and a fair line through these points is the required pitch line of the first half revolution.

As the slider is to rest during the next 60° rotation, draw the radial Ca_7 making 60° with a_6c, and draw the tangent

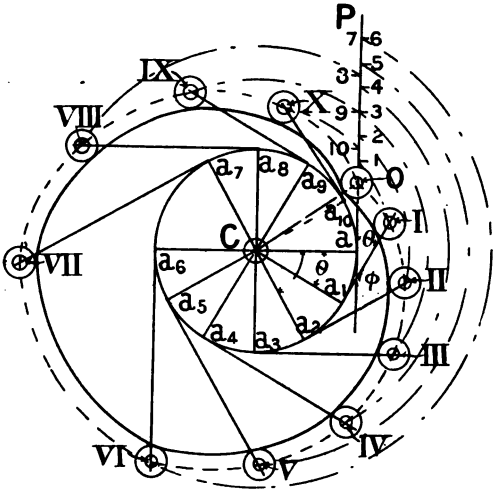


FIG. 211.

at a_7. Describe a circular arc from centre C through VI meeting the tangent from a_7 in VII. This arc is the pitch line for the period of rest.

Divide the stroke line 7 O, and the arc a_7a each into four equal parts at 8, 9, 10, a_8, a_9, a_{10}. With centre C, describe arcs from 8, 9, 10, meeting tangents of the circle from a_8, a_9, a_{10} in VIII, IX, X. A fair line through these points completes the pitch line.

Make the various points on the pitch line centres of circles of diameter $\frac{5}{8}$ in., and draw the interior touching line, or envelope, of all these circles. This is the required outline of the cam.

198. Many problems are easily solved by loci methods.

Ex. In Fig. 212 *the line* WV *represents the face of a vertical wall.* VE *represents the top surface of a sloping roof which abuts against the said wall. In making extensions to the building, it is desired to construct a span roof having both sides equally inclined to the vertical; its ridge on the line* VE, *and eaves at* C *and* D *respectively. Determine the position of the ridge.*

This problem may be rendered into geometrical terms thus:—Given two points C and D, and lines VE and WV; find a point Q on the line VE such that lines from C and D to the point Q shall make equal angles with WV.

Draw C*s*, DS parallel to WV. With centres D and C, and any radius, describe arcs intersecting C*s*, DS in *u* and U respectively. Draw D*u*, CU which make equal angles with the direction WV, and if (*n*), the point in which they intersect, falls on VE the problem is solved. In the

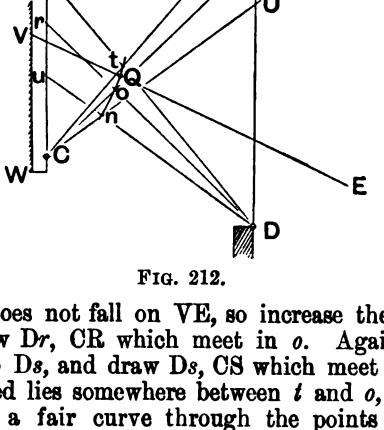


FIG. 212.

diagram *n* does not fall on VE, so increase the radius to D*r*, and draw D*r*, CR which meet in *o*. Again increase the radius to D*s*, and draw D*s*, CS which meet in *t*. The point required lies somewhere between *t* and *o*, it is found by drawing a fair curve through the points *t*, *o*, *n* to intersect VE in Q.

Ex. 2. *A brick pier with the load it supports weighs*

30. *tons. The dimensions of the pier at the base are 8 ft. × 3 ft. 9 in. The base of the pier is to be extended equally on all sides so that the load on the supporting earth shall not exceed 2·5 tons per square foot. Make a diagram showing the extended base.*

Draw a rectangle ABCD, Fig. 213, with sides in the ratio 3 : 3·75, *i.e.*, in the ratio of the sides of the pier. Through the centre O of the figure draw lines OM, ON parallel to the sides of the rectangle. The total area of the supporting surface may be found from the equation

$$\text{Area} = \frac{\text{Total load in tons}}{\text{Safe load per foot super. in tons}}.$$

$$\textit{i.e.}\ \text{Area} = \frac{80}{2{\cdot}5} = 32 \text{ sq. ft.}$$

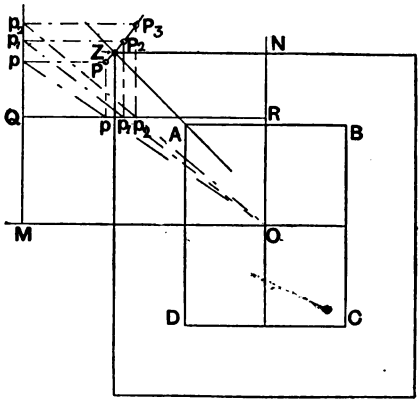


FIG. 213.

One fourth of this area, or 8 sq. ft., forms a rectangle having adjacent sides along OM, ON. The corner opposite to O lies in the straight line which bisects the angle DAR (since the increase is to be equal on all sides of the pier). The exact position of the corner in this bisector may be found as follows:—Draw any rectangle as OMQR having an area of 8 ft. From the centre O draw radials cutting the sides MQ, QR (produced if necessary in pp; p_1p_1

p_2p_2, etc. From the points p and p draw lines parallel to OM, ON meeting in P; repeat this for the points p_1p_1 . . . and p_2p_2, obtaining P_2, P_3 . . . Draw a fair curve through the points P, P_2, P_3 . . . which crosses the bisector of the angle DAR in Z. Then Z is a corner of the required base. Complete by symmetry. (Observe that the area of any rectangle having adjacent sides in OM, ON and the corner opposite to O in the curve PP_2P_3 is equal to OM . MQ.

Exercise.

Draw a rectangle, sides 3 in. and $1\frac{3}{4}$ in. Suppose this rectangle to represent a garden. It is required to trace round the garden a path of *uniform width*, leaving in the centre a rectangular plot one-half the area of the garden. Draw this plot. (B. E.)

Hint. Draw through the centre of the given rectangle lines parallel to the sides. And by the method of problem 343 plot a rectangle in one quarter of the figure having an area equal to $\frac{3 \times 1\frac{3}{4}}{8}$. Complete by symmetry.

199. Lines tangent, or normal, to most of the mathematical curves met with in the constructive arts can be drawn from known properties of the curve. If the known data is insufficient to provide a direct solution of the problem, a *locus* may be used.

In modern geometry, a curve is considered as made up of an infinite number of points placed in order along the curve.

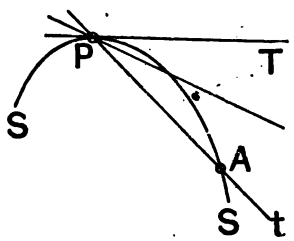


FIG. 214.

Any one of these elemental points is indefinitely near to the next consecutive point.

A *right* line drawn through two consecutive points in a curve is called a **tangent.**

An illustration may help to make this definition clear. Let a line P*t* (Fig. 214) intersect the curve SS in points A and P. Imagine the point A to move along the curve toward the fixed point P, and the line to pass always through P. The moving line changes position continually as the point A approaches the fixed point P, and ultimately, when A coincides with P, the line P*t* lies in the tangent PT.

Problem. *Through a fixed point* P *draw a tangent to a given curve mn.*

First Case. Fig. 215 (a). *The point* P *is on the curve.*

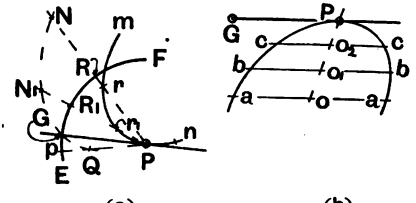
With centre P and any radius describe a circular arc EF. Draw PR intersecting the curve in *r* and the arc EF in R.

Make RN on PR produced = Pr. Through a second point r_1 on the curve draw Pr_1 meeting the arc EF in R_1. Make R_1N_1 = Pr_1. Draw lines from P through other points on the curve, one of which, n, is situated on the right of P. Set off along these lines, measured from the arc EF, lengths equal to the respective chords of the curve mn, and to the right or left of the arc EF according as the line intersects mn on the right or left of P. Thus in the Fig. pQ = Pn. Draw a fair line through the points N, N_1 . . . Q, which intersects the arc EF in G. The point G is in the tangent of P.

For, imagine NP to rotate about the fixed point P in the curve, and let PN, PN_1, Pp represent different positions of the moving line. Then the curve NN_1GQ is the locus of points which are in the moving line, and at a distance from the circle EF equal to that of P from the second and variable point r in which the line intersects the curve, and on the left or right of the arc, according as NP cuts the curve on the left or right of P. Consequently the point G where the locus NN_1GQ crosses EF corresponds to that position of the moving line where it does not cut the curve on either side of P. Hence, GP is tangent at P.

Second Case. Fig. 215 (b). *Let* G *be the given point not on the curve.*

Place a pencil point on G, slide a straight edge against the pencil, and rotate so as to bring the edge as near as



(a) (b)

FIG. 215.

the eye can judge in contact with the curve. Draw a line along the edge ; this is the required tangent. The point of contact P may be found thus : Draw parallels to the tangent meeting the curve in aa, bb, cc ; bisect aa in o, bb in o_1, cc in o_2. Draw a fair line through o, o_1, o_2, and produce it to

meet the curve in P, which is the point of contact of the tangent from G.

For, the curve oo_1o_2 is the locus of the mid-points of chords parallel to the tangent, and P marks the position where the gradually shortening chord is reduced to two consecutive points in the curve.

200. **Problem.** Fig. 216. *To draw a normal of a curve* SS *to pass through a fixed point.*

First Case. Let Q be the point, and *not on the curve.* With Q as centre describe circular arcs intersecting the curve in SS, VV, UU. Bisect these arcs (or their chords) in o, o_1, o_2. Draw a fair line through these points and produce to meet the curve SS in P, which is a point in the normal from Q; for, the circle centre Q and radius QP passes through P and not through any other point in the curve SS.

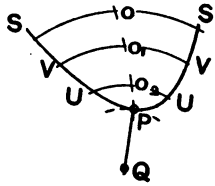


FIG. 216

Second Case. If it is required to draw a normal from a point *situate on the curve*, determine first the tangent of the point (Art. 199), and draw a perpendicular thereto.

201. **Involute** and **Evolute.** Fig. 217. If a tangent PE roll, without sliding, along the fixed curve OO_2, any fixed point P in the tangent has a *definite* locus P, P_1 .. . P_3, which is called an **involute** of the fixed curve. The fixed curve is called the **evolute** of P, P_1 . . P_3.

Since each point in the rolling tangent has a definite locus, there may be any number of involutes of a given curve.

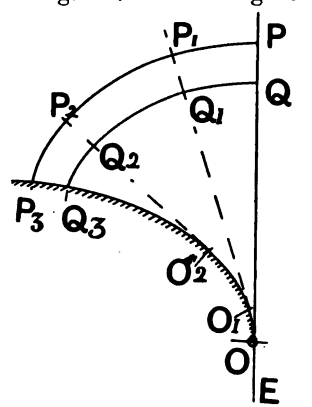


FIG. 217.

Problem. *Given a fixed curve* O, O_1 . . O_2 *and a fixed point* P, *draw an involute of the curve to pass through* P.

First Method. Draw PE

tangent to the curve and determine the point of contact O.* Draw OP on a piece of tracing-paper. Insert a pricker at O and rotate the tracing through a *small* angle, then prick through the new position of P. Transfer the pricker to the new point of contact O_1, and further rotate the tracing and prick through P. Continue in this way until the line on the tracing measures itself along the curve O . . O_2. A fair line drawn freehand (or aided by French curves) through the different positions occupied by P is the required involute.

Second Method. To draw the involute by continuous motion.

Determine O the point of contact of the tangent from P as before. Insert pins close together along the curve O . . O_2, or cut a templet to fit the curve. Fasten a flexible and inextensible thread to the drawing-board at O. Make this thread taut in the direction OP and attach a pencil to the thread at P. If this pencil be moved in an anti-clockwise direction, the thread, if kept taut, is wrapped against the curve O . . O_2, and the pencil traces the involute P, P_1 . . P_3.

202. If we consider the curves P . . P_3, Q . . Q_3, Fig. 217, as the paths of definite points P, Q in the tangent PE as it rolls along the fixed curve O . . O_2, we see that the direction of the curves at corresponding points—P_2, Q_2—must be the same as that of the generating points P, Q when they occupy those particular positions.

Now clearly at any particular instant the generating points P and Q can move in one direction only, viz., at right-angles to the direction of EP at the particular instant. Thus the curves PP_3, QQ_3 have the same direction and are consequently parallel. From this we conclude that *all involutes derived from a fixed evolute are* **parallel** *lines.*

Since the rolling tangent of the *evolute* is always normal to the *involute*, the evolute may be defined as the locus of the intersection of normals at consecutive points in a curve. As there can be but one such line for any particular curve, it follows that *any plane curve has a definite evolute, and one only.*

203. In Fig. 218, APQR is the involute of the triangle

* Employ the construction of Art. 199 if knowledge of the evolute provides no easier method.

ABC. This curve consists of three circular arcs: AP, centre B, radius BA ; PQ, centre C, radius CP = CB + BA, and QR, centre A, radius AQ = AC + CB + BA, or the perimeter of the triangle. The *involute* of *any* rectilinear figure is made up of tangential arcs, and conversely, every curve comprised of arcs of circles has a rectilinear evolute.

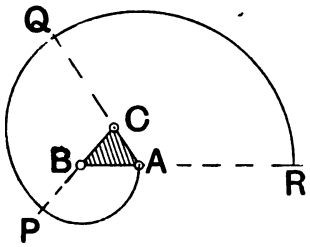


FIG. 218.

204. **Curvature.** The curvature of a line is its deviation from a straight line. The curvature of a circle is the same at every point, and in order that the curvatures of different circles may be compared, curvature is measured by the reciprocal of the radius of the circle, *i.e.*, by $\frac{1}{r}$ where r is the radius. It follows that, of two circles that which has the greater radius has the smaller curvature.

The curvature of curves other than circles is not constant, but varies continually. The amount of curvature at any point is equal to that of the circle which passes through the point and has the most intimate contact with the curve at that point. In Art. 413, we have seen that any plane curve and its tangent have two consecutive points in common. Consider now the curve APB, Fig. 315A, and its tangent PT. Clearly any circle having centre on the normal PO and passing through P is tangent to the curve AB, and has two consecutive points in common with the curve AB. An infinite number of such circles can exist; there must,

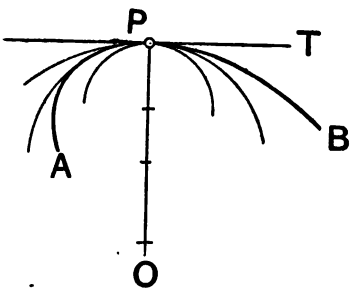


FIG. 219.

however, exist one circle which has closer agreement with the curve AB at P than any other circle can possibly have. This particular circle is considered as passing through the point P and two other points one on either side of, and infinitely close to, P. This circle is said to **osculate** the curve at P, and is called the **osculatory circle**,* and its curvature is equal to that of the curve at the point P. The centre and radius of the osculatory circle are respectively the **centre of curvature**, and **radius of curvature** of the curve at the point P.

Since the osculatory circle of a point P passes through three consecutive points in the curve, it follows that its centre must be at the meeting point of normals drawn from points consecutive at P ; but, the locus of all such points is the evolute of the curve, hence the centre of curvature of any point in a plane curve must lie on the evolute of the curve.

To find the locus of centres of curvature of any plane curve. In the case of a known mathematical curve, an algebraic expression for the radius of curvature at definite points can, in general, be found by analysis. These expressions may often be interpreted graphically. When a curve is given and nothing is known of its properties, its evolute may be found by drawing normals at different points, and then drawing the envelope of these normals. The evolute is the locus required.

To determine the centre of curvature at a given point on a curve. Draw from the point a tangent to the evolute and find its point of contact which is the required centre.

The problem may be solved without drawing the evolute, as shown in Fig. 220. Let APM be the given curve and P the point for which the centre of curvature is required. Draw PN the normal at P (Prob. 200), from P draw any

* In a curve of varying curvature, the curvature at a point on one side of a given point P is, *in general*, greater than at P, whilst at a point on the opposite side of P the curvature is less than at P. Hence the osculatory circle of a point P always crosses the curve at P unless the curvature is equal on both sides of the point, in which case there is a cusp in the evolute at the centre of curvature of the point. Examples of this type occur at the ends of the axis of a conic. See Figs. 247 (a) (b), p. 244.

chords PM, PD, PQ. Draw the perpendicular bisector of PM which intersects the normal in e; Draw eE perpendicular to PN and equal PM. Apply similar construction to all the chords obtaining points F . . . H. A fair line through these points will determine in its intersection with PN at O the centre of Curvature of the point P.

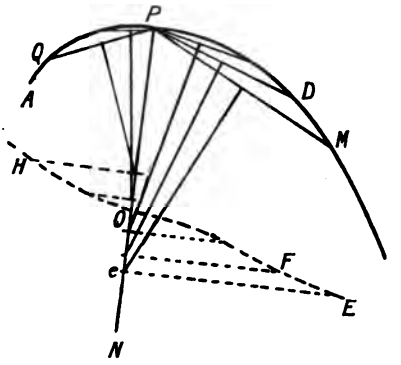


FIG. 220.

For the point E in the curve H . . . E corresponds to the chord PM and is at a distance from PN equal to the length of chord PM, similarly for any other chord, the corresponding point on H . . . E will be distant from PN equal to that chord and to the left or right according as the chord be taken on the left or right of the normal, and clearly the point O in which H . . . E crosses PN corresponds to that chord which has no length and is thus the point in which consecutive normals at P intersect, hence O is centre, and OP radius of curvature at P.

Exercise.

A circle of radius $1\frac{1}{2}$ in. rolls internally on the circumference of a fixed circle the radius of which is 3 in. By the method of Problem 196 verify :—

(a) Any point on the circumference of the rolling circle describes a diameter of the fixed circle.

(b) Any point in the plane of the rolling circle (except the centre and points on the circumference) describes an ellipse.

The methods illustrated in Art. 191 may all be deduced from this principle.

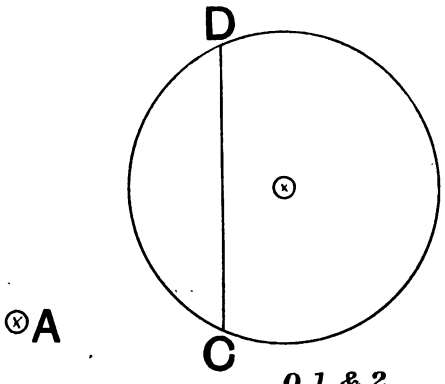
Exercises.

(1) Draw a line from A to cut the circle (centre O) such that the part of the line without the circle shall be equal to the part within. (A locus may be used.)

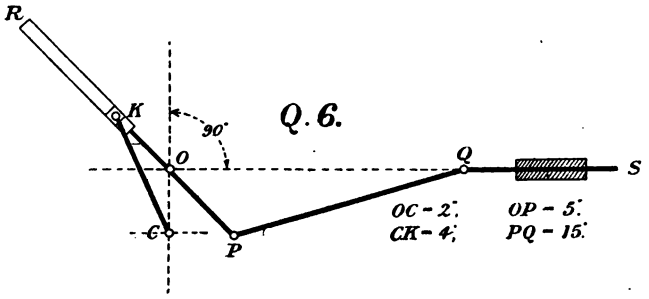
(2) CD is a chord of the circle, centre O, shown in the diagram. From a given point A draw a line cutting the circle, such that the portion of it lying within the circle shall be bisected by the chord CD. (B.E. 1900.)

(3) The centre of a circle of 1 in. radius is a point $1\frac{3}{4}$ in. from a line. Find the locus of the centres of all circles which will touch the circle and the line. (B.E. 1900.)

(4) Describe a circle of 2 in. radius. Draw a diameter on which mark a point A distant 1 in. from one end. Through A draw a chord BAC such that AB : AC = 3 : 2.



Q.1 & 2.



Q. 6.

Measure and state the lengths of AB and AC. Multiply these numbers together by arithmetic and give their product. What should the product be exactly? (B.E. 1901.)

(5) Two lines meet at an angle of 60°. Find the locus of points in the interior of the angle such that the sum of their distances from the two given lines is constant, and equal to $2\frac{3}{8}$ in. (B.E. 1897.)

(6) In the mechanism shown the slide S, which oscillates with a "quick return" motion, is actuated as follows :—

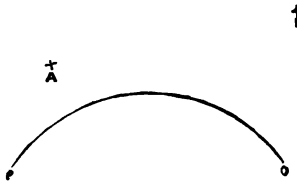
CK is a crank turning uniformly about C, its end K moving in the slot of the link ROP which rotates about O at a varying speed. The slide S is connected to the link by the rod PQ.

Set out a line diagram of the mechanism to the given figured dimensions, scale 1 cm. to 1 in., for the position in which the crank angle OCK = 32°. (B.E. 1910.)

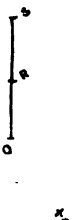
(7) PQ is a portion of the circumference of a circle of which O is the centre ; A and B are two given points. Find, *by a geometrical construction*, a point *c* on the circumference such that lines drawn from A and B to *c* shall make equal angles with the radius O*c*. (B.E. 1896.)

(8) A circle of $2\frac{1}{4}$ in. diameter rolls on a fixed circle of $1\frac{3}{4}$ in. diameter, the smaller circle being *inside* the larger one. Draw the curve traced by a point moving with the larger circle, and $1\frac{1}{8}$ in. from its centre, during one complete revolution. (B.E. 1894.)

(9) Obtain the form of cam plate to be keyed on to a shaft whose centre is O, and impart motion to a point Q which travels in the given



Q. 7.



Q. 9.

Q. 10.

line QRS. In one-third of a revolution of the cam shaft, Q is to be raised at uniform speed to R. It is then to remain at rest for one-sixth of a revolution, and after that to rise to S at a uniform rate in one-quarter of a revolution, and then fall back from S to Q in one-quarter of a revolution. (B.E. 1887.)

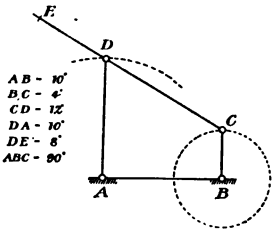
(10) Determine the locus of a point which moves so that the lines joining it to two given points A, B, make equal angles with the given straight line CD. What is this locus? (B.E. 1892.)

(11) Draw the locus of a point which moves in such a manner that its distance from a fixed line MX is always *equal* to its distance from a point S distant 1 in. from MX. Also the locus when the distance from MX is (1) $\frac{2}{3}$ and (2) $\frac{3}{2}$ times the distance from S.

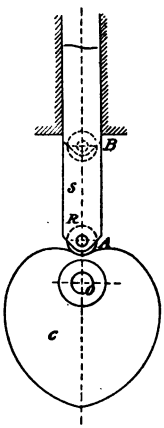
(12) In the given mechanism, the crank BC and lever AD are centred at A and B, as shown, and are connected at their ends by the rod CD, which is extended to E. Set out the figure to the given dimensions, scale $\frac{1}{4}$. Plot the curve traced by E during one complete rotation of BC. Show the angle through which AD oscillates. (B. E. 1912.)

(13) A slider S provided with a roller R has the latter resting on a cam C, which rotates uniformly about O. During successive half-turns of the cam, the slider rises and falls at uniform rates between the limits of A and B for the centre of the roller. The shape of the cam is shown, incorrectly, and you are required to set out the true shape.

Take OA = $1\frac{1}{4}$ in., AB = $1\frac{1}{2}$ in., diameter of roller = 1 in., and do not copy the diagram. (B. E. (2.) 1908.)



Q. 12.



Q. 13.

CHAPTER XVII.

CONIC SECTIONS.

PARABOLA, ELLIPSE, HYPERBOLA.

205. These important curves—called **conic sections** from the fact that every plane section of a cone is one or other of these forms—cannot be drawn with compasses, because no part of either curve conforms to any circle. Curves which look very like a *parabola,* or an *ellipse,* or a *hyperbola,* can be built up of circular arcs of different radii (*see* Art. 266) ; but, no matter how close the resemblance may be, the geometry of such curves differs radically from that of the conic sections.

There are in existence many mechanical devices for drawing conic sections by continuous motion, but, with few exceptions, such devices are costly and suitable only for large drawings. The student of geometry has to draw the curves freehand through a number of predetermined points.

206. **The Parabola** (Fig. 221). A parabola is a plane curve every point of which is equally distant from a fixed point and a fixed straight line.

The fixed point S is called the **focus** of the parabola; the fixed straight line MM′ is called the **directrix.** The straight line AX passing through the focus and perpendicular to the directrix MM′ is called the **axis** of the parabola. The point A where the axis intersects the curve is called the **vertex.** A parallel to the axis from any point Q in the curve is called a **diameter.** The point Q in which the diameter meets the curve is called the **vertex of that diameter.** Any straight line Qq terminating at each end on the curve is called a **chord.** The chord LL′ perpendicular to the axis and passing through the focus is called the **latus rectum.**

207. **Equation of the Parabola.** If the vertex A of the parabola be taken as origin, and AX be taken as the axis of abscissæ, the equation* of the parabola is :

$$y^2 = 4ax \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

or

$$x = \frac{y^2}{4a} \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

where the constant a = AS, the distance of the focus from the vertex, and x, y are co-ordinates of the point which traces the parabola.

208. If we write the equation of the parabola in the form

$$y = \pm 2 (ax)^{\frac{1}{2}},$$

then for every positive value of x we obtain two values for y equal in magnitude but opposite in sign ; and the greater

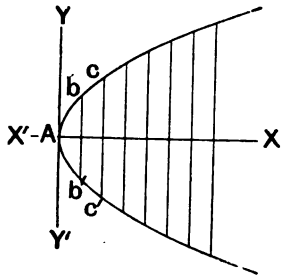


FIG. 221.

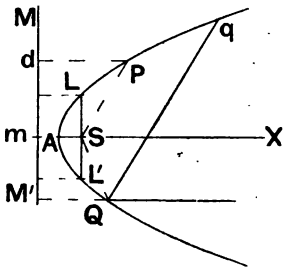


FIG. 222.

the value we give to x the greater the value we get for y. Hence, *the parabola is symmetrical about its axis, and extends to an infinite distance on each side of the axis.*

209. Again, for any value of y positive or negative the corresponding value of x will always be positive, since the negative value of y when squared will be positive ; therefore *the parabola extends only to the right of the vertex* A.

210. **Problem.** Fig. 221. To trace the curve whose equation is $y^2 = 4ax$, having given $a = 1$. If we write the equation in the form $y = \pm 2 (ax)^{\frac{1}{2}}$ and assume for x

* See p. 264.

different numerical values from 0 to 7, we obtain corresponding values of y as given below, when

$x = 0, y = 0$	$x = 4, y = \pm 4$
$x = 1, y = \pm 2$	$x = 5, y = \pm 4{\cdot}47$
$x = 2, y = \pm 2{\cdot}83$	$x = 6, y = \pm 4{\cdot}90$
$x = 3, y = \pm 3{\cdot}46$	$x = 7, y = \pm 5{\cdot}30$

In the diagram, the point (o, o) is represented by A, the origin: the point $(1, +2)$ is represented by b, the point $(1, -2)$ by b', and so on, etc.

211. *The squares of ordinates are to each other as their corresponding abscissæ.* For $y^2 = 4ax$, and if we designate any two ordinates by y', y'' and the corresponding abscissæ by x', x'', we have $\frac{y'}{y''^2} = \frac{4ax'}{4ax''} = \frac{x'}{x''}$.

212. Fig. 222. *To find the distance of any point on the curve from the focus.*

$$SP = Pd = x + Am,$$

but, $Am = AS$ (definition of parabola),

$$\therefore SP = x + a.$$

Cor. 1. *The vertex* A *bisects* mS, *and* $Am = a$.

Cor. 2. $SL = Sm = 2a$. Hence LL_1, *the latus rectum* $= 4a$.

213. **Problem.** Fig. 223. *Construct a parabola given the axis* aX, *vertex* a, *and focus* S.

A construction is derived from the definition of the curve. Thus make am on the axis $= a$S; and through m draw mM perpendicular to the axis; then mM is the directrix. Since all points on the curve are equidistant from the focus and directrix, take any points 1, 2, 3 on the axis, and with the focus S as centre and $m1$, $m2$, $m3$ as radii draw arcs meeting the ordinates through 1, 2, 3 in the points n, p, q. These are points on the curve.

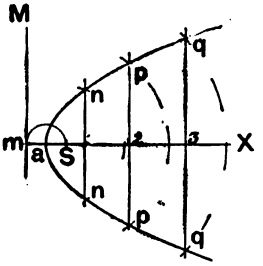


FIG. 223.

214. **Tangent and Normal.** Fig. 224. Let x', y' be co-ordinates of a point P on the parabola, the vertex being the origin. The line PT drawn to touch the curve at P is the **tangent**

at P. A perpendicular from P on to the axis meets the axis in N; NT is called the **sub-tangent** of P. A line PG perpendicular to PT is called the **normal** of P, and NG is called the **sub-normal.**

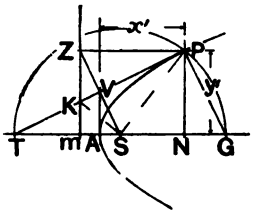


FIG. 224.

215. *The equation to the tangent at any point x', y' of a parabola is* $yy' = 2a(x + x')$.

216. *To determine the point in which the tangent intersects the axis of x*: Make $y = 0$, the equation then reduces to $x = -x'$, that is $AT = -AN = NA$,

hence, *the sub-tangent is bisected at the vertex* A.

217. *To determine the point in which the tangent intersects the axis of y.* Make $x = o$, and we have

$$y = \frac{2ax'}{y'} = \frac{y'^2}{2y'} = \frac{y'}{2},$$

that is $AV = \frac{1}{2}PN$.

Cor. 1. $TV = \frac{1}{2}TP$. Since AV $\|$ NP.

218. $ST = SP$.

For let SV produced meet the directrix in Z.

then $\because AS = \frac{1}{2}mS \therefore VZ = VS$.

and the Δs TVS, PVZ are congruent.

$\therefore ST = PZ = SP$.

Cor. 1. *The tangent of a point* P *bisects the focal angle,* SPZ.*

For Δs ZPV, SVP are congruent.

Cor. 2. *The angle* SVP *is a right-angle.*

For the four angles at V between TP, SZ are equal to one another.

Cor. 3. The angle KSP *is a right-angle.* For KSP = KZP and KZP is a right-angle. Hence, *the intercept* PK *of a tangent subtends a right-angle at the focus.*

219. **Problem.** Fig. 224. *Given the axis of a parabola and a tangent with its point of contact* P, *determine the focus, vertex, and directrix of the parabola.*

Produce the tangent (if necessary) to intersect the axial

* This property is also true for the ellipse and the hyperbola.

line in T. Draw VS the perpendicular bisector of TP, meeting the axis in S, which is the focus. From V drop a perpendicular on to the axis at A, which point is the vertex. Make $Am = AS$. Then the directrix passes through m.

220. *The equation to the normal at any given point* P (x', y') *of a parabola is*

$$y - y' = -\frac{y'}{2a}(x - x').$$

221. *To find the point in which the normal intersects the axis of* x.

Make $y = o$, and the equation reduces to $x - x' = 2a$, that is, $AG - AN = 2a$; hence

NG, *the sub-normal, is constant and equal to the semi-latus rectum.*

Cor. 1. $SG = SP$.

For $SG = SN + NG = x' - a + 2a = x' + a$.

$\therefore SG = SP$.

222. **Problem.** Fig. 224. *To draw a tangent and a normal to a parabola at a given point* P *on the curve.*

The preceding properties supply several solutions.

The following is a good one. With the focus S as centre, and radius SP describe a semicircle intersecting the axis in T and G. PT is tangent and PG is normal at P.

223. **Problem.** Fig. 225. *To draw tangents to a parabola through an external point* O.

With centre O and radius OS, describe a circle intersecting the directrix in K, W. Draw KP, WQ $\parallel$ the axis meeting the curve in P and Q. Then OP, OQ are the required tangents, for OP bisects angle KPS, and OQ bisects angle WQS.

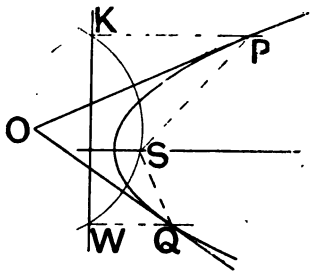


FIG. 225.

224. **Problem.** Fig. 226. *To draw a normal to a parabola through a point* Z.

Determine Q the foot of the normal by Problem 200.

225. **Problem.** Fig. 226. *To draw a curve parallel to a given parabola at a distance x from it.*

Obtain a number of normals to the parabola (Problem 222). Mark off from A along the axis outwards the distance Aa = x, and from the points 1, 2, 3, etc., the same distance along the respective normals, obtaining points b, c, d, e, f. A fair line drawn through these points will be the required curve.

226. **Some useful Properties.** Figs. 227 and 228.

(1) *If pq is the chord joining points of contact of two*

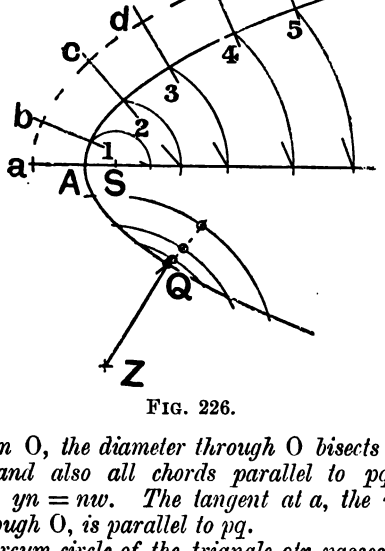


FIG. 226.

tangents from O, *the diameter through* O *bisects the chord of contact pq and also all chords parallel to pq. That is, pe* = *eq and yn* = *nv. The tangent at a, the vertex of the diameter through* O, *is parallel to pq.*

(2) *The circum-circle of the triangle otr passes through the focus* S.

(3) *The tangent at a bisects the tangents op, oq, and is bisected in a by the diameter* On. That is, *ot* = *tp* ; *or* = *rq* ; *ra* = *at.*

(4) (3) is a particular case of the general theorem that *two fixed tangents are cut proportionally by the other tangents.*

Thus, in Fig. 228, where fixed tangents OY, OZ are cut by tangents ss', tt', uu', $\frac{st}{s't'} = \frac{su}{s'u'} = \frac{tu}{t'u'} = \frac{tY}{t'O} = \frac{OY}{ZO}$.

Let the diameter bisecting YZ in i meet the curve in a.

Let p, q be points in the curve, and ap, aq be drawn to

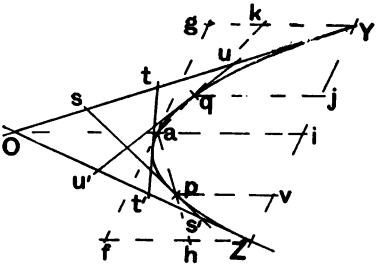


FIG. 228.

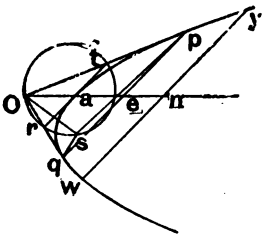


FIG. 227.

intersect the sides of the parallelogram, YZ fg in h and k. Let pv, qj, parallel to ai, meet YZ in v and j.

(5) Then, $\frac{Yk}{Yg} = \frac{Yj}{Yi}$, and $\frac{Zh}{Zf} = \frac{Zv}{Zi}$.

(6) *The two tangents from a point* O *subtend equal angles at the focus.* That is angle osp = angle osq. Fig. 227.

227. **Problem.** Fig. 229. *Given the curve of a parabola; to determine its axis, focus, latus rectum, and equation.*

Draw any two parallel chords PP', QQ', and the bisector er of these chords, Draw $Pp \perp er$, meeting the curve again in p, and bisect Pp in N. Draw NM $\parallel$ er meeting the curve in A. Draw AV $\perp$ AN and make AV equal

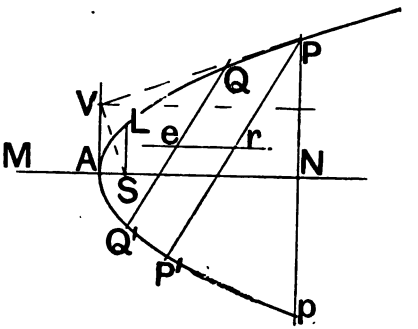


FIG. 229.

$\frac{1}{2}$ PN. Join VP and draw the ⊥r VS, meeting AN in S. Then AN is the axis, A is the vertex, and S is the focus of the parabola.

The latus rectum is 2 SL = 4 AS. The equation of the curve is

$$PN^2 = 4 \cdot AS \cdot AN.$$

228. **Problem.** *Given two tangents* OP, OQ *to a parabola at* P *and* Q. *Construct the curve.*

Fig. 230 (a). A solution based on Art. 226 (4). Divide OP, OQ into a number of equal parts at 1, 2, 3, I, II, III.

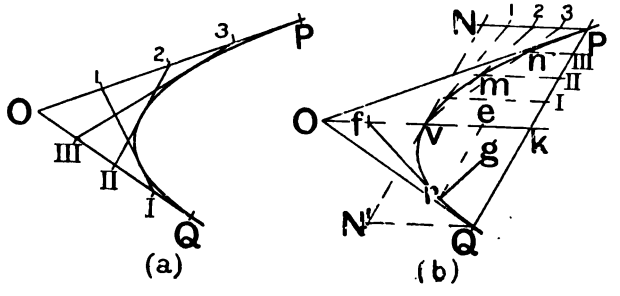


FIG. 230.

Join 1I, 2II, 3III. These lines are tangents to the parabola, which may be drawn by freehand.

Fig. 230 (b). A solution based on Art. 226 (5). Join PQ, and draw the diameter *vk* through O. Draw NN′ ∥ PQ. Divide *k*P, NP each into a number of equal parts at 1, 2, 3, I, II, III. Radials *v*1, *v*2, *v*3 meet parallels to *vk* from corresponding points I, II, III in points as *m*, *n*, which are on the curve.

229. **Problem.** Fig. 230 (b). *To draw a tangent and normal at a point r, when the axis and focus are not available.*

Draw through *r* a line *re* ∥ PQ meeting the diameter *vk* in *e*. Make *vf* = *ve*, then *fr* is tangent at *r*, and *rg* ⊥r to *rf* is normal at *r*.

230. **Problem.** Fig. 231. *Two lines* OP, OQ *from a*

point O *are tangents to a parabola at* P *and* Q. *Find the focus.*

Draw OC ⊥r OQ to meet the ⊥r bisector of OP in C. Draw OC′ ⊥r OP to meet the ⊥r bisector of OQ in C′. With centre C, and radius CO, describe a circle. With centre C′, and radius C′O, describe a circle to meet the circle previously drawn in S.

Then S is the focus.

For, angle QOp = angle in the segment OSQ.

Also, angle POq = angle in the segment OSP.

But, angle QOp = angle POq ∴ angle OSQ = angle OSP. Hence S is the focus. Art. 226 (6).

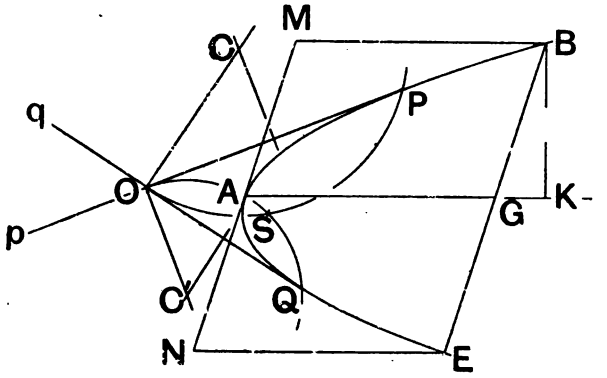


FIG. 231.

231. The **area of a segment of a parabola** cut off by any straight line BE is ⅔ of the circumscribing parallelogram BENM.

i.e., Area BAE = ⅔ AG 2 BK

if BK be perpendicular to AG.

232. Curves resembling the parabola may be built up of circular arcs.

Since the parabola is symmetrical about its axis, the circle drawn through the vertex A must have its centre on the axis. Arcs for segments above the axis are struck from centres below the axis, and *vice versâ*.

Ex. *Given the ordinate* QN, *and abscissa* aN *of a point* Q, *construct an approximate parabola with circular arcs.*

First Method. Fig. 232. Bisect QN in e, and draw $ev \parallel a$N to meet the perpendicular from a in v. Join Qv and draw QC′ perpendicular to Qv. Make aC on the axis = av. Also make Qf on QC′ = av. Draw the perpendicular bisector of Cf which intersects QC′ in C′. From C′ draw a perpendicular to the axis and make OC″ = OC′. Then

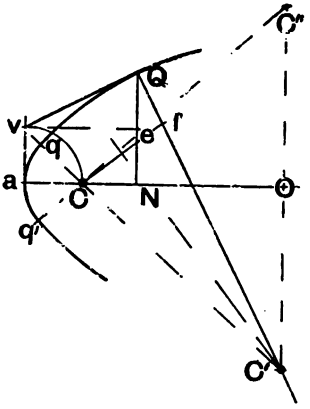


FIG. 232.

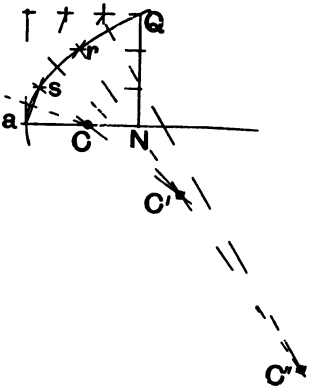


FIG. 233.

C, C′, C″ are centres of circles. The points of contact q, q' lie on C′C, C″C.

Second Method. Fig. 232. If a curve is required to lie close to the parabola, a large number of circles must be used; the closer the agreement required, the more numerous the arcs must be. In Fig. 233, three centres are used to determine the arc a, s, r, Q. The construction is as follows:—Determine points in the parabola, as r, s. Draw the ⊥r bisector of as which crosses aN in C. Draw the ⊥r bisector of sr which intersects sC in C′. Draw the ⊥r bisector of rQ which intersects rC′ in C″, etc. The points C, C′, C″ are the required centres.

The Ellipse.

233. **Experiment.** Fig. 234. Insert pins in any two points S, S_1, and place a looped thread over the pins. Place a pencil within the loop and make taut the thread. The pencil will take up some position as shown in the diagram and the thread will be brought against the pins. If the pencil be moved subject to the limitation of the looped thread, it will trace a curve called an *ellipse*.

The two fixed points S, S′ are called *foci*.

The points A, A′ in which the ellipse crosses the straight line through S, S′ are called **vertices**. The line AA′ is called

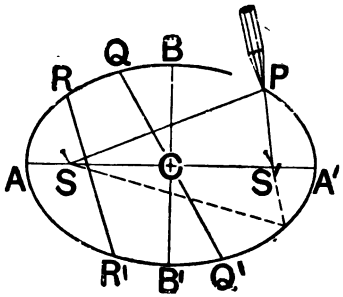


FIG. 234.

the **major** or **focal axis**. The mid-point C of the major axis is called the **centre**. A line BB′ through C at right-angles to AA′ is called the **minor** axis. Any straight line QQ′ passing through the centre and terminating on the ellipse is called **a diameter**.

A straight line RR′ terminating on the ellipse but *not* passing through the centre is called a **chord**.

234. Evidently, for any point P on the ellipse, SP + S′P = AA′. The foci are equidistant from the centre, and SB = S′B = AC.

235. *The equation of the ellipse.* If the axes of the ellipse

be taken as axes of reference with C as origin, the equation of the ellipse* is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

$$\text{or } y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad . \quad . \quad . \quad \text{(ii.)}$$

Where a = semi-major axis, b = semi-minor axis, and x, y are co-ordinates of the point P which traces the ellipse.

236. *To determine the points in which the curve intersects the axes*, take

1st. $y = 0$ and we obtain

$$x = \pm a, = \text{CA or CA'}.$$

2nd. $x = 0$ then

$$y = \pm b, = \text{CB or CB'}.$$

237. To *determine intermediate points*, reduce the equation to the form

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2},$$

from which the value of y corresponding to any assumed value of x may be computed.

238. **Ex.** Fig. 235. *Trace the ellipse whose equation is*

$$\frac{x^2}{81} + \frac{y^2}{36} = 1.$$

Solving the equation for y, we have

$$y = \pm \frac{6}{9}\sqrt{81 - x^2}.$$

Assuming for x numerical values from 0 to 9, we obtain corresponding values for y as given below:—

When $x = 0, y = \pm 6.$	When $x = 5, y = \pm 4{\cdot}99.$
$x = 1, y = \pm 5{\cdot}96.$	$x = 6, y = \pm 4{\cdot}47.$
$x = 2, y = \pm 5{\cdot}85.$	$x = 7, y = \pm 3{\cdot}77.$
$x = 3, y = \pm 5{\cdot}66.$	$x = 8, y = \pm 2{\cdot}75.$
$x = 4, y = \pm 5{\cdot}38.$	$x = 9, y = \pm 0.$

When $x = 0, y$ equals $\pm$ 6, which gives two points, b and b', one above and one below the axis of x. When $x = 1$, $y = \pm 5{\cdot}96$, which gives the points e, e', and so on until $x = 9$, when $y = \pm 0$. If x be taken greater than 9, the value of y will be the square root of a negative quantity; hence the

* See p. 265.

curve does not extend beyond $x = 9$. If the values of x be taken *negative*, we obtain points on the left-hand side of the minor axis ; and the curve does not extend beyond $x = -9$. *The ellipse is therefore a closed curve symmetrical about both axes, and thus the two axes divide the figure into four equal parts.*

239. When $b = a$ the equation of the ellipse reduces to

$$x^2 + y^2 = a^2,$$

which is the equation of a circle (Art. 357) ; hence *the circle may be regarded as an ellipse having equal axes.*

240. Fig. 236. Circles described on the major and minor

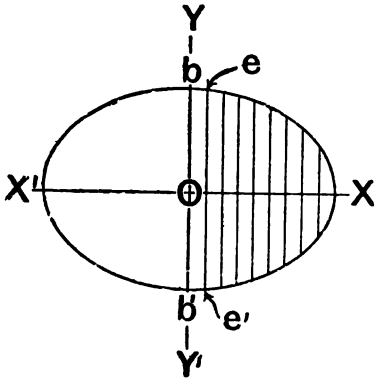


FIG. 235.

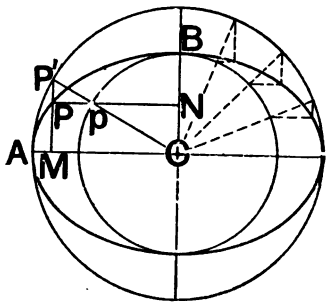


FIG. 236.

axes respectively of the ellipse are called the **major** and **minor auxiliary circles.**

Let PN ⊥r to BC meet the *minor* circle in p, and let PM ⊥r AC meet the *major circle* in P'. The points P, p and P' are called *corresponding points* on the ellipse and its minor and major circles.

The sides of Δ P'Pp lie in the corresponding sides of the similar Δ P'MC ; hence P', p, C are co-linear and

$$\frac{P'M}{PM} = \frac{CP'}{Cp} = \frac{a}{b}$$

also

$$\frac{PN}{pN} = \frac{CP'}{Cp} = \frac{a}{b}$$

241. **Problem.** *To draw an ellipse when the axes* AA', BB' are given.

First Method. Fig. 236. *A construction based on the preceding property.*

Describe circles on AA', BB' as diameters. Draw CP' meeting the larger circle in P' and the smaller circle in p. Draw P'P ∥ BC and pP ∥ CA meeting in P. Then P is a point on the ellipse. Other points are found in a similar manner. A fair line through all the points is the required ellipse.

Second Method. An expeditious method of finding points on the ellipse, or of drawing the curve mechanically, is based

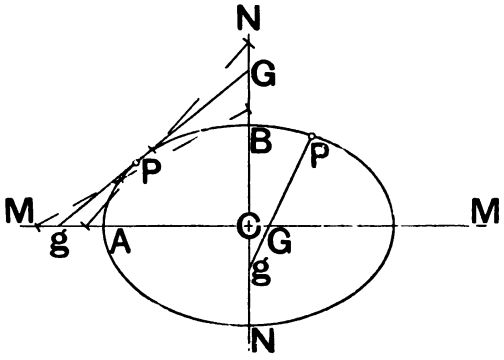


FIG. 237.

on the principle illustrated in Art. 191, which may be expressed as follows :—

Fig. 237. *If two fixed points* Gg *in a straight line slide along two fixed straight lines, any point* P *in the moving line describes an ellipse.*

If the directing lines MM, NN are at right angles to each other, the axes of the ellipse lie along the lines as in the Fig. We can show—when the generating point P lies between the points Gg—

$$Gg = AC + BC.$$

When P lies on gG prolonged

$$Gg = AC - BC.$$

Exercise. Take major and minor axes AA′, BB′ 10 *cm. and* 6 *cm. respectively.* Draw a straight line on tracing-paper, and set off along this line GP = AC, Pg = BC, and determine points in the ellipse by pricking through different positions occupied by P when G and g are on the minor and major axes. Also, repeat the operation when G and g are both on the same side of P.

242. **Problem.** Fig. 238. *Given one axis of an ellipse in magnitude and position and a point* P *on the curve, determine the other axis.*

Fig. 237 suggests a solution—

1st. Let the major axis AA′ be given. Draw BB′ the ⊥r bisector of AA′. With centre P and radius AC, describe

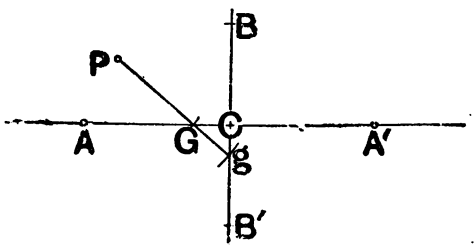


FIG. 238.

arc meeting CB in g. Join Pg intersecting AA′ in G. PG = *semi-minor axis.*

2nd. Let the minor axis BB′ be given. Draw the ⊥r bisector of BB′. With centre P and radius BC describe an arc meeting AC in G. Draw PG and prolong to intersect BC in g. Pg = *semi-major axis.*

243. **Area of Ellipse.** We know from Art. 434 that the projections of parallel straight lines are parallel, and that the ratio of the lengths of the projections is equal to the ratio of the lengths of the lines. Hence (Art. 240) *an ellipse may be regarded as the projection of its major circle, or, as the projection of its minor circle.*

Now, the area of any projection of a plane figure is equal to the area of the figure multiplied by the cosine of

the angle between the plane of the figure and the plane of projection. Hence, if we denote the angle between the plane of the major circle and the plane of the ellipse by α we have,

$$\text{Area of ellipse} = \pi a^2, \cos. \alpha.$$
$$= \pi a^2, \frac{b}{a}.$$
$$= \pi ab.$$

244. *The* **circumference** *of the ellipse* $= 2\pi \sqrt{ab}$ *approx.*

245. Fig. 239. *If a square* DEFG *with its inscribed circle be projected on to any plane neither parallel nor perpendicular to its own plane the circle will project into an ellipse while the projection of the square is a parallelogram circumscribing the ellipse.*

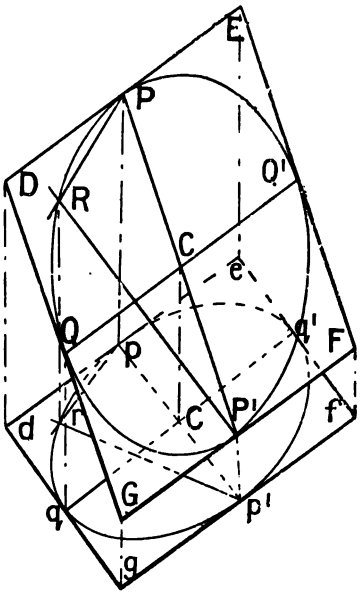


FIG. 239.

If PP', QQ' be diameters of the circle which join points of contact of the circle and square, and PR, P'R, chords through any point R in the circle, the projections of P, P', Q, Q' will be diameters of the ellipse and will join corresponding points of contact of the ellipse and its circumscribing parallelogram ; whilst the projections of PR, P'R will be corresponding chords of the ellipse.

246. Let HACB (Fig. 240) be a square having sides equal to the radius of a circle centre C. Let BC and HB be divided into equal parts at 1, 2, 3. Lines radiating from A and D through corresponding points 1, 1,

2, 2, etc., meet in points at n, r, s which lie on the circle whose diameter is AD.

For, the right-angled Δs DCf, DAv, AHy are similar. And in the Δ DnA, $\angle$ DAn = 90° − $\angle$ HAy = 90° − $\angle$ nDA.

$\therefore$ $\angle$ DAn + nDA = 90°, and DnA = 90°. Hence n is a point on the semicircle of diameter AD.

247. **Problem.** Fig. 241. *To inscribe an ellipse in a parallelogram to touch the sides of the parallelogram at the mid-points.*

Draw RR, NN the bisectors of parallel sides meeting in c. Divide cN, KN each into the same number of equal parts

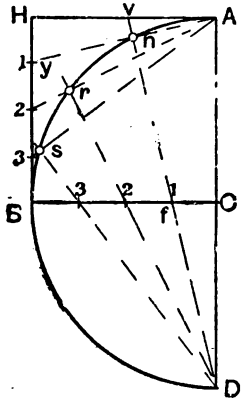


FIG. 240.

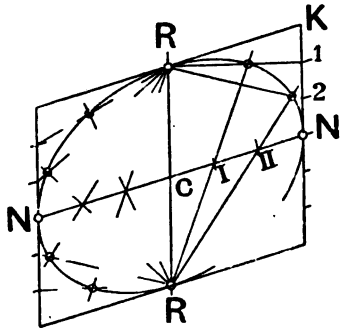


FIG. 241.

(3 in the Fig.) and number as shown. Lines drawn from R, R through corresponding points I 1, II 2, intersect in points of the curve. Treat all the quadrants in a similar manner. The construction will be evident if the ellipse and parallelogram be regarded as the projection of a circle inscribed in a square.

248. Suppose one or more chords of the circle (Fig. 239) parallel to one diameter PP′, or QQ′, these chords will be bisected by the other diameter, and the projections of the chords will be corresponding chords of the ellipse parallel to one diameter and bisected by the other. Also, either

diameter in the ellipse is parallel to the tangents at the extremities of the other.

Two diameters of an ellipse are called **conjugate diameters** when one is parallel to tangents drawn from the extremities of the other.

249. **Problem.** No Fig. *Given two conjugate diameters of an ellipse to draw the curve.*

At the extremities of each diameter draw lines parallel to the other diameter, thus forming a parallelogram which circumscribes the ellipse. The problem is now reduced to that immediately preceding.

250. **Important property of conjugate diameters.** *The area of the parallelogram formed by the tangents at the*

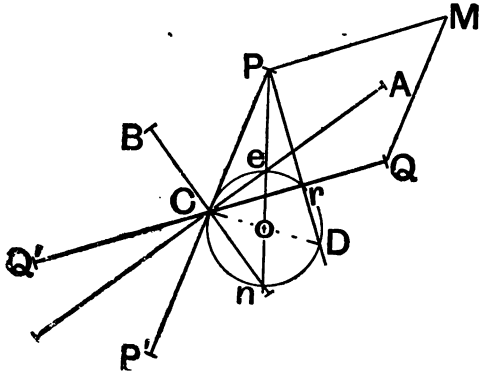


FIG. 242.

extremities of any two conjugate diameters of an ellipse is constant.

251. **Problem.** Fig. 242. *Given two conjugate diameters* PCP', QCQ' *of an ellipse, to find the major and minor axes.*

From P draw the ⊥r PD on Q'Q and make PD = CQ ; join CD ; and on CD as diameter describe a circle centre O ; join PO, cutting the circle in points *e*, *n*. Draw C*e* and produce to A, making CA = P*n*. Draw *n*C and produce to

B, making CB = Pe. Then CA, CB are the semi-axes of the ellipse.*

Note.—It is a determinate problem to construct an ellipse, which shall have any two mutually bisected lines for conjugate diameters; but if one diameter is fixed in an ellipse, the conjugate diameter cannot be chosen arbitrarily.

252. **Problem.** No Fig. *Given an ellipse and a diameter* PP′ *to determine the diameter conjugate to* PP′.

Draw any chord of the ellipse parallel to PP′; bisect PP′ and this parallel chord.

A line through the mid-points is the diameter required.

253. Fig. 243 (a). Any chord through a focus of the ellipse is called a **focal chord.** The focal chord ⊥ to the major axis is called the **latus rectum.**

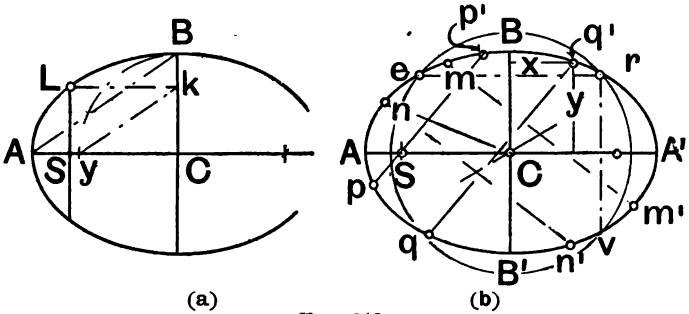


FIG. 243.

Important property of focal chords.

Any focal chord of an ellipse is a third proportional to the major axis and the diameter parallel to that chord.† Hence, the semi-latus rectum SL, the semi-minor axis BC, and the semi-major axis AC, are continual proportionals.

$$\frac{SL}{BC} = \frac{BC}{AC}. \quad \therefore SL = \frac{BC^2}{AC} = \frac{b^2}{a}.$$

254. **Problem.** Fig. 243 (b). *Given the curve of an ellipse to determine its centre, axes, foci, latus rectum, and its equation.*

Draw any two parallel chords pp' qq' and the bisector

* CA . CB = nP . eP = DP . rP = CQ . rP = parallelogram CPMQ.

† This is a property of the hyperbola also.

of these chords. Similarly draw the bisector of another two parallel chords mm', nn'. The two bisecting lines meet in C, the centre of the ellipse. With centre C describe any circle intersecting the ellipse in points e, r, v, q. Through C draw $AA' \parallel er$, and $BB' \parallel rv$. The longest of these lines AA' is the major axis, and the shortest BB' is the minor axis.

Make BS = CA. The point S is a focus. To find the

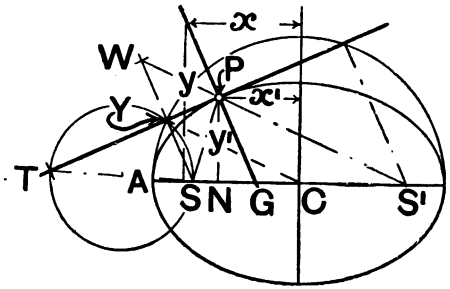


FIG. 244.

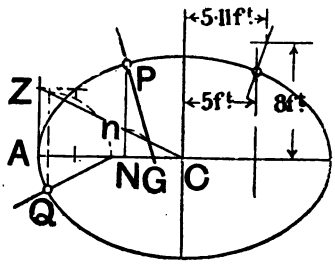


FIG. 245.

latus rectum. See Fig. (a). Draw a ⊥r SL through S, then SL is the semi-latus rectum.

To find the length of SL *independently of the curve:* Make Cy = CB and draw $yk \parallel$ AB. Then Ck = SL.

The equation of the ellipse is $\frac{x^2}{CA^2} + \frac{y^2}{CB^2} = 1$.

255. **Normal and Tangent.** Fig. 244. *The equation to the normal at any point of an ellipse is*

$$y - y' = \frac{a^2y'}{b^2x'}(x - x') \quad . \quad . \quad . \quad \text{(i.)}$$

or

$$x - x' = (y - y')\frac{b^2x'}{a^2y'} \quad . \quad . \quad . \quad \text{(ii.)}$$

where x and y are the general co-ordinates of the normal line, and x', y' the co-ordinates of the point on the ellipse.

If we take a point at an extremity of the major axis, y' vanishes, and Equation i. gives $y = 0$.

Hence the normals at the extremities of the major axis lie along that axis.

If we take a point at an extremity of the minor axis, x' vanishes and Equation ii. gives $x = 0$.

Hence the normals at the extremities of the minor axis lie along that axis.

The normal at any point of an ellipse, except at the ends of the axes, does not pass through the centre of the figure.

256. *To find the point* G, *in which the normal of a point* P *cuts the major axis*, make $y = 0$ in Equation ii., and we have, after reduction,

$$x = \frac{a^2 - b^2}{a^2} x'.$$

Now $x = CG$, and $x' = CN$.

If we subtract the value of x determined as above, from x' we have

$$NG = x' - \frac{a^2 - b^2}{a^2} x' = \frac{b^2 x'}{a^2},$$

which forms the proportion.

$$NG : NC : : b^2 : a^2.*$$

257. **Problem.** Fig. 245. *To draw the normal and the tangent at a point* P *on an ellipse.*

First Method. A solution based on the preceding property.

Draw AZ ⊥r AC and make AZ = semi-latus rectum (Art. 254). Join ZC and draw PN ⊥r AC, meeting ZC in n. Make NG = Nn, and join PG, which is normal at P.†

The normal at any other point, as Q, is found in a similar manner.

Alternative solution. The normal at any point P. may be found by bisecting the angle SPS′ (Fig. 244). The tangent bisects the supplementary angle. Refer to Art. 218.

258. Fig. 244. *The feet of perpendiculars from the foci on to a tangent are on the major circle.*‡ Join PS, PS′ and

* This property is true for the hyperbola also.

† When the point P is at an extremity (A) of the major axis, we have CN = CA

$$\therefore \text{Sub-normal} = \frac{CB^2}{CA} = \text{semi-latus rectum.}$$

From similar Δ's, CNn, CAZ ; $CN/Nn = CA/AZ$

and $\because NG = Nn$

$$\therefore CN/NG = CA/AZ = CA^2/CB^2.$$

‡ This property is true for the hyperbola.

draw SW $\perp^r$ to the tangent intersecting the tangent in Y and S'P produced in W.

$$\therefore \text{SP} = \text{PW and SY} = \text{YW}.$$

In the triangles YPS, YPW, YP is common, angles PYS, PYW are equal, and angles YPS, YPW are equal.

In the triangles S'WS, CYS, the angle at S is common.

$$\text{SS}' = 2\,\text{SC}\,;\ \text{SW} = 2\,\text{SY},$$

$$\therefore \text{S}'\text{W} = 2\,\text{CY}$$

$$\text{and CY} = \tfrac{1}{2}\,\text{S}'\text{W} = \tfrac{1}{2}\,(\text{S}'\text{P} + \text{SP}) = \text{CA}.$$

259. **Problem.** Fig. 244. *To draw a tangent to an ellipse through a given external point* T.

First Method. The tangent may be drawn by adjusting a straight edge to touch the curve and pass through the given point T.

To find the point of contact.

From S draw a $\perp^r$ to the tangent and make YW = YS. Join WS', which crosses the tangent at P, the point of contact.

Second Method. The following construction determines the tangent independently of the curve of the ellipse.

Join T to a focus S. On TS as diameter describe a circle intersecting the major circle in Y. Then TY is the required tangent. The point of contact is determined as before.

Two tangents to an ellipse can always be drawn from an external point.

260. In certain cases—for example, in the setting out of large elliptical bridges—we may wish to determine a point in the normal of a given point at a definite height above the major axis.

Ex. Fig. 245. *The semi-axes a and b of an ellipse are* 20 *ft. and* 7 *ft.; determine the point in which the normal of a point whose abscissa is* 5 *ft. intersects a line parallel to the major axis and* 8 *ft. above same.*

Here we have $a = 20$, $b = 7$, $x' = 5$.

The ordinate y' corresponding to x' may be measured from a drawing, or computed by substituting the given values in Equation, Art. 237.

$$y' = \pm \frac{b}{a}\sqrt{a^2 - x'^2}.$$

Then solving for y' we obtain $y' = 6{\cdot}78$. Knowing the position of the point on the curve at which the normal is drawn, we use equation ii., Art. 255.

$$x - x' = (y - y')\frac{b^2 x'}{a^2 y'}.$$

Substituting known values, and solving for x, we have $x = 5{\cdot}11$ ft. Hence, the required point of intersection is 5·11 ft. to the right of the minor axis.

261. **Problem.** No Fig. To draw a normal to an ellipse from a given point not on the curve.

This may be done as in Problem 416.

262. *Curves parallel to an ellipse.* **No two ellipses are parallel.**

Problem. Fig. 246. *To draw a curve parallel to a given ellipse at a distance x therefrom.*

By the first method of Art. 257 draw normals at numerous points on the ellipse, and set off along these normals the given length x. Through the points thus found draw a fair line which will be parallel to the ellipse.

In the diagram two lines are shown, one inside and one outside the ellipse. Both lines satisfy the given conditions, and both are parallel, although the inner line does not much resemble the other two.

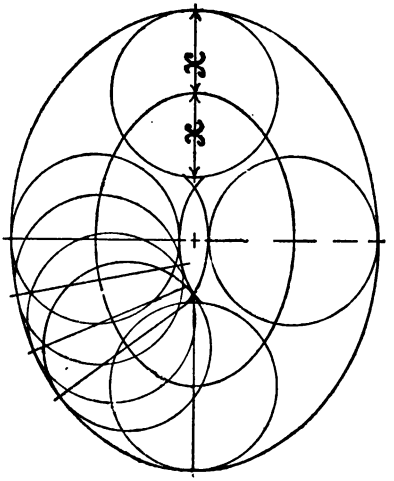


FIG. 246.

263. **Radius of Curvature of a Conic Section.** The radius of curvature of a conic section is equal to *the cube of the normal* PG *divided by the square of half the latus rectum* SL. Thus, in Fig. 247 (a), if O′ is the centre

of the circle which osculates the conic at P, and G the point in which the normal of P intersects the focal axis, then $O'P = \frac{PG^3}{SL^2}$.

The centre of curvature of a point P of any conic may be found graphically as follows: Draw the normal PG, and a line from P through a focus S. Draw GK ⊥ PG meeting PS in K. Draw KO′ ⊥ PS meeting the normal in O′.

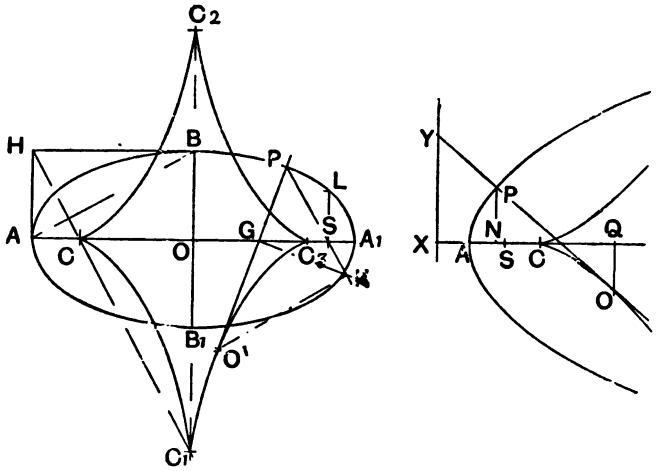


FIG. 247.

Then O′ is centre, and O′P radius, of curvature of the point P.

An easy method of finding the centre of curvature of a point P in a parabola is shown in Fig. 247 (b)*: Extend the normal of the point to meet the directrix in Y, make PO = 2 PY. Then O is centre, and OP is radius, of curvature at P. The point O may be found independently of the directrix, by making NQ = 2 NX = 2 (NA + SA),

* This construction cannot be used for the ellipse and hyperbola.

and drawing QO perpendicular to AQ meeting the normal in O.

The radius of curvature at the vertex A of a parabola is AC = 2 AS.

264. **Evolute of a Parabola.** The locus of centres of curvature is the evolute of the parabola, and since the parabola is symmetrical about its axis, the evolute is also symmetrical about the axis and has a cusp on the axis at C, the centre of curvature of A. Fig. 247 (b).

265. **Evolute of the Ellipse.**—Fig. 247 (a). The evolute is the locus of centres of curvature of the ellipse. The construction for finding the centre of curvature given above fails when applied to the ends of the axes. The centre of curvature at these points may be found thus: Construct a rectangle on the semi-axes AO, OB. Join AB and draw from H a perpendicular to AB intersecting the major axis in C, and the minor axis produced in C_1. Then C, C_1 are centres of curvature at A and B respectively. The centres for A_1 and B_1 are the *symmetriques* of C and C_1; they are shown at C_3, C_2. The evolute of the whole ellipse consists of four equal branches, as shown in the diagram, with cusps at C, C_1, C_2, C_3. Observe that the circle centre C, radius CA lies wholly within the ellipse, the circle centre C_1, radius C_1B is wholly without the ellipse, whilst for any intermediate point O as centre, the osculatory circle cuts the ellipse at the point of osculation. Refer to Art. 204. Observe that from any point situate in the space enclosed by the evolute of the ellipse lines can be drawn tangent to each of the four branches of the evolute, whereas for any point outside that space lines can be drawn tangent to two branches only. This shows that *from any point within the space enclosed by the evolute of an ellipse four normals to the ellipse can be drawn, and from any point outside two normals can be drawn.*

The employment of an ellipse in building work often necessitates the use of one or more curves parallel to the ellipse. In setting out details full size, the evolute of the ellipse enables lines parallel to, and perpendicular to, the ellipse to be drawn mechanically through any points whatsoever.

False Elliptic Curves.

266. In many cases in which it is proposed to use an elliptic curve, a curve built up of arcs of circles is substituted. Such curves are called *false elliptic curves.*

If pins are inserted in points n, c_1, c_2, m on the evolute of the ellipse, Fig. 248, then a pencil attached to a thread fastened at n will describe a curve, consisting of circular arcs, and this curve will very closely resemble the ellipse provided the pins are fairly numerous and placed in correct position.

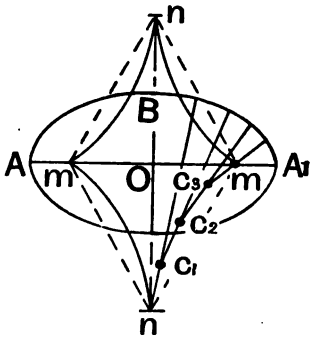


FIG. 248.

Again, a closed curve somewhat resembling the ellipse may be obtained by using four pins only, two on each axial line. In this case, there will be only four circular arcs in the complete curve.

267. *The number of centres cannot be less than four.*

In the ellipse, tangents at the extremities of the axes are perpendicular thereto; it follows that no matter how many circles are used in the false ellipse, the centre of the circle through each end of the major axis must lie on such axis, and the centre of the circle through an extremity of the minor axis must lie on the line of the minor axis. And, since OA is greater than OB, the radius of the circle through A must be less than OB, and therefore less than OA: whence there must be separate centres for the circles through each end of each axis.

Hence *the minimum number of circular arcs for the false ellipse is four.*

When only four circles are used, the circle through B has internal contact with the equal circles drawn through A and A_1.

268. If the number of arcs exceeds four, the additional

centres must lie within the lozenge *mnmn* (Fig. 248), and the increase must be made in each of the four quadrants, hence the number of arcs must always be some multiple of four.

269. If more than four centres are used, the circles through A and B are not in contact, but are connected by the series of arcs drawn from the additional centres (see Fig. 251, p. 252). The circle through A_1 has external contact with the circle centre c_3, which last has similar contact with the circle (c_3) next in order; and so on until the circle through B is reached, which has internal contact with the adjacent arcs on both sides of OB. The complete curve thus consists of a series of circles of varying radii so arranged that every circle, excepting those through the ends of the axes, have internal contact with the adjoining and smaller circle, and external contact with the adjoining and larger circle.

270. Like the ellipse, these curves are symmetrical about the *axes*, so that the centres for one quadrant only need be discussed; corresponding centres for the other quadrants are found by symmetry.

271. **The general Case of Four Arcs—the minimum Number.** Fig. 249.

If there are four centres for the complete curve, there will be two different radii. The curve at A is described from a centre C′ on AO, and the curve at B is described from a centre C on BO produced. If R denote the large radius BC, and r denote the small radius AC′, and a, b respectively denote AO, BO, we have $AC' = r$; $OC' = a - r$; $OC = R - b$; $CC' = CN - r = R - r$.

And
$$CC'^2 = OC'^2 + OC^2 \text{ (right-angled triangle)}$$
$$= (a - r)^2 + (R - b)^2$$
$$\therefore \quad (R - r)^2 = (a - r)^2 + (R - b)^2$$

hence
$$R = \frac{a^2 + b^2 - 2ar}{2(b - r)} \quad . \quad . \quad . \quad \text{(i.)}$$

Similarly,
$$r = \frac{a^2 + b^2 - 2bR}{2(a - R)} \quad . \quad . \quad . \quad \text{(ii.)}$$

If we take $r = b$, R is infinitely great, hence it is necessary to make r less than b to render the problem possible.

Again, if $R = \frac{a^2 + b^2}{2b}$, the small circle vanishes and the circle of radius R will pass through A; hence R must be less than $\frac{a^2 + b^2}{2b}$. Within these limits we may assume a value for either R or r, and determine the corresponding value of the other radius, either graphically or by computation.

272. **Ex.** Fig. 249. *To draw a false ellipse when a =*

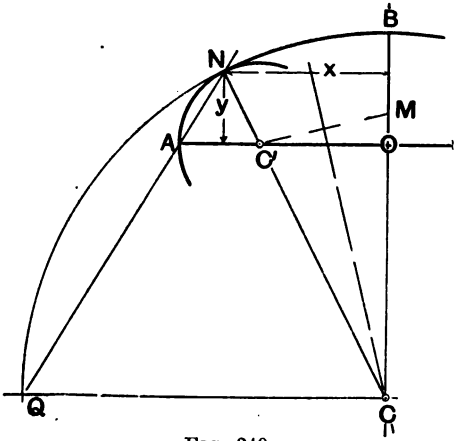


FIG. 249.

60 ft. and b = 32 ft. having given (i.) *the short radius r =* $22\frac{2}{3}$ *ft.*, (ii.) R = *102 ft.*

Graphic Method. (i.) Select some convenient scale—say 0·1 in. = 2 ft. Draw straight lines ⊥ to each other meeting in O. Make OA = 60 ft.; OB = 32 ft. Make AC′ on AO = $22\frac{2}{3}$ ft. Make BM on BO = AC′. Draw C′M and its ⊥r bisector which intersects BO produced in C. Draw CC′. Then C′, C are the centres for one quadrant. The centres for the other quadrants are found by symmetry.

With centre C′ and radius C′A, describe an arc meeting CC′ produced in N. With centre C and radius CB or CN, describe an arc NB.

The Δ CMC′ is similar to the isosceles Δ CBN. Hence, circle centre C touches the small circle at N.

(ii.) The large radius is given as 102 *ft.* Make BC on BO = 102 ft. (to scale). With centre C and radius CB describe a quadrant BQ. Draw QA and produce to intersect the quadrant in N. Join NC intersecting AO in C′. Then Δs QNC, ANC′ are similar.

$$\therefore \quad C'N = C'A.$$

hence C′ is the centre of the small circle and N is the point of contact.

(i.) *To calculate* R.

Substituting the given values in Equation i., Art. 271, we have

$$R = \frac{60^2 + 32^2 - (2 \times 60 \times 22\frac{2}{3})}{2 \times (32 - 22\frac{2}{3})} \text{ ft.}$$

$$= 102 \text{ ft.}$$

(ii.) *To calculate r.*

Substituting the given values in Equation ii., Art. 271, we have

$$r = \frac{60^2 + 32^2 - (2 \times 32 \times 102)}{2 \times (60 - 102)} \text{ ft.}$$

$$= 22\frac{2}{3} \text{ ft.}$$

273. Where curves of this type are used in large arches they may be drawn with triangular trammels (see Art. 146).

To do this, we must know the position of N relative to the axial lines.

Calling the distance of N from OB = x, and from OA = y, we have

$$\text{From similar } \Delta\text{s}, \quad x = R\frac{a - r}{R - r}.$$

$$\text{Also,} \qquad y = r\frac{R - b}{R - r}.$$

Substituting the known values in this example, we have

$$x = 102 \times \left(\frac{60 - 22\frac{2}{3}}{102 - 22\frac{2}{3}}\right)$$

$$= 48 \text{ ft.}$$

for y we have $y = 22\frac{2}{3}\left(\frac{102 - 32}{102 - 22\frac{2}{3}}\right)$

$$= 20 \text{ ft.}$$

274. We now give a construction which, owing to its extensive use, merits special notice. In Fig. 250, the ratio $\frac{R}{r}$ is such that the angle NCB subtended at the centre C by the arc BN is equal to the angle BAO $= \alpha$. In this case, the triangles AC'K, ABO, CBK are similar and have each a right-angle.

It can be shown that $AK = \frac{\sqrt{a^2 + b^2} - (a - b)}{2}$.

I.e., 2 AK *is equal to* AB *minus the difference between the semi-axes. Hence the construction.* Fig. 250. OA, OB are respectively the semi-major and semi-minor axes. Make OD = OA. Join AB, and mark off on this line from B, BP=BD. Bisect AP with the perpendicular NC which by its intersection with AO and BO determines the required centres.

$$r = AC' = \frac{AB}{AO} \cdot AK = \frac{\sqrt{a^2 + b^2}}{a} \left\{ \frac{\sqrt{a^2 + b^2} - (a - b)}{2} \right\}$$

$$R = CB = \frac{AB}{OB} \cdot BK = \frac{\sqrt{a^2 + b^2}}{b} \left\{ \frac{\sqrt{a^2 + b^2} + (a - b)}{2} \right\}$$

from similar triangles.

275. If it is desired to get a curve which shall lie very close to an ellipse a greater number of arcs must be employed. In such cases, it is best to determine points in the ellipse and construct circles of varying radii passing through these points. The following example will make the method clear :—

Fig. 251 illustrates a case in which 12 centres are used to describe the complete false ellipse. As there are to be 12 centres in the complete curve there will be $\frac{12 - 4}{4} = 2$ intermediate centres and 2 principal centres in each quadrant, *i.e.*, 4 centres and 4 arcs in each quadrant.

Construct the rectangle AEBO on the semi-axes. Divide AO, AE into four equal parts (one for each arc) and by Problem 502 determine points P, Q, D, on the true ellipse.

Join PB and draw its perpendicular bisector which intersects BO in C. Join CP, and draw the perpendicular bisector of QP which intersects CP in C_1. Join C_1Q, and draw the perpendicular bisector of DQ which intersects C_1, Q in C_2. The points C, C_1, C_2 are three of the four centres required. The remaining centre C_3 must lie in AO and will not necessarily fall on C_2D. We require to locate the centre point in AO from which a circle can be described which shall have external contact with the circle centre C_2, radius C_2Q, and shall also pass through the point A. This problem is solved as in Art. 272, Case ii. The centre is at C_3, and T is the point of contact of the two circles.

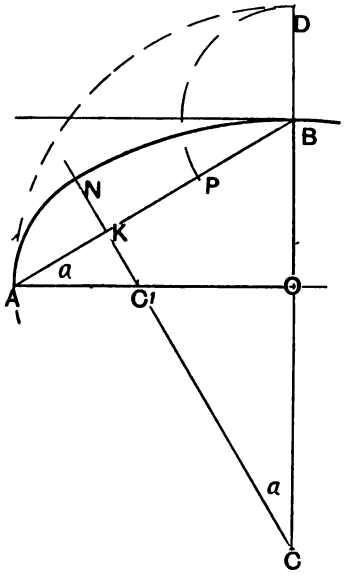


FIG. 250.

276. *Rectangular co-ordinates of points* P, Q, *etc.*, Fig. 251. Let x = PM, the abscissa of the point P, and y = PN, the corresponding vertical ordinate of the point. Produce B_1P to meet BE in p, and draw $pm \parallel$ PB intersecting B_1B in m.

From similar Δs B_1IO, B_1pB ; $Bp = 2\ OI$.

„ Bpm, $EB1$; $Bm = \frac{Bp \cdot E1}{BE} = \frac{2OI \cdot E1}{a}$

$$B_1m = B_1B + Bm = 2b + \frac{2OI \cdot E1}{a}$$

Again $$x = PM = \frac{B_1B \cdot Bp}{B_1m} = \frac{2b \cdot 2\ OI}{2b + \frac{2\ OI \cdot E1}{a}}$$ Eq. (i.)

Also, from similar Δs PNI, B_1OI,

$$y = PN = (PM - OI)\frac{B_1O}{OI}$$

$$= (x - OI)\frac{b}{OI} \quad . \quad . \quad . \quad \text{Eq. (ii.)}$$

Numerical Example. Determine x and y of point Q in Fig., having given $OB = b = 8$ *ft., and* $OA = a = 16$ *ft.*

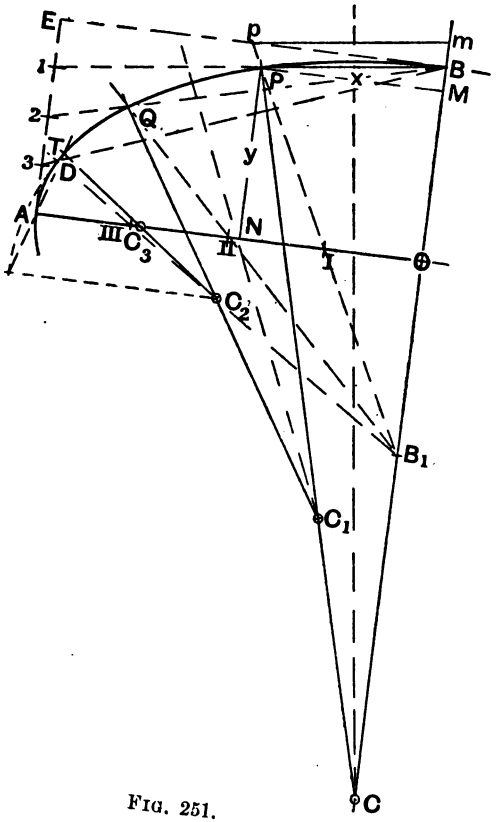


FIG. 251.

Since points II and 2 bisect OA and EA respectively,

Equation i. $$x = \frac{2b \cdot 2\,\mathrm{OI}}{2b + \frac{2\,\mathrm{OI} \cdot \mathrm{E}\,1}{a}},$$

becomes $$x = \frac{2b \cdot 2\,\mathrm{OII}}{2b + \frac{2\,\mathrm{OII} \cdot \mathrm{E}\,2}{a}}$$

$$= \frac{2ba}{2b + \frac{b}{2}}$$

$$\therefore \quad x = \frac{2 \times 8 \times 16}{2 \times 8 + 4} = 12\tfrac{4}{5} \text{ ft.}$$

For y, we have Equation ii.

$$y = (x - \mathrm{OI})\frac{b}{\mathrm{OI}}, = (x - \mathrm{OII})\frac{b}{\mathrm{OII}},$$

which, on substituting numerical values, reduces to

$$y = 12\tfrac{4}{5} - 8 = 4\tfrac{4}{5} \text{ ft.}$$

Exercise. Determine x and y co-ordinates of the point P.

Ans. $x = 7{\cdot}53$ ft.; $y = 7{\cdot}06$ ft.

277. In all the foregoing examples of false ellipse curves, the centres from which the curves are described will serve as centres of curves parallel to the false ellipse.

The Hyperbola.

278. Fig. 252. Insert pins in any two points S, S', and attach a thread to the point S and to a straight edge at Q. Let the straight edge be placed against the point S' and so as to make the thread taut. Then place a pencil against the thread at Q and rotate the edge without sliding about the fixed point S' as a pivot. During this rotation let the

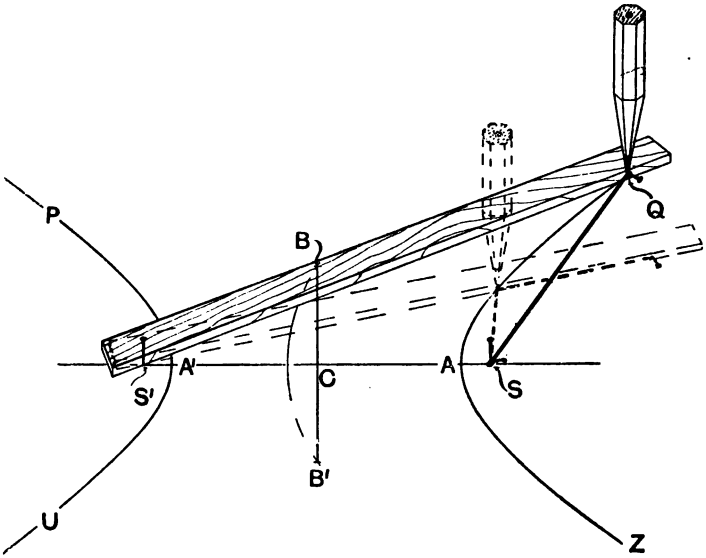


FIG. 252.

thread be kept taut by sliding the pencil along the edge. The path traced by the pencil is a curve QAZ. If the same thread be now fixed at the point S' and the same point in the straight edge be made to rotate on the point S, a pencil constrained to move as above described will trace the curve PA'U. The two curves traced by the moving pencil are the two branches of a **hyperbola.**

The two fixed points S, S' are called **foci.** The points

A, A′, in which the two branches of the curve cut the straight line joining the foci are called the **vertices**. AA′ is called the **transverse axis.** The mid-point C of AA′ is the **centre** of the hyperbola.

A line BB′ through C perpendicular to AA′, and whose extremities lie on the circle of radius CS′ and centre A, is called the **conjugate axis** of the *hyperbola.*

279. The equation to the hyperbola* referred to its centre as origin, and transverse axis as axis of x is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

or

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2) \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

where a, b are the semi-transverse and semi-conjugate axes.

If we make $y = 0$ in Equation i., we have

$$x = \pm a = \text{CA or CA}'.$$

Hence the curve cuts the axis of x in two points equidistant from, and on opposite sides of, the centre C, and since $\text{AA}' = 2a$, it follows that the difference of the *two lines drawn from any point of a hyperbola to the foci is equal to the transverse axis*, that is, $\text{QS}' - \text{QS} = \text{AA}'$.

280. If we reduce Equation ii. to the form

$$y = \pm \frac{b}{a}(x^2 - a^2)^{\frac{1}{2}}$$

and give to x any value less than a, the corresponding value of y is the square root of a negative quantity. Therefore the curve does not approach nearer to the axis y than $\pm a$. For any value of x, positive or negative, greater than a, the equation gives for y two values equal in magnitude but opposite in sign, and the greater the numerical value of x the greater the corresponding values of y. Hence, the hyperbola extends indefinitely on both sides of the transverse axis, and both to the right and left of C.

281. *To construct a hyperbola from its equation*, we reduce the equation to the form

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

and compute values of y corresponding to assumed values of x.

* See p. 265.

Ex. Fig. 253. *Trace the curve whose equation is*

$$\frac{x^2}{81} - \frac{y^2}{36} = 1.$$

Solving for y, we have

$$y = \pm \frac{6}{9} \sqrt{x^2 - 81.}$$

Assuming for x numerical values from 9 upwards, we obtain the corresponding values of y as given below:—

When $x = 9, y = 0.$	When $x = 12, y = \pm 5{\cdot}29$
$x = 10, y = \pm 2{\cdot}91.$	$x = 13, y = \pm 6{\cdot}25$
$x = 11, y = \pm 4{\cdot}21.$	$x = 14, y = \pm 7{\cdot}47$

If we give a negative sign to each of the above values of

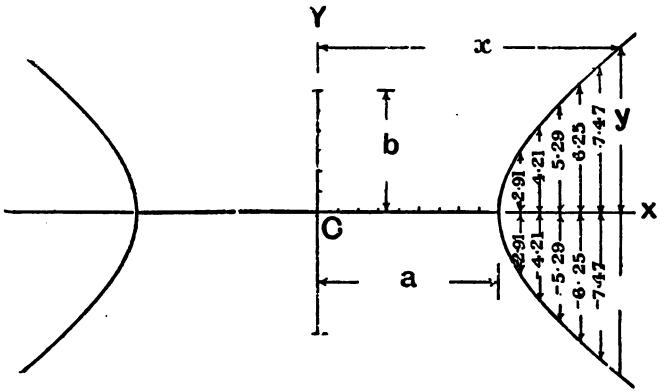


FIG. 253.

x we obtain values for y equal to those in the above table and a curve on the left of CY equal to that on the right.

282. **Conjugate Hyperbola.** Fig. 254. A hyperbola which has BB′ for its transverse axis and AA′ for its conjugate axis is called the **conjugate hyperbola.**

If $y^2 = \frac{b^2}{a^2}(x^2 - a^2)$ be the equation of the first hyperbola, then the equation of its conjugate is $y^2 = \frac{b^2}{a^2}(x^2 + a^2)$.

That is, the equation of the conjugate hyperbola is obtained from that of the original by writing $-a^2$ for a^2, and $-b^2$ for b^2.

283. **Asymptotes.** A line toward which a curve continually approaches in one direction without actually touching, but such that the distance of the curve from the line is ultimately less than any assignable quantity is called an **asymptote.**

An asymptote may be conceived as meeting the curve at infinity, and in this sense an asymptote is said to be tangent to a curve at a point which is infinitely distant.

284. Fig. 254. The diagonals CT, Ct, of the rectangles of which the semi-axes of a hyperbola are adjacent sides are asymptotes of the hyperbola.

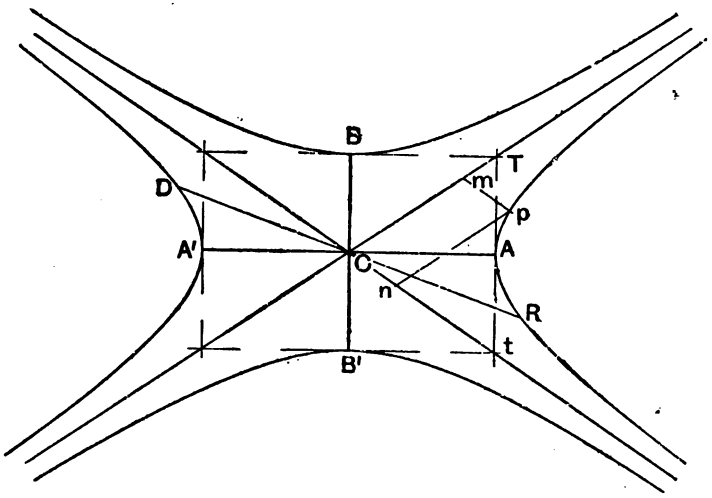


FIG. 254.

285. If the axes AA', BB' are equal, CT and Ct are at right angles to each other, and the curve is called a **rectangular hyperbola.**

In the case of the rectangular hyperbola, Equation (2), Art. 279, reduces to

$$y^2 = x^2 - a^2.$$

286. The rectangular hyperbola is to the common hyperbola what the circle is to the ellipse. Refer to Art. 239.

287. Fig. 239. The equation of the hyperbola referred to its asymptotes as axes of reference, is $xy = \frac{a^2 + b^2}{4}$, where a and b are the semi-axes, and the x and y, co-ordinates are measured parallel to the asymptotes.

288. The equation of the conjugate hyperbola is found by writing $-a^2$ for a^2, and $-b^2$ for b^2 and is

$$xy = \frac{-a^2 - b^2}{4}.$$

Exercise.—Trace the curve whose equation is $xy = 12$. *First* when the axes are rectangular ; *second* when they include an angle of 65°.

289. The equation $xy = \frac{a^2 + b^2}{4}$ shows that for any point P on the hyperbola *the product of the abscissa pm and ordinate pn is constant.*

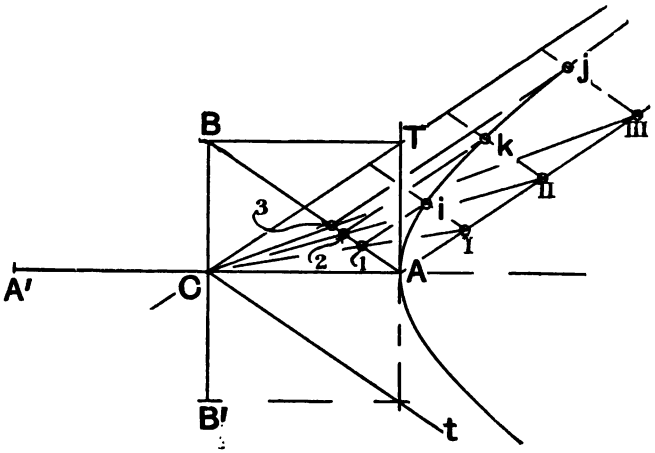


FIG. 255.

290. **Problem.** Fig. 255. *Given* AA', BB' *the axes of a hyperbola, construct the curve.*

We use the preceding property.

Draw the rectangle CBTA, having the given semi-axes for adjacent sides. Draw the diagonals of this rectangle, one of which CT is an asymptote of the hyperbola, and the other BA is parallel to the other asymptote C*t*.

Through the vertex A draw lines parallel to the asymptotes. Through C, the centre, draw radials cutting these parallels in points 1, 2, 3, etc., I, II, III. Draw parallels to the asymptotes through corresponding points 1, I ; 2, II ; etc., intersecting in *i*, *k*, *j*, which are points on the hyperbola. In a similar manner, the other branch of the curve may be constructed. A similar construction applied to the vertices B and B′ gives points in the conjugate hyperbola.

291. *Tangent and Normal.* Fig. 256. The equation to the tangent at any point x', y' of a hyperbola, referred to the asymptotes as axes of references, is

$$yx' + xy' = 2\,x'y' = \frac{a^2 + b^2}{2}.$$

If we put $y = 0$ in this equation, we have

$$xy' = 2\,x'y',$$

or $$x = 2\,x'.$$

Thus, if P be the point whose abscissa and ordinate are x', y', and T the point in which the tangent at P intersects the asymptote CX, we have

$$CT = 2\,CN$$

If we put $x = 0$ in the equation to the tangent, we have

$$yx' = 2\,x'y'$$

or $$y = 2y'$$

That is, C*t* = 2 CM.

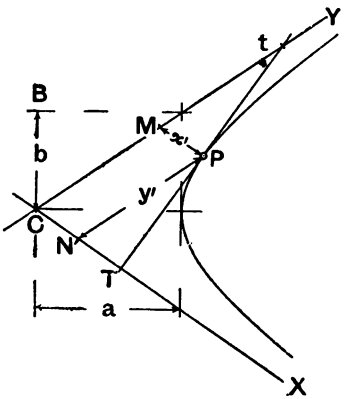


FIG. 256.

292. From similar Δs NPT, M*t*P ; T*t* is bisected in P.

293. **Problem.** Fig. 257. *To draw a tangent and a normal at a point* P *on a hyperbola.*

First Method. We may make use of the asymptotes and the preceding property. Draw P*n* ‖ YC meeting CX in *n*.

Make $nt = Cn$. Draw tP, which is tangent at P. The normal PG is $\perp^r$ to tP.

Alternative Solution. In the hyperbola, as in the ellipse, it can be shown that NG : NC : : $b^2 : a^2$.

Hence, draw AZ $\perp^r$ AC and make AZ = SL (the semi-latus rectum). Join ZC and draw PN $\perp^r$ AC, meeting ZC

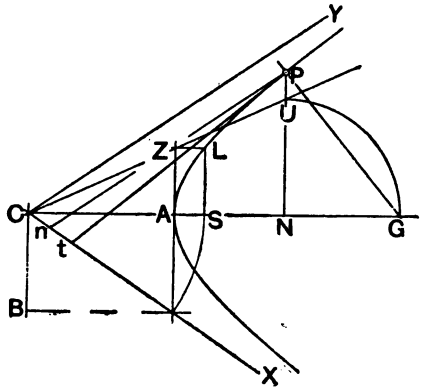


FIG. 257.

produced in U. Make NG = NU and join PG, which is normal at P.*

This is an excellent method when normals at several points are required.

Third Method. The tangent bisects the focal angle SPS'. *The normal bisects the supplementary angle.*

Refer to Art. 218.

294. Fig. 254. Any straight line DR drawn through the centre of a hyperbola and terminating on the two branches of the curve is called a diameter.

As in the ellipse, *every diameter of a hyperbola is bisected by the centre.*

Also, *any diameter produced bisects all chords parallel to the tangent at its extremity.*

* The proof is similar to that given in Art. 257.

295. Two diameters, each of which bisects the chords parallel to the other, are called **conjugate diameters.**

296. **Problem.** Fig. 258. *Given the curve of a hyperbola, to determine its centre, axes, focus, and the equation of the curve.*

Draw any two parallel chords *vp*, *n'g*, and *jj* the bisector of these chords. Similarly, draw *ff* the bisector of other two parallel chords *vh*, *n'w*. The two bisecting lines meet in C, the centre of the hyperbola. With centre C describe a circle intersecting the hyperbola in points *o*, *o*.

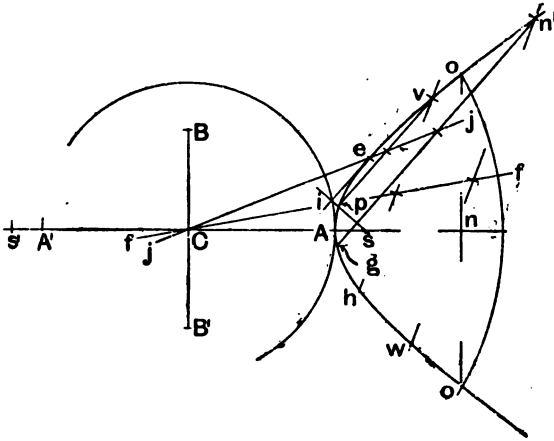


FIG. 258.

Through C draw C*n* ⊥r to *oo* meeting the curve in A. With centre C and radius CA describe a circle. Draw *ei* ∥ *vp* intersecting the previously drawn circle in *i*. Draw *is* ⊥r *ei* to intersect CA prolonged in *s*. Make C*s'* = C*s*, and CA' = CA. Make AB' = C*s*. Then AA', BB' are transverse and conjugate axes. The points *ss'* are foci.

The equation of the curve is

$$\frac{Cn^2}{CA^2} - \frac{on^2}{CB^2} = 1.$$

297. **Problem.** Fig. 259. *To draw a tangent to a hyperbola through an external point* T.

A construction based on Art. 258.

Join T to the nearest focus S. On TS as diameter describe a circle intersecting the auxiliary circle in q. Tq produced will touch the hyperbola. To determine the point of contact Q. Extend Sq to R, making $qR = Sq$. Join S'R and produce to meet Tq in Q, which is the point of contact.

A second tangent TP is found in a similar manner.

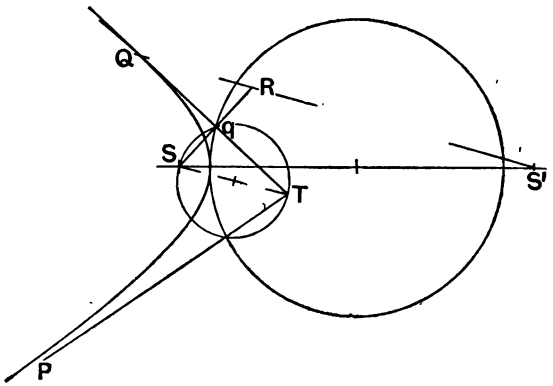


FIG. 259.

298. **Problem.** Fig. 260. *Given a double ordinate pq of a hyperbola and the transverse axis aa'. Construct the curve.*

Construct the rectangle *pqrn*, having base *pq* and altitude *va*. Divide *pn* and half *pq* each into the same number of equal parts at I, II, III; 1, 2, 3. Radials from *a'* and *a* through corresponding points 1, I; 2, II, etc., meet in points on the curve.

This construction, as also those shown in Figs. 228, 241, are *radial projections* or *perspectives* of Fig. 240.

The construction is equally applicable to all the *conics*, when we have given as data *any chord, and the diameter conjugate to that chord.*

(In the case of the parabola, the axis is indefinitely long; hence the lines in Fig. 228 corresponding to one set of radials in Figs. 241, 260 are parallel.)

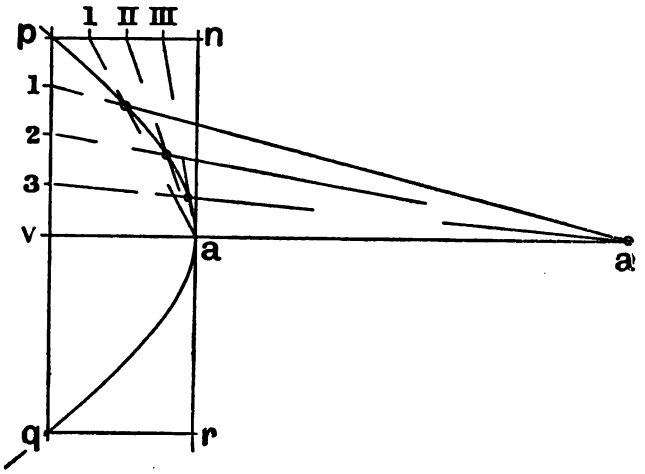


FIG. 260.

An illustration of the general application of this principle is shown in the solution of the following:—

299. **Problem.** Fig. 261. *To describe a conic to pass through a given point a and touch two given lines* OH, OG *at the points* H *and* G.

Draw through *a* a line *nn* ‖ HG and *av* ‖ OG. Divide *av* into a number (3) of equal parts. Similarly, divide *an*. Radials from G and H through corresponding divisions meet in points on the arc *a*H. In a similar manner, the arc *a*G may be determined.

Note.—Let perpendiculars from *n* and *a* on to HG meet same in E and F.

If FE = EH the curve is a *parabola.*
If FE > EH „ „ „ an *ellipse.*
If FE < EH „ „ „ an *hyperbola.*

300. **Eccentricity of a Conic Section.** From the

definition (Art. 206) we see that a parabola is a plane curve traced by a point which moves in such a way that its distance from a fixed point (the focus) is in a constant ratio to its perpendicular distance from a fixed straight line (the directrix).

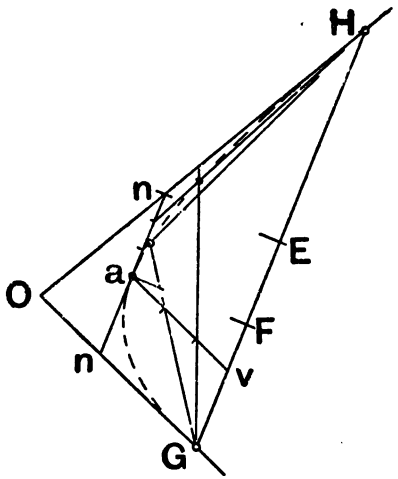


FIG. 261.

This definition may be extended to an ellipse and a hyperbola, and the curve traced out by a point moving as above will be a parabola, or an ellipse, or a hyperbola, according as the *constant ratio* is equal to unity, less than unity, or greater than unity.

The constant ratio is called the **eccentricity** of the conic. An ellipse and a hyperbola have each two directrices corresponding to their two foci.

Similar Conics.—Two conics are similar if corresponding axes in the two figures are in the same ratio (Art. 280). Hence if two conics have equal eccentricity, the figures are similar: and since the eccentricity of all parabolas is unity, it follows that all parabolas are similar. If two hyperbolas are similar, their asymptotes make equal angles, and conversely.

Notes.

To find the equation of the parabola. Figs. 221, 222.
Let AX, AY be axes of reference, A being the origin.
Let $Am = a$, and let x, y be co-ordinates of P.
Then $SP = Pd$; $\therefore\ SP^2 = Pd^2$.
Because $SA = Am$ $\quad y^2 + (x - a)^2 = (x + a)^2$,
$\therefore\ y^2 = 4ax$.

To find the equation of the ellipse and of the hyperbola. Figs. 234 and 252.

In both Figures, let AA_1, BB_1 be axes of reference with c as origin. Let S, S_1, be foci and P any point on the curve. Draw PM $\perp$r AA_1. Let x and y denote the co-ordinates of P, and a, b, denote CA, CB, respectively, and let c denote CS.

From right-angled triangles we have

$$SP^2 = PM^2 + MS^2 = PM^2 + (CM - CS)^2$$

or
$$d^2 = y^2 + (x - c)^2 \quad . \quad . \quad (1)$$

Also $S_1P^2 = PM^2 + MS_1^2 = PM^2 + (CM + CS)^2$

i.e.,
$$d_1^2 = y^2 + (x + c)^2 \quad . \quad . \quad . \quad (2)$$

Adding Equations (1) and (2) we obtain

$$d^2 + d_1^2 = 2\,(y^2 + x^2 + c^2) \quad . \quad . \quad . \quad (3)$$

Subtracting Equation (1) from (2) we obtain

$$d_1^2 - d^2 = 4\,cx,$$

or
$$(d_1 + d)\,(d_1 - d) = 4\,cx \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But from the mode of generation of the ellipse we have

$$d_1 + d = 2\,a \quad . \quad (5)$$

Dividing Equation (4) by (5) we have

$$d_1 - d = \frac{2\,cx}{a}.$$

Combining this with Equation (5) we find

$$d_1 = a + \frac{cx}{a} \quad . \quad (6)$$

$$d = a - \frac{cx}{a} \quad . \quad (7)$$

If we square these values and substitute them in Equation (3) we find

$$a^2 + \frac{c^2x^2}{a^2} = c^2 + x^2 + y^2,$$

or

$$a^2y^2 + (a^2 - c^2)\,x^2 = a^2(a^2 - c^2). \quad (8)$$

Now $BC^2 = BS^2 - CS^2$

or
$$b^2 = a^2 - c^2. \quad (9)$$

On substitution in Equation (8) we find

$$a^2y^2 + b^2x^2 = a^2b^2 \quad . \quad (10)$$

or
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad (11)$$

which is the equation of the ellipse referred to its axes.

It is sometimes written

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad . \quad (12)$$

But, from the mode of generation of the hyperbola, we have

$$d_1 - d = 2\,a.$$

Substituting this value in Equation (4) we obtain

$$d_1 + d = \frac{2\,cx}{a}.$$

Solving these two equations, we find

$$d_1 = a + \frac{cx}{a} \quad . \quad (5)$$

$$d = -a + \frac{cx}{a} \quad . \quad (6)$$

If we square these values and substitute them in Equation (3) we find

$$a^2 + \frac{c^2x^2}{a^2} = c^2 + x^2 + y^2$$

or

$$(c^2 - a^2)\,x^2 - a^2y^2 = a^2(c^2 - a^2) \quad (7)$$

Putting $b^2 = c^2 - a^2$, the equation reduces to

$$b^2x^2 - a^2y^2 = a^2b^2 \quad . \quad (8)$$

or
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad (9)$$

which is the equation of the hyperbola referred to its centre and transverse axis.

It is sometimes written

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2). \quad . \quad (10)$$

CHAPTER XVIII

SURFACES AND VOLUMES OF SOLIDS.

301. Any geometrical figure that has length, breadth, and thickness is called a *solid.*

In Fig. 262, we have a sketch of a figure bounded entirely by planes, viz., the square DBFI, and the triangles ABD, ADF, AFI, AIB. This figure is called a *pyramid.* The square is called the *base.* The triangles which meet in A

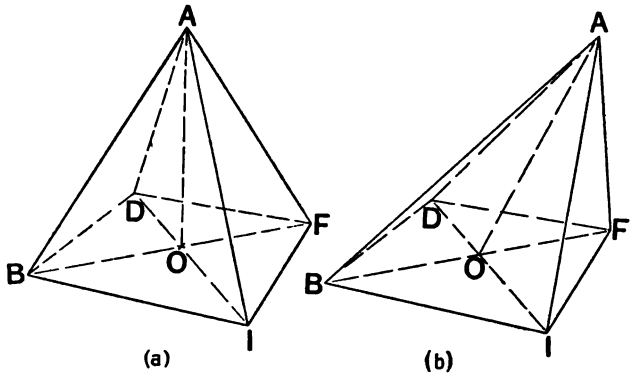


FIG. 262.

are called *sides* or *faces.* The boundaries of the faces are *edges.* The point (A) in which all the sloping edges intersect is called the *apex.* The line AO joining the apex to the centre of the base is called the *axis.*

A pyramid may have any polygon for base. If the base is a triangle, it is called a *triangular pyramid;* if a square, it is called a *square* pyramid; if a pentagon, it is called a *pentagonal* pyramid, and so on.

A pyramid is said to be **right**, or **oblique**, according as its axis is, or is not, perpendicular to the base. See Fig. 262, (*a*) and (*b*).

When the base of a *right* pyramid is a *regular* polygon the figure is called a *regular pyramid;* and all the sloping faces are *congruent* triangles.

302. If we regard a circle as a regular polygon of innumerable sides which have no finite length and imagine lines drawn from the apex A, in Fig. 263, to points on the circumference of the circle centre O, and close together, we

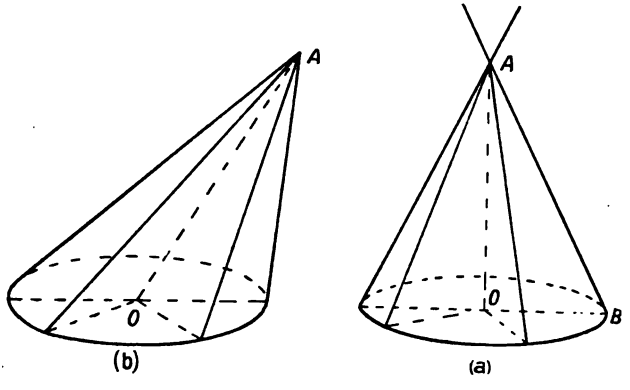


FIG. 263.

obtain a figure having a sloping surface which is not made up of a number of triangles, but which is one regular curved surface upon which no straight lines can be drawn except from the apex to points on the circular base. This figure is called a *cone.*

A cone is said to be *right* or *oblique* according as its axis is, or is not, perpendicular to the base. See Fig. 263, (*a*) and (*b*).

Another conception of the right cone. Fig. 263 (*a*). Let one side OA of a set-square occupy a fixed position, and let the set-square revolve on this line as an axis through a complete turn. Then the side OB describes a circle in a

plane perpendicular to the line OA. The locus of B is the circumference of the circle, and the hypotenuse BA generates the surface of the cone, which has OA for axis and the circle centre O, radius OB, for base.

303. If parallel lines of equal length be drawn from each corner of the square DBFI, Fig. 264, and the extremities of these lines be joined in proper order, we obtain a figure bounded entirely by planes, viz., the two parallel squares DBFI, DBFI, and four parallelograms DBBD ; BFFB ; FIIF ; IDDI. This figure is called a prism. The squares

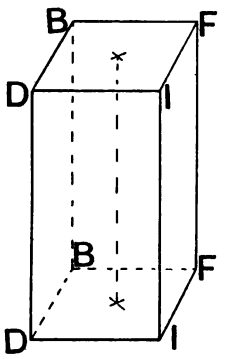


FIG 264A.

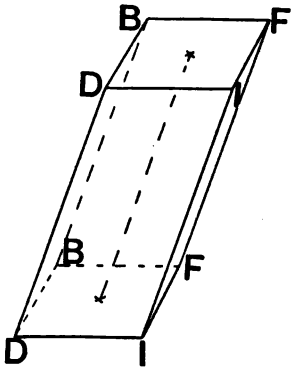


FIG. 264B.

are called the *ends*, and either end may be called the *base* of the prism. The parallelograms are called *sides* or *faces*. The line joining the centres of the two ends is called the *axis*.

A prism is said to be *right* or *oblique* according as its axis is, or is not, perpendicular to the base. Fig. 264A shows a *right* prism ; Fig. 264B shows an *oblique prism*.

In a right prism, all the faces or sides are rectangles.

Prisms are named according to the shape of their base or end, *i.e.*, triangular, square, pentagonal, or hexagonal, according as their base is a triangle, square, pentagon, or hexagon, etc.

A cube is a solid bounded by 6 *square faces.*

304. If the sides of the polygonal base of a prism be continually increased in number, and reduced in length, then finally the plane faces disappear and are replaced by a regular curved surface on which there are no straight lines except those joining corresponding points on the periphery of the two ends of the solid, *i.e.*, parallel to the axis of the prism. See Fig. 265. This figure is called a *cylinder*. When the base is a circle and the axis is perpendicular to the base, the cylinder is called a *right circular cylinder*.

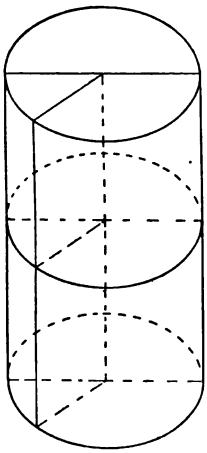


FIG. 265A.

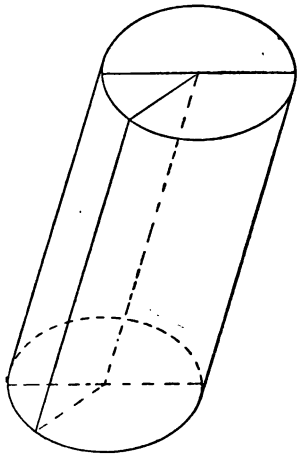


FIG. 265B.

305. **Superficial areas of solid figures.**

This part of our subject is illustrated by showing how the various solids may be built up of stout paper. The student should copy the diagrams given in the book to a larger scale ; cut the whole diagram from the paper with a sharp knife and cut half through along the dotted lines about which the paper is folded to form the model. The narrow strips attached are intended to wrap around the edges and to be gummed or pasted to secure the correct form of the model.

Apart from the immediate purpose the making of these

models will serve, they will be of great assistance to the student and save him much mental labour when engaged on any problem connected with the particular solids.

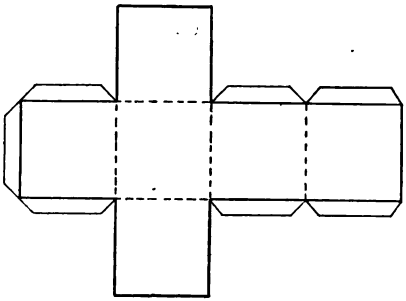


FIG. 266.

The superficial area of a solid is equal to the total area of the separate faces.

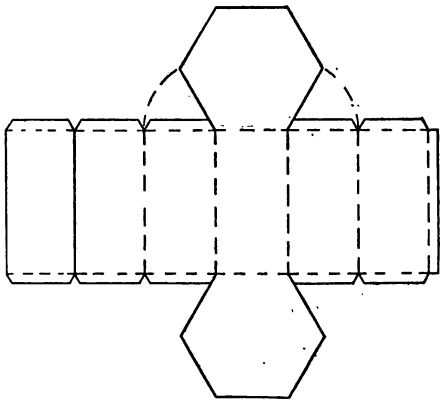


FIG. 267.

306. **The cube.** *The surface of a cube consists of* 6 *squares.* The diagram, Fig. 266, if cut half through along the dotted lines and folded, will make a cube.

If a in. is the length of the edge of the cube, then the area of each face $= a^2$ in. And hence

the superficial area of cube $= 6a^2$ sq. in.

307. **Right prism.** *The superficial area of any right prism is equal to the area of the two ends added to the rectangles which form the faces or sides of the prism. There are as many rectangular faces as there are sides to the polygonal base.*

The diagram, Fig. 267, if cut half through along the dotted line, and folded, will make a hexagonal prism.

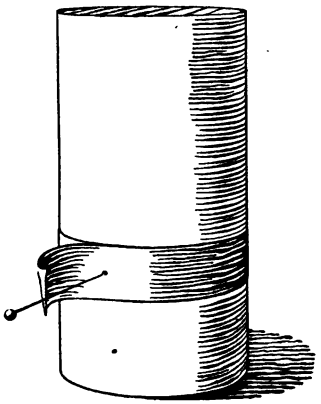


FIG. 268.

308. **Right cylinder.** *The superficial area is equal to the area of the two circles forming the ends added to that of the curved surface.*

Suppose we have a short length of a round iron bar or other cylindrical object, placed on end, as in Fig. 268. Wrap a strip of paper tightly round the object and insert a pricker where the paper overlaps. When this paper is unfolded, the distance between the two pin-holes is the circumference of the circle which forms the base of the cylinder. If we take a strip of paper of width equal to the height of the cylinder and length equal to the distance

between the two pin-holes in the diagram, we get a rectangle having sides respectively equal to the height of the cylinder, and the circumference of the circular base. This rectangle, if applied to the curved surface of the cylinder, will be found to cover it exactly. Hence for a cylinder of diameter d in., and height h in.,

Area of each end $= \frac{\pi}{4}d^2$ sq. in.

Area of curved surface = circumference of cylinder × height
$= \pi d \,.\, h$. sq. in.

$\therefore$ Superficial area $= \frac{\pi d^2}{4} + \frac{\pi d^2}{4} + \pi dh$

$= \pi d \left(\frac{d}{2} + h\right)$ sq. in.

Ex. Find the cost of painting the exterior of a closed cylindrical tank which is 10 *ft. in diameter, and* 15 *ft. high at* $1\frac{1}{2}$*d. per sq. ft.*

Area of each end $= \left(\frac{\pi}{4} \times 10^2\right)$ sq. ft.

Area of curved surface $= (\pi \times 10 \times 15)$ sq. ft.

Superficial area $= 2\left(\frac{\pi}{4} \times 100\right) + (\pi + 150)$
$= 628$ sq. ft.

Or, using above formula :—

Superficial area $= \pi d \left(\frac{d}{2} + h\right) = \pi \times 10\,(5 + 15) = 628$ sq. ft.

$$\text{Cost} = \frac{1{\cdot}5 \times 628}{12 \times 20} = £3\ 18s.\ 6d.$$

309. **Micrometer screw caliper.** Fig. 269 shows a common form of micrometer gauge used in making exact measurements of the thickness of metal plates, the diameter of wire, and other objects. The spindle c is attached to the thimble E, and is threaded to fit a screw on the inside of the hollow cylindrical sleeve D, which forms part of the frame A. As the thimble is turned between the thumb and finger, the spindle revolves with it, moving through the threaded sleeve approaching or receding from the anvil B, according to the direction in which the thimble is being revolved.

In order to allow of the distance between the anvil and the end of the spindle being read off with facility, scales are engraved on the sleeve D of the frame and on the bevelled edge of the thimble. These scales are not the same on all micrometers. Some tools are graduated to measure millimetres and fractions of the millimetre. The tool shown in Fig. 269 is made to measure thousandths of an inch. The pitch of the screw-threads on the spindle is $\frac{1}{40}$ in. One complete

revolution of the spindle therefore moves it along the sleeve $\frac{1}{40}$ (or 0·025) of an inch. The sleeve is marked with lines $\frac{1}{40}$ inch apart so that each division on the sleeve corresponds to one complete turn of the spindle and represents a movement of the spindle equal to 0·025 in. Each fourth line, representing $\frac{4}{40}$ (or 0·1) of an inch, is longer than the others. The zero (0) mark on the sleeve coincides with the position of the edge of the bevel on the thimble when the micrometer is closed. Hence, when the anvil and spindle are in contact no part of the scale on the sleeve D is visible. Backward rotation of the spindle through one complete revolution brings the edge of the thimble to the first line on the sleeve reading from *o*. Further rotation through one more complete revolution brings the edge of the thimble to the second line from *o*, thus exposing to view *two* of the scale divisions on the sleeve; and so on. And, since every part of the spindle and thimble recede equally from the anvil B, *the distance between the anvil and spindle is always equal to the visible portion of the scale on the sleeve.*

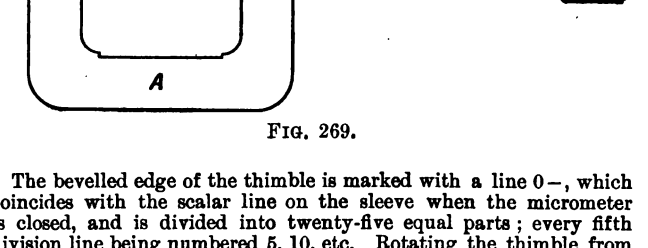


FIG. 269.

The bevelled edge of the thimble is marked with a line 0—, which coincides with the scalar line on the sleeve when the micrometer is closed, and is divided into twenty-five equal parts; every fifth division line being numbered 5, 10, etc. Rotating the thimble from one of these marks to the next moves the spindle along the sleeve $\frac{1}{40}$ of 0·025 in., or *one-thousandth of an inch.* A rotation of two divisions moves the spindle *two-thousandths* of an inch and so on. Rotation through a complete revolution corresponds to a lateral movement of 0·025 (or $\frac{1}{40}$) of an inch. The fraction of a complete revolution through which the spindle has been rotated, is indicated by the particular mark on the thimble scale coinciding with, or nearest to, the long scale line on the sleeve. Hence, to read the micrometer we have the **rule**: Multiply the number of divisions visible on the sleeve scale by 25, and add the number of divisions on the thimble scale, counting from 0 to the line which coincides with the scale line on the sleeve.

Ex. The object being measured in Fig. 269 is seen to be twelve divisions of the sleeve scale and a fraction of the next division,

i.e., (12 × 0·025 in.) + a fraction of 0·025 in.

The scale line on the sleeve coincides with the *third mark* from 0 on the thimble scale, hence the measure of the fraction above is $\frac{3}{25}$, or three-thousandths of an inch.

The object measures (12 × 0·025 in.) + 0·003
= 0·303 in.

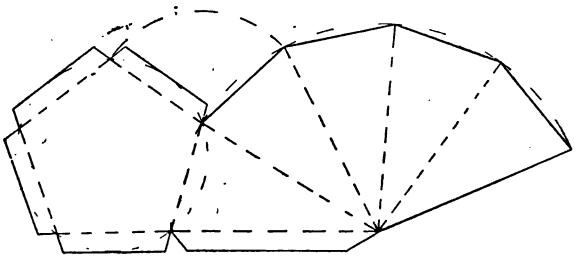
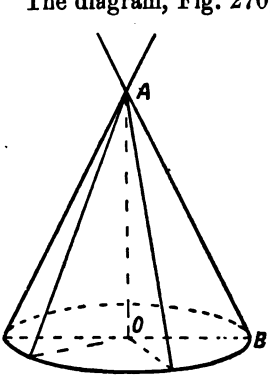


FIG. 270.

310. **Right pyramid.** *The superficial area of any pyramid is obtained by adding the area of the base to that of the triangular faces.* There are as many of these triangles as there are sides to the polygonal base.

The diagram, Fig. 270, if cut and folded along the dotted lines will form a right pentagonal pyramid.



(a)

FIG. 271.

311. **Circular cone.** If a piece of paper be wrapped, without crumpling or tearing, round the curved surface of a cone (Fig. 271), and cut with a knife along the edge of the base and the line AB, and then folded out, the paper will be a sector of a circle (Fig. 272) of radius AB equal to the slant height of the cone, and the length of the arc BD will be equal to the circumference of the base of the cone.

If we know the perpendicular height h, and radius of base r; then since angle AOB is a right-angle, the slant height $AB = \sqrt{r^2 + h^2}$.

The arc $BD = 2\pi r$.

The angle DAB in radians $= \frac{\text{length of arc}}{\text{radius}} = \frac{2\pi r}{AB}$.

If $l = AB$, we have—

Area of sector $= \frac{1}{2}(\text{radius})^2 \times (\text{angle in radians})$. See p. 136.

$$= \tfrac{1}{2}l^2 \times \frac{2\pi r}{l} = \pi r l.$$

Ex. Find superficial area if a cone of radius of base $r = 3$ in., and slant height $l = 5$ in.

(Superficial area = area of base + area of curved surface.)

$= \pi r^2 + \pi r l.$

$= \pi r(r + l)$ sq. in.

$= \pi 3(3 + 5).$

$= 24\pi$ sq. in. $= 75{\cdot}36$ sq. in.

Note. — The angle DAB, Fig. 272, of the developed surface of a cone is $\frac{2\pi r}{l}$ radians, or $\left(360° \times \frac{r}{l}\right)$ degrees, when r = radius of base and l = slant height AB of the cone.

Let θ denote the base angle of the cone. Then $\frac{r}{l} = \cos\theta$, hence the angle DAB $= 360° \cos\theta$ degrees. Thus, if $\theta = 60°$, $\cos\theta = \frac{1}{2}$, and the angle DAB for a cone having a base angle 60° = $360° \times \frac{1}{2} = 180°$.

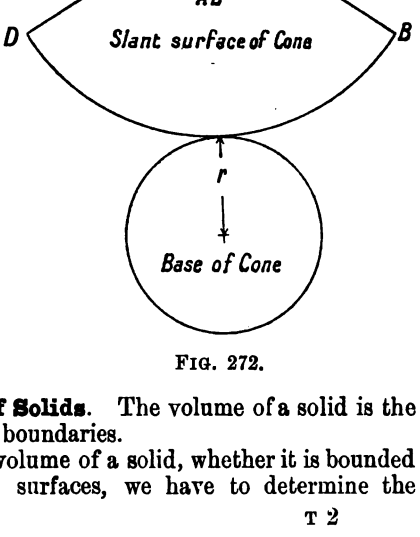


FIG. 272.

312. **Volumes of Solids.** The volume of a solid is the space enclosed by its boundaries.

In measuring the volume of a solid, whether it is bounded by plane or curved surfaces, we have to determine the

number of times it contains a particular volume, which we call a *unit of volume.* All units of volume are cubes the faces of which are squares of unit area, and the edges corresponding units of length. If each edge of a cube is 1 in. long, the volume is a cubic inch; similarly, a cube having an edge 1 ft. long is a cubic foot, and one having an edge 1 yd. long is a cubic yard.

313. To find the volume of any prism or cylinder having parallel ends.

Rule.—*Multiply the area of the base by the height.* This is expressed symbolically

$$V = A\ h,$$

where V represents volume, and A, h respectively represent the area of the base, and the height of the prism, or cylinder. This formula is general for all prisms and cylinders; we give below equivalent expressions for some simple solids.—

A **cube**, length of edge a ft.
$V = a^2 \times a = a^3$ cub. ft.
A **square prism**, height h ft., edge of base a ft.
$V = a^2h$ cub. ft.
A **triangular prism**, height h ft., sides of base a, b, and c ft.
$V = \left\{\sqrt{s(s-a)(s-b)(s-c)} \times h\right\}$ cub. ft.
where $s = \dfrac{a+b+c}{2}$.

A **hexagonal prism**, height h ft., sides of base a ft.
$V = 2{\cdot}598a^2h$ cub. ft.
A **circular cylinder**, radius of circle r ft., height h ft.
$V = \pi r^2 h$ cub. ft.

314. To find the Volume of a pyramid or cone.

Rule.—*Multiply the area of the base by $\frac{1}{3}$ the height.*

$$\text{Symbolically} \quad V = A\,\frac{h}{3}.$$

Where A represents the area of the base and h the height of the solid.

Thus the formulæ for volumes of prisms given in Art. 313 will express the volumes of corresponding pyramids if each

is divided by 3. For example, volume of a square pyramid side of base a ft. and height h ft. is $\frac{1}{3}a^2h$ cub. ft.

Similarly, for the volume of a right circular cone, given

r = radius of base of a cone.
h = perpendicular height.

Then volume = area of base $\times \frac{1}{3}$ height

$$= \pi r^2 \times \frac{1}{3} h$$

$$= \frac{\pi r^2 h}{3} \text{ cub. in.}$$

If the cone is 4 ft. high and has a base of radius 3 ft.

$$\text{Volume} = \frac{\pi \times 9 \times 4}{3} = 37{\cdot}68 \text{ cub. ft.}$$

315. **To find the volume of a frustrum of any prism made by two planes inclined to each other at any angle.**

Rule.—Multiply the *mean length* of all the parallel edges by the *cross section* of the prism.

$$\text{Symbolically } V = A \frac{h_1 + h_2 + h_3 + \ldots + h_n}{n}$$

Where A = area of cross section and h_1, h_2 . . . . h_n respectively represent the parallel edges of the frustrum.

Ex. A loft is formed in a roof of which Fig. 273 is a sketch. The trapezium ABCD is the form of the roof at eaves level, ER is the

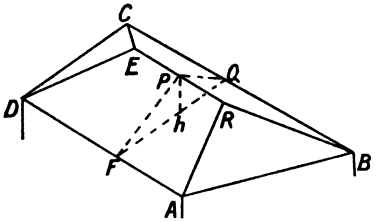


FIG. 273.

ridge and parallel to DA, CB. PQF is a cross section. Determine the capacity of the loft, having given, FQ = 35 ; Ph = 11·25 ; AD = 42 ; BC = 51·5 ; ER = 24·75 feet.

$$\text{Formula } V = A \cdot \frac{h_1 + h_2 + h_3 + \ldots + h_n}{n}$$

$$\text{becomes } V = A \cdot \frac{h_1 + h_2 + h_3}{3}.$$

$$\text{Hence capacity} = \frac{FQ \cdot Ph}{2}\left(\frac{AD + BC + RE}{3}\right).$$

$$= \frac{(35 \times 11{\cdot}25)}{2} \cdot \frac{42 + 51{\cdot}5 + 24{\cdot}75}{3}$$

$$= 7{,}760 \text{ cubic feet.}$$

316. **The "Prismoidal Formula."** If a solid bounded by parallel end planes is divided into 6 slabs of thickness h by other planes parallel to those of the ends and the areas of the plane sections are denoted by A_1, A_2, A_3, A_4, A_5, A_6, A_7., then substituting "area" for "ordinate" in Simpson's Rule, Art. 155, we obtain an expression for the volume, thus :—

$$\text{Volume} = \frac{h}{3}\left\{A_1 + A_7 + 4\,(A_2 + A_4 + A_6) + 2\,(A_3 + A_5)\right\} \text{ (i.)}$$

If three sections only are taken, the formula reduces to

$$\text{Volume} = \frac{h}{3}\left\{A_1 + A_3 + 4\,A_2\right\} \text{ (ii.)}$$

$$= l\left\{\frac{A_1 + A_3 + 4\,A_2}{6}\right\} \text{ Where } l = \text{total length.}$$

The **average section** is $\frac{\text{volume}}{\text{length}} = \frac{\text{volume}}{2h}$,

$$= \frac{h}{3 \times 2\,h}\left(A_1 + A_3 + 4\,A_2\right),$$

$$= \frac{1}{6}\left\{\text{first section} + \text{last section} + 4 \text{ central section}\right\}.$$

These are called the *prismoidal formulæ.* Formula (ii.) above suffices to determine the volume of many solids.

Ex. *Fig.* 274 *shows by plan and elevation a solid the base of which is a square, and top face a regular octagon. Determine the volume having given the following dimensions. Edge of base* b = 3·5 *ft.; edge of top face* 1 *ft.; height of solid* 6 *ft.*

By the prismoidal formula II.

$$V = l\,\frac{(A_1 + A_3 + 4A_2)}{6},$$

here $l = 6$; $A_1 = (3{\cdot}5)^2 = 12{\cdot}25$; $A_3 = 4{\cdot}828$.*

The mean section (A_2) is an eight-sided figure the alternate sides of which are equal to each other. Let x denote the larger side, and d and D respectively denote the diameter of the *top* and the mean sections.

Reference to the plan shows

$$x = \frac{3{\cdot}5 + 1}{2} = 2{\cdot}25\,;\ d = 1 + \sqrt{2},$$

$$D = \frac{b + d}{2} = \frac{3{\cdot}5 + 1 + 1{\cdot}414}{2}$$

$$= 2{\cdot}957.$$

The mean section is made up of the rectangle of sides x and D, and *two* trapeziums having parallel sides equal to x and D, and height equal $\frac{1}{2}(D - x)$.

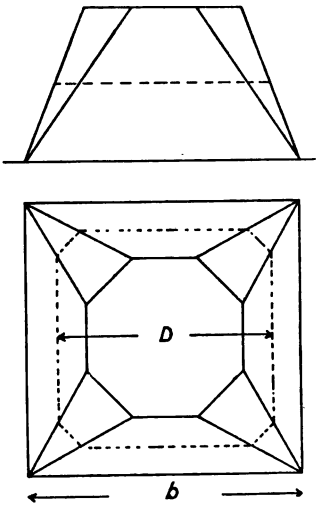


FIG. 274.

$$\therefore A_2 = Dx + \left(\frac{D + x}{2} \cdot D - x\right)$$

$$= Dx + \frac{D^2 - x^2}{2}$$

$$= (2{\cdot}957 \times 2{\cdot}25) + \left(\frac{2{\cdot}957 \times 2{\cdot}957) - (2{\cdot}25 \times 2{\cdot}25}{2}\right)$$

$$= 8{\cdot}49.$$

$$\therefore V = 6\left(\frac{12{\cdot}25 + 4{\cdot}828 + (4 \times 8{\cdot}49)}{6}\right)$$

$$= 51{\cdot}05 \text{ cubic ft.}$$

Ex. 2. *Fig.* 275 *shows plan and elevation of a vault formed by the intersection of two semi-cylindrical arches. Determine the weight of the arch and the infilling up to a level of* 5 *ft. above springing, having given span of arch* 8 *ft., and each cubic foot of material weighs* 125 *lbs.*

The weight = volume in cubic feet × 125 lbs. Volume equals the difference in the volume of the square prism and that of the vault.

For the former we have $8 \times 8 \times 5 = 320$ cubic feet.

* Let a denote side of octagon and d its diameter. Then

$$A_3 = 8 \cdot \frac{a}{2} \cdot \frac{d}{2} = 2ad.$$

Now $a = 1$, and $d = 1 + \sqrt{2}$.

Hence $A_3 = 2 \times 1 \times (1 + 1{\cdot}414)$

$= 4{\cdot}828.$

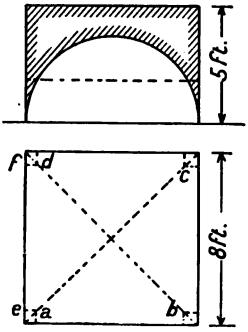


FIG. 275.

For the volume of the vault we have

$$V = l\left(\frac{A_1 + A_3 + 4A_2}{6}\right).$$

Here $A_1 = 8^2 = 64$; $A_3 = 0$; $l = 5$.

Area of mean section, A_2 = A_1 − 4 times the small square on *ea*.

$= 8^2 - 4\,(ea^2)$.

Now $ea = 4 - \sqrt{4^2 - 2^2}$.

$= 4 - 3{\cdot}464 = {\cdot}536$.

$\therefore A_2 = 64 - (4 \times {\cdot}536^2)$.

$= 63{\cdot}713$.

$$V = 4\left\{\frac{64 + (4 \times 63{\cdot}713)}{6}\right\}$$

$= 212{\cdot}57$.

Volume of arch $= 320 - 212{\cdot}57$.

$= 107{\cdot}43$ cub. ft.

$\therefore$ Weight $= 125 \times 107{\cdot}43$

$= 13{,}429$ lbs.

317. **The area of surface and the volume of a sphere of radius *r*.**

The surface of a sphere is equal to that of its circumscribing cylinder, hence surface $= 2\pi r \,.\, 2r$,

$= 4\pi r^2$.

The volume of a sphere is *two-thirds* the volume of its circumscribing cylinder,

hence volume $= \frac{2}{3}\pi r^2 h$,

$= \frac{4}{3}\pi r^3$.

A Zone of sphere lying between two parallel planes. Fig. 276.

Let r = radius of sphere.

r_1 and r_2 = radii of circular faces of zone

and h = height of zone.

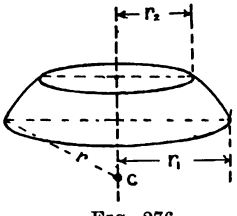


FIG. 276.

Then volume of zone $= \frac{\pi h}{6}\{3\,(r_1^2 + r_2^2) + h^2\}$

If $r_2 = 0$ we have segment of sphere of height h and volume

$= \frac{\pi h}{6}\left\{3r_1^2 + h^2\right\}$

Surface of zone $= 2\pi rh$
$=$ area of surface of circumscribing cylinder.

Lune of sphere lying between two meridian planes. Fig. 277.

Let angle between planes be θ degrees.

Then

superfical area : $4\pi r^2$ :: θ : 360

hence superficial area

$$= \frac{\pi r^2 \theta}{90}.$$

Volume of Lune.

Volume : $\frac{4}{3}\pi r^3$ :: θ : 360

hence volume $= \dfrac{\frac{4}{3}\pi r^3 \theta}{360}$

$$= \frac{\pi r^3}{270}.$$

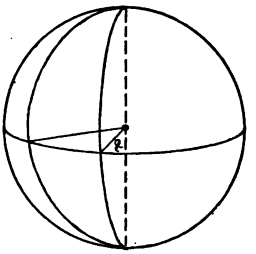


FIG. 277.

Exercises.

(1) A reservoir is 50 ft. long, 30 ft. wide, and 8 ft. deep, and it is filled with water. What weight of water does it contain? How many gallons does it contain?

(2) A swimming bath is 30 yds. long and 12 yds. wide, and its depth varies gradually from 7 ft. at one end to 3 ft. at the other. How many gallons and how many tons of water does it contain?

(3) A column is 5 ft. diameter and 12 ft. high. Find its superficial area (both ends included) and its weight if each cubic foot weighs 100 lb.

(4) Find the superficial area, volume, and weight of a hexagonal pyramid of height 10 ft., the side of hexagon being 3 ft. long. Each cubic inch of the material weighs 0·26 lb. The length of the line joining the centre of the base line of each triangular face to the apex must be found graphically.

(5) If one brick occupy a space 9 ins. by $4\frac{1}{2}$ ins. by 3 ins., how many would be required for a wall 100 ft. in length, 10 ft. in height, and a brick and a half in thickness.

(6) The base of a cone is a circle of 3·25 in. diameter. The vertical height is 5·24 in. It is made of cast iron, which weighs 0·26 lb. per cubic inch. What is the weight of the cone? (B.E. 1909.)

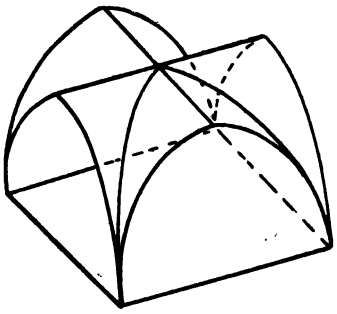
(7) A hexagonal bar of 3 in. side and 1 ft. long has a circular hole $1\frac{1}{2}$ in. diameter bored through it longitudinally. Find its weight if each cubic inch weighs 0·26 lb.

(8) A hollow iron column is 12 in. external diameter, and the metal is $1\frac{1}{4}$ in. thick. It carries a load of 125 tons. Find the compressive stress, *i.e.*, the load upon each square inch of sectional area of the column.

(9) A sphere of stone is 3 ft. in diameter, the weight of the stone per cubic foot is 120 lb.; what is the weight of the sphere? (B.E. 1907.)

(10) A regular hexagonal prism is cut by a plane inclined to the base at an angle of 45° and the distance of the line of intersection of the cutting plane with the plane of the base from the centre of the base is three times its perimeter; find the volume of the frustrum, a side of the base being 2 ft.

(11) Find the number of cubic yards of earth excavated from a railway-cutting made through ground the original surface of which was an inclined plane running in the same direction as the rails; the length of the cutting being 4 chains, 15 links, the breadth at bottom 30 ft., the breadth at top at one end 75 ft., and at the other 135 ft., and the depths of these ends 20 ft. and 46 ft. respectively.



(12) Determine the volume of the vault shown in the sketch if the vault at springing level is a square of 22 ft. side, the curves of the arches being arcs of circles of radius 22 ft.

(13) A groined surface is generated by a variable square moving with the mid points of its sides along the circumference of two equal and vertical semicircles and remaining always horizontal. Determine the cubical contents of the groin when the diameter of director circles is 12 ft.

(14) Substitute parabolas with axes vertical for the semicircles in Q. 17.

Answers.

(1) 747,600 lb.; 74,760 galls.
(2) 450·5 tons; 1,009,260 galls.
(3) 227·6 sq. ft.; 23,562 lb.
(6) 3·767 lb.
(7) 67·44 lb.
(8) 2·96 tons per square inch.
(9) 1,696 lb.

CHAPTER XIX.

GRAPHICAL ALGEBRA.

318. Lines are measured in terms of some unit length, surfaces in terms of some unit area ; volumes in terms of some unit volume, hence all geometrical magnitudes may be expressed by numbers which give either exactly or approximately the measure of the magnitude in terms of the unit of measure.

319. **Use of Signs in Geometry.**—When we have points as A, B, C in a straight line, Fig. 278, the segment AB—reading from left to right—is supposed to be generated by a point moving from A to B. The segment BA by a point moving from B to A. All measurements from left to

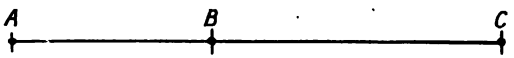


FIG. 278.

right are considered *positive*, thus AB, BC are both *positive:* and all measurements from right to left *negative;* thus BA, CB are both negative.

If we represent the segment between the points A, B by a, and that between points B, C by b, we have AB $= a$; BC $= b$; BA $= -a$; CB $= -b$.

Also AC $= a + b$; CA $= -b - a$.

Construction of Simple Algebraic expressions.

320. **Problem.** Fig. 279. *To construct the expression* $x = a + b$ *when a and b are* 4·5 *and* 3 *linear units respectively* (linear unit ¼ in.).

On a straight line OM set off from O towards M, OB $= a$ (4·5 units) and BC $= b$ (3 units). Then OC represents $a + b$.

321. **Problem.** Fig. 279. *To construct the expression* $x = a - b$

(*magnitudes as before*). Make OB = a, and from B lay off the distance BC = b in the direction BO. Then the difference OC = $a - b$.

322. **Problem.** Fig. 280. *To construct the expression* $x = a - b + c + d - e$. *When* $a = 4$, $b = 3$, $c = 2$, $d = 4{\cdot}5$, $e = 5$.

Write the expression

$$x = a + c + d - e - b.$$

On a straight line OM lay off OA = a, AB = c, BC = d. Then OC = $a + c + d$. Lay off from C toward O, CD = e, DE = b. Then OE = $a + c + d - e - b$.

In a similar manner we may construct any series of known linear magnitudes connected by the signs + and −.

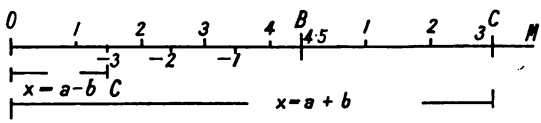


FIG. 279.

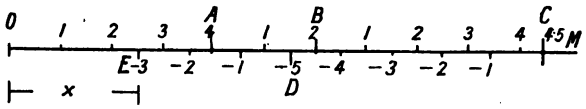


FIG. 280.

323. The expression ab denotes a surface and is represented by a rectangle having adjacent sides, respectively equal to a and b linear units.

If we wish to represent this magnitude by a straight line, we must reduce this rectangle to an equivalent rectangle having one side equal to the linear unit. If we denote the other side of the new rectangle by x we have $ab = x \times 1$ which forms the proportion $x : b :: a : 1$.

324. The expression abc denotes a volume and is represented by a rectangular prism having adjacent edges respectively equal to a, b, and c linear units.

If we wish to represent this magnitude by a straight line, we must reduce this prism to an equivalent prism having *two* edges each equal to the linear unit. Denoting the other edge by x we have $x : c :: ab : 1$.

Thus we see that in order to represent by a straight line

the result of multiplying two or more terms together, we must construct a line proportional to the given lengths.

325. **Problem.** *The three lines* (A, B, C) *represent the three terms* A, B, C, *of the proportion* A : B : C : X. *Determine* X.

Draw any two intersecting lines as OM, ON, Fig. 281. Set off

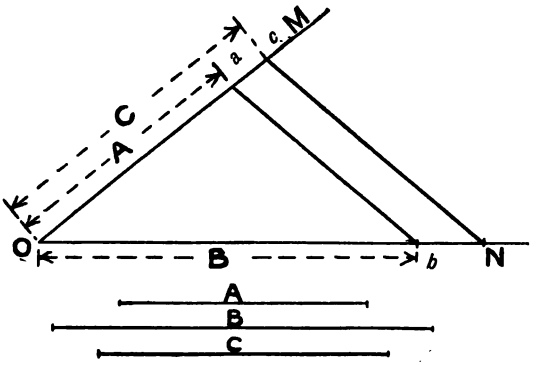


FIG. 281.

along OM from O the given lengths of A and C. Set off along ON from O the given length of B. Join ab, and draw $cN \parallel ab$ meeting ON in N.

Then, A : B : : C : ON

$\therefore$ ON is the required term X.

326. **Problem.** *To construct the expression* $x = \frac{bc}{a}$. This expression may be written $a : b :: c : x$, hence x is a fourth proportional to the three given lines a, b, c and corresponds with ON in Fig. 281.

327. **Problem.** No Fig. *Construct the expression* $x = \frac{a^2}{c}$. This may be written $x = \frac{a \times a}{c}$, thus x is a fourth proportional to the lines c, a, and a.

328. **Problem.** Fig. 282. *Construct the expression* $x = \frac{abc}{de}$. Write it in the form $x = \frac{ab}{d}, \frac{c}{e}$.

Let $m = \frac{ab}{d}$. The expression thus becomes $x = \frac{mc}{e}$; m is a fourth proportional to lines d, b, a, and is constructed as in Problem 325. It

is shown at OM. x is a fourth proportional to lines e, m, c, and is determined in a similar manner; it is shown at OX.

Note.—The magnitudes e, c have been set off along a line other than OD to prevent overcrowding.

In a similar manner the expression $x = \frac{a\,b\,\ldots\,n}{d\,e\,\ldots\,n}$ may be constructed.

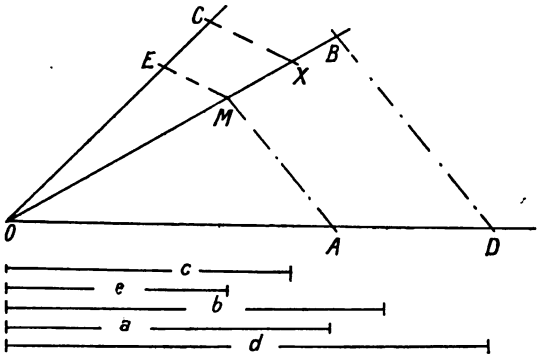


FIG. 282.

329. In a right-angled triangle, if a perpendicular be drawn from the right-angle on to the hypotenuse, it can be

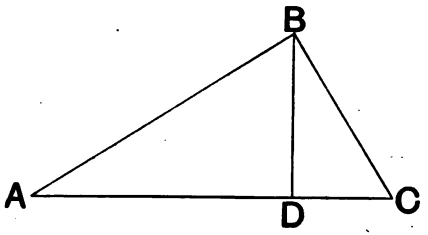


FIG. 283.

shown that, the perpendicular divides the figure into two triangles similar to the original and to each other.

From this proposition some useful facts can be deduced.

Let ABC, Fig. 283, be a triangle having a right-angle at B.
Let BD be the ⊥r on AC.

Then (i.) $\frac{AD}{DB} = \frac{DB}{DC}$ or $DB^2 = AD \cdot DC$

i.e., DB is a mean proportional between AD and DC.

(ii.) $\frac{AD}{AB} = \frac{AB}{AC}$ or $AB^2 = AD \cdot AC$

i.e., AB is a mean proportional between AD and AC.

(iii.) $\frac{DC}{BC} = \frac{BC}{AC}$ or $BC^2 = DC \cdot AC$

i.e., BC is a mean proportional between DC and AC.

(iv.) Since, $AB^2 = AD \cdot AC$, and $BC^2 = DC \cdot AC$

$$AB^2 + BC^2 = AC(AD + DC) = AC^2$$

i.e., if squares be constructed on the hypotenuse, and the two sides of a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the two sides.

We have thus available some useful constructions.

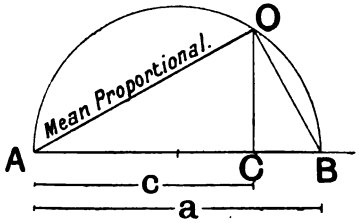


FIG. 284.

Ex. To find a mean proportional (b) *to two given straight lines* a *and* c, *i.e., find a line* b, *such that* $\frac{a}{b} = \frac{b}{c}$.

First method. Make AB, AC, Fig. 284, on a straight line equal to *a* and *c* respectively. Draw a semicircle on diameter AB. Draw CO perpendicular to AB to meet the semicircle in O. Draw AO, then triangles AOB, ACO are right-angled at O and C respectively, and have a common angle at A. The triangles, therefore, are similar and $\frac{AB}{AO} = \frac{AO}{AC}$ ∴ AO is the required mean proportional.

Second method. Set off *a*, *c*, Fig. 285, on opposite sides of A on a straight line. On BC describe a semicircle and draw AO perpendicular to BC, meeting the semicircle in O. Then AO is the required mean proportional; for in the right-angled triangles BAO, OAC,

angle AOB = 90° − angle AOC = angle ACO;

∴ △s BAO, OAC are similar,

and $\frac{AB}{AO} = \frac{AO}{AC}$.

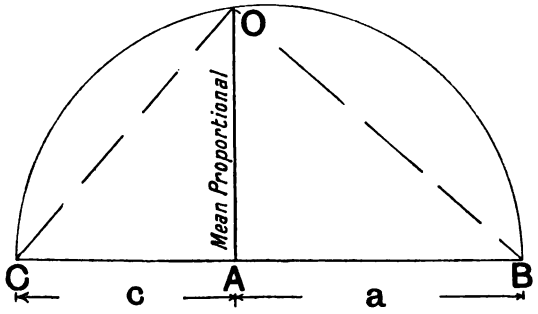


FIG. 285.

330. *To find (graphically) a third proportional to two given lines* a *and* b, *i.e., to find a line* c *such that* a : b : : b : c.

Make OA, OB′, Fig. 286, on a straight line OM equal to the given

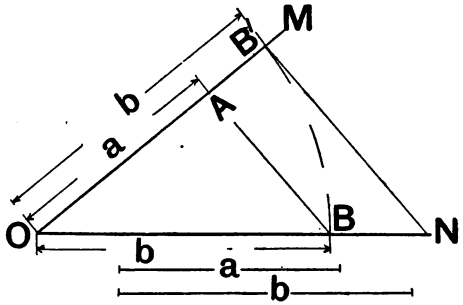


FIG. 286.

lines *a* and *b*. Draw any line ON, and make OB on this line equal OB′. Join AB, and draw B′N ∥ AB. Then ON is the line required.

$$\text{For, OB} = \text{OB}', \text{ and } \frac{\text{AO}}{\text{OB}} = \frac{\text{OB}'}{\text{ON}}.$$

331. If in the above figure we take OA = 1, we have

$$1 \times \text{ON} = \text{OB} \cdot \text{OB}', \text{ and since OB} = \text{OB}'$$
$$\therefore \text{ON} = \text{OB}^2.$$

Hence the construction may be employed to raise a line of given length, or a given number, to *any power*.

Ex. To find (graphically) the values of (i.) $(1{\cdot}7)^2$; (ii.) $(1{\cdot}7)^3$.

Draw two intersecting lines AB, AC, Fig. 287. Select some convenient unit—say $\frac{1}{2}$ in.—and make AE on AC = 1·7 units. Make AD on AB = 1 unit. Join DE; make A*e* on AB = AE; draw *e*F ∥ DE. Then

Since $\frac{AF}{Ae} = \frac{AE}{AD}$ or AF × AD = AE × A*e*; and AD = 1 we have

$$AF \times 1 = AE \times Ae$$
$$\therefore AF = AE^2 \quad \text{(since AE = A}e\text{)}$$
$$= (1{\cdot}7)^2.$$

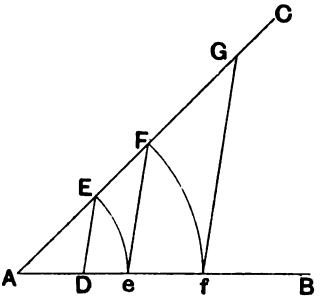


FIG. 287.

332. The **square root** of a given number may be obtained graphically by constructions based on the theorem of Art. 329.

Ex. Find graphically the square root of each of the following numbers 2, 3, 5. Fig. 288.

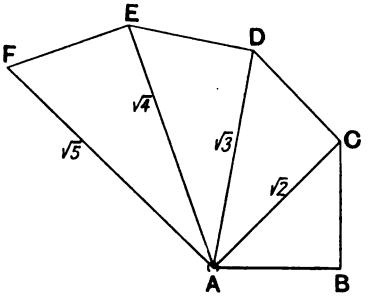


FIG. 288.

(i.) Select some convenient unit, say, 1 inch.

Draw AB = 1 unit. At B erect a $\perp^{r}$ BC = AB. Join AC.

Then since $AC^2 = AB^2 + BC^2$ and AB = BC = 1,

$\therefore$ AC = $\sqrt{2}$.

(ii.) At C erect CD perpendicular to AC and = BC.

Join AD, then AD = $\sqrt{AC^2 + CD^2} = \sqrt{3}$.

(iii.) At D erect DE perpendicular to AD and = DC.

Join AD, then AE = $\sqrt{4}$.

Proceeding in the same way, determine F.

Then AF = $\sqrt{5}$.

333. **Problem.** *To construct the expression* $x = \sqrt{ab}$. This equation forms the proportion $a : x :: x : b$. x is thus a mean proportional to lines a and b and is found as in Fig. 284 or 285.

334. **Problem.** (1) *To construct the expression* $x = \sqrt{a^2 + b^2}$. x is the hypotenuse of the right-angled triangle which has lines a and b for sides.

(2) The *value* x *in the expression* $x = \sqrt{a^2 - b^2}$ is one side of a right-angled triangle which has lines a and b for hypotenuse, and side.

335. **Problem.** Fig. 289. *Construct the expression* $x = \frac{a^3b^2c}{d^2 . e . f}$. If we write this, $x = \frac{a}{d} \times \frac{a \times a}{d} \times \frac{b \times b}{e} \times \frac{c}{f}$ and let $m = \frac{a \times a}{d}$, then m is the fourth proportional to lines d, a, a.

Thus from a point O draw lines OD, ON inclined at any angle. Make OD = d (to scale); make OA = a, also make $OA_1 = a$. Join DA and draw $A_1M \parallel DA$ meeting ON in M. Then OM = $m = \frac{a \times a}{d}$.

Let $n = \frac{b \times b}{e}$. It is then a fourth proportional to lines e, b, b.

Draw any line OB_1 from O and mark thereon OE = e, $OB_1 = b$, also make OB = e. Join EB and draw $B_1N \parallel EB$ meeting ON in N. Then ON = $n = \frac{b \times b}{e}$.

We now have $x = \frac{a}{d} \times \frac{m \times n}{1} \times \frac{c}{f}$ or $\frac{am}{d} \times \frac{nc}{f}$.

Let $p = \frac{am}{d}$. It is the fourth proportional to lines d, a, m.

Make OM_1 on OD = OM, draw $M_1P \parallel DA$. Then OP = $p = \frac{am}{d}$.

Let $q = \frac{nc}{f}$. It is the fourth proportional to lines f, n, c.

Make OF = f, and OC = c. Join FN and draw CQ $\parallel$ FN meeting ON in Q. Then OQ = $q = \frac{nc}{f}$.

We now have $x = \frac{pq}{1}$.

i.e., $1 : q :: p : x$, and x is constructed at OX.

336. **Problem.** *To construct the expression* $x = \sqrt{a^2 + b^2 - c^2 - d^2}$. Put $a^2 - c^2 = m^2$, and construct m as in Problem 334 (2). We shall then have $x = \sqrt{m^2 + b^2 - d^2}$. Put $b^2 - d^2 = p^2$, and construct p as in Problem 334 (2).

We now have $x = \sqrt{m^2 + p^2}$, which is constructed as in Problem 334 (1).

In a similar manner we may construct the expression $x = \sqrt{a^2 + b^2 - c^2 + d^2 - e^2 - f^2 +}$, etc.

337. **Problem.** *To construct the expression* $x = a^2 + ab$

Put $a^2 = d$. Then $1 : a :: a : d$. The line d is constructed as in Fig. 287.

Put $ab = e$. Then $1 : a :: b : e$. The line e is constructed as in Fig. 281.

338. **Problem.** *To construct the expression* $x = \sqrt{a^2 - bc}$. Put $bc = d^2$. Then d is a mean proportional to lines b and c. And we have $x = \sqrt{a^2 - d^2}$. The line n is constructed as in 334 (2).

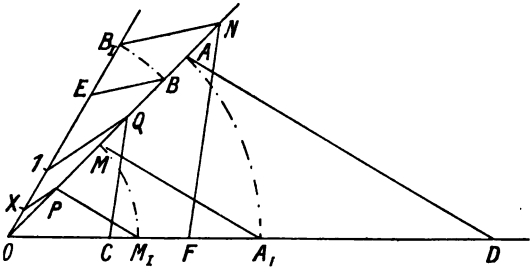


FIG. 289.

CHAPTER XX.

USE OF SQUARED PAPER.

339. If we wish to fix definitely a point in space so that at any future time we can again find the exact point, we must make certain measurements. Consider a small electric lamp in a room, as shown in the sketch, Fig. 290 ; suppose that this lamp is to be removed, and that the ceiling from which the lamp hangs is to be renewed, so that the point from which the wires are suspended will be obliterated. After the renewal of the ceiling we wish to replace the lamp in **exactly the same position** in the room.

Before removing the lamp we must measure (1) its distance from the ceiling, say 6 ft. ; (2) its perpendicular distance from the back wall, say 7 ft. ; (3) its perpendicular distance from the side wall, say 5 ft. The lamp may now be removed, and when we have to replace it, we first find the point on the ceiling from which the lamp hangs by measuring 7 ft. from the back wall and 5 ft. from the side wall. If we now attach the lamp to this point by a wire 6 ft. long, the lamp is back in its old position.

340. The student will see from this illustration that *in order to fix a point in space we require to know its distance from each of three fixed planes* (the two walls and the ceiling). If we knew only two of these distances, say the two distances from the walls, we could not again find the exact position of the lamp, although we could readily find the point on the ceiling from which the lamp hangs. Thus, *to fix the position of a point in a plane* (the ceiling) *we require its distance from each of two fixed lines in the plane* (the two lines in which the walls meet the ceiling).

341. Now consider a sheet of drawing-paper. We can fix

the position of any point thereon when we know its distance from two edges of the paper, say the left-hand edge and the bottom. Again, consider a sheet of squared paper ruled

FIG. 290.

with squares of $\frac{1}{10}$-in. side. If we know that a point P is 30 divisions from the left OY, Fig. 291, and 40 divisions above the bottom OX, we can at once locate the position of the point by counting 30 divisions horizontally along OX, and then counting 40 divisions vertically to the point P.

This method of locating the point P is called **plotting a point.**

The two edges or fixed lines OX and OY, from which the measurements are made, are called the **Axes of Reference.** Their point of intersection O is called the **Origin,** and the two measurements 30 and 40 which locate the point P are called **co-ordinates.** The point P is referred to as the point whose co-ordinates are 30, 40, and this is denoted by

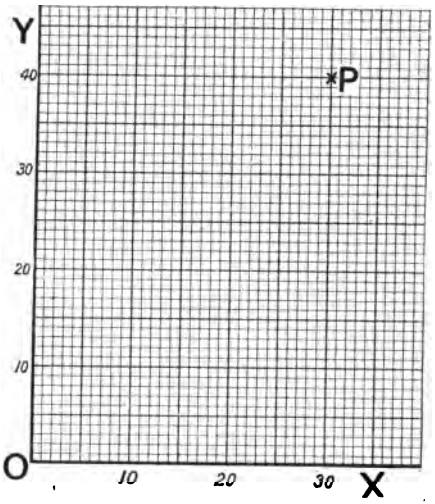


FIG. 291.

$P_{(30, 40)}$; the co-ordinate which is given first is always the distance measured horizontally or parallel to the axis of reference OX, and that given second is the vertical measurement parallel to the axis of reference OY.

The horizontal axis OX is often called the *axis of* x, and the measurements parallel to this axis are called **abscissæ** or x *co-ordinates.* Similarly, the vertical axis is often called the *axis of* y, and the measurements parallel to this axis are called **ordinates** or y *co-ordinates.*

In general language, $P_{(x, y)}$ is the point whose co-ordinates are x and y respectively.

342. If we take the two axes as shown in Fig. 292 along the two centre lines of the paper, and we require a point 20 divisions from the horizontal axis and 30 divisions from the vertical axis, each of the four points P, Q, R, S will agree with these two dimensions. There is thus an ambiguity as to which of the four points is the one required.

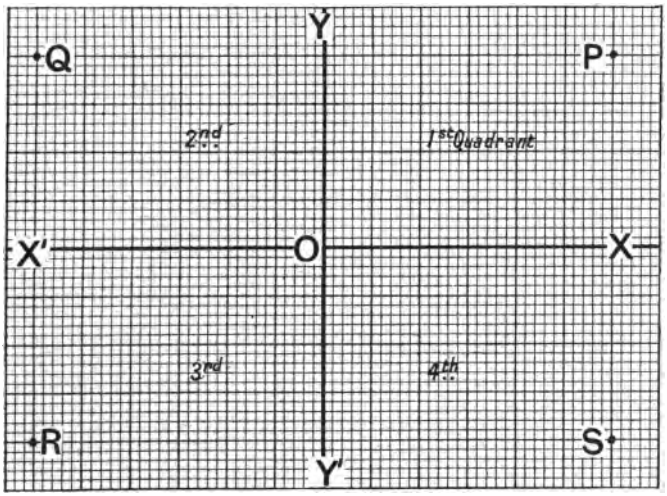


FIG. 292.

In order to remove this ambiguity, we call those *distances measured upwards* from the horizontal axis, and those measured to the *right of the vertical axis, positive;* while those *measured downwards* from the horizontal axis and those measured *to the left of the vertical axis* are *negative.* All distances are measured from one or other of the two axes, which are now lettered XOX′ and YOY′, thus dividing the paper into four quadrants, called the 1st, 2nd, 3rd, and 4th quadrants respectively as shown in Fig. 292.

The distances or co-ordinates of P are thus + 30, + 20.
" " Q " − 30, + 20.
" " R " − 30, − 20.
" " S " + 30, − 20.

In each case, the horizontal distance or x co-ordinate is given first.

The line or curve which joins up a series of plotted points is called a **graph.**

The following examples should be carefully plotted by the student and the result of each example should be particularly noted :—

Ex. 1. Plot the points (5, 7), (− 5, 7), (− 5, − 7), (5, − 7), and find the area enclosed by the four lines joining the points.

Ex. 2. Plot the points (5, 2), (5, 4), (5, 6), (5, − 2), (5, − 4), (5, − 6), and note that they all lie on a line parallel to, and distant 5 units from, YOY'.

Ex. 3. Plot the points (− 6, 3), (− 4, 3), (− 2, 3), (0, 3), (2, 3), (6, 3), and note that they all lie on a line parallel to, and distant 3 units from, XOX'.

Ex. 4. Plot the points (0, 2), (0, 4), (0, − 2), (0, − 4), also the points (3, 0), (1, 0), (− 2, 0), (− 4, 0), and note that all these points lie on one or other of the two axes of reference.

Ex. 5. Plot the points (4, 4), (2, 2), (0, 0), (− 2, − 2), (− 4, − 4), and note that all these points lie on a line passing through the origin and inclined at 45° to the axis XOX'.

Ex. 6. Plot the points (4, 0), (2, 3·5), (0, 4), (− 2, 3·5), (− 4, 0), (− 2, − 3·5), (0, − 4), (2, − 3·5), and note that a circle with centre at the origin and radius equal to 4 units practically passes through all these points.

Ex. 7. Plot the points (4, 0), (2, 2·6), (0, 3), (− 2, 2·6), (− 4, 0), (− 2, − 2·6), (0, − 3), (2, − 2·6). Observe that all the plotted points lie on an ellipse.

343. Graphs are commonly used to show the relationship between two sets of connected quantities which may, or may not, be of the same *kind*, and may, or may not, differ much in *magnitude*. In the latter case, the scales for ordinates and abscissæ are not alike, *e.g.*, in Fig. 293, the scale for abscissæ is five times as large as that for ordinates. If we know that a graph when plotted will be a straight line, and have data to plot two points on the line, we can at once determine the graph.

For example, corresponding values of inches and centimetres give a straight line graph.

We know that 1 in. is equal to 2·54 cm., hence 10 in. = 25·4 cm., and we also know that 0 in. = 0 cm.; thus, in order to draw a graph showing the relationship between inches and centimetres, we plot the point P, Fig. 293, the co-ordinates of which are 10 and 25·4, and join this point to the origin, for the co-ordinates of the origin are 0, 0. From this graph, we can readily convert inches into centimetres, and *vice versâ*.

For example, 5 in. = 12·7 cm.
also 20 cm. = 7·9 in.

Exercises.

The student should draw the following graphs and keep them for

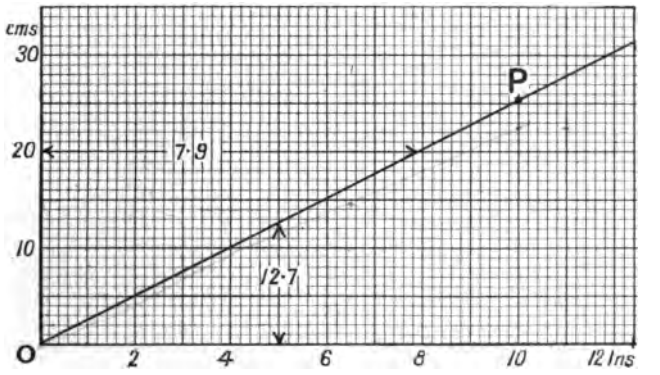


FIG. 293.

reference in his work. They are all straight lines, and can be found by plotting one point on each and joining to the origin.

(1) 100 lb. = 45·36 kilos. Plot a graph from which corresponding values of lbs. and kilograms may be read off, and find from the curve the number of kilograms equivalent to 65 lb.

Answer, 29·5.

(2) A pint of water at 4° C. is equal to 0·57 litres. Plot a graph from which corresponding values of pints, quarts, or gallons and litres of water can be read off.

(3) 1 litre of water = 61·025 cub. in. Plot a graph showing corresponding values of litres and cubic inches.

(4) 1 radian = 57·3 degrees. Plot a graph showing corresponding values of radians and degrees.

(5) The circumference of a circle = 3·1416 times the diameter. Show this relationship by a graph which will read for all circles up to 25 in. diameter.

(6) 33,000 ft.-lbs. (mechanical units of work) = 746 watts (electrical units). Draw a graph for watts and ft.-lbs.

(7)

1 cub. ft. of water weighs		62·3 lb.
1 cub. in. of cast iron weighs		0·26 lb.
„	wrought iron weighs	0·28 lb.
„	steel „	0·29 lb.
„	brass „	0·301 lb.
„	aluminium „	0·092 lb.
„	copper „	0·319 lb.
„	lead „	0·414 lb.
„	gold „	0·69 lb.
„	silver „	0·38 lb.
„	tin „	0·27 lb.

In each of the above cases, draw a graph showing corresponding weights and volumes.

(8) A knot (nautical mile) is 6,080 ft. Plot a graph showing the relation between miles and knots. What velocity in miles per hour corresponds to 27·5 knots per hour ?

(9) 1 mile = 80 chains, and 1 chain = 22 yds. Draw graphs for miles and chains, and yards and chains. Convert 175 chains into miles, and 85 yds. into chains.

344. **Graphical method of solving a simple equation.** If we take *any simple equation whatever* and place all the terms (whether they contain the unknown quantity or not) on one side of the equation, then the other side of the equation is = 0.

Thus, $x - 5 = 3x + 2$.

Transposing x we get $\quad -5 = 3x - x + 2$.

Transposing -5 (*i.e.*, add 5 to each side) we get $\quad -5 + 5 = 3x - x + 2 + 5$

or $0 = 2x + 7$.

Now put y in place of 0, and we get $y = 2x + 7$. If in this equation we put $x = 0, 1, 2, 3, 4$, etc., we can calculate a series of corresponding values of y.

Thus, when $x = 0$, $\quad y = 2x + 7$
$= (2 \times 0) + 7$
$= 0 + 7$
$= 7$.

Again, when $x = 1$, $\quad y = (2 \times 1) + 7$
$= 2 + 7$
$= 9$.

Repeating this process we get the Table shown below :—

x	0	1	2	3	4	5
y	7	9	11	13	15	17

If we now plot these values of y and x on squared paper, putting values of y vertically and values of x horizontally, as shown in Fig. 294, we get a **graph which is a straight line.** By producing this graph until it crosses the horizontal axis of reference XOX′ on either side of the origin, we

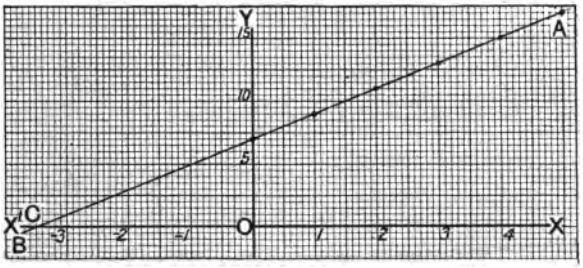


FIG. 294.

can find *the value of x at the point at which the graph intersects the axis, and this value is the solution of the equation.* It must be remembered that if the graph cuts the horizontal axis on the right of the origin, the value of x is positive, and if on the left it is negative.

Thus, in the above example, the graph AB cuts the axis at a point C, $3\frac{1}{2}$ divisions to the left of the origin, and hence $x = -3\frac{1}{2}$ is the solution of the equation. By solving the equation algebraically, we also get $x = -3\frac{1}{2}$.

In solving the equation $y = 2x + 7$ graphically, we have found that when the graph crosses the axis XOX′, *i.e.*, when $y = 0$, the corresponding value of x gives the required solution of the equation. Now, our original equation reduced to $0 = 2x + 7$, and in solving this by the algebraic method, we transpose the terms thus :—

$$2x = -7$$

hence $x = -3\frac{1}{2}$.

If in the equation $y = 2x + 7$, we put $x = 3\frac{1}{2}$, then

$$y = (-3\frac{1}{2} \times 2) + 7$$
$$= -7 + 7$$
$$= 0.$$

Thus, the value of x which makes the value of $y = 0$ corresponds to the value of x which is the solution of the equation, and hence the above rule for solving simple equations graphically.

Ex. *Solve* $5x + 10 = 3x + 22$ *graphically.*

We have $5x - 3x + 10 - 22 = 0$,

or $2x - 12 = 0$.

Put $y = 2x - 12$.

Our Table is now :—

x	0	8
y	-12	4

and the graph is DE, Fig. 295.

This cuts the axis at F at which $x = 6$.

Hence $x = 6$ is the solution required.

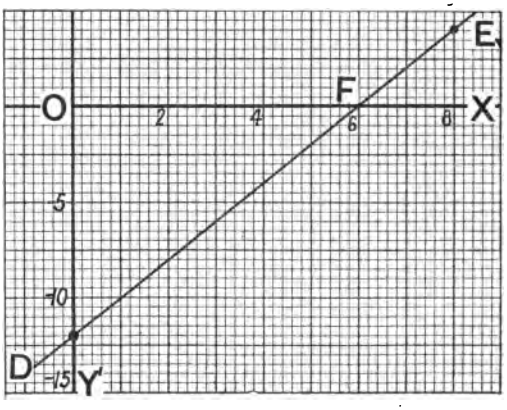


FIG. 295.

345. Simple equations are often called **linear equations,** by reason of the fact stated above—that if we take *any simple equation whatever* and plot it on squared paper,

we get a *graph which is a straight line.* This graph is called the **graph of the equation.** As the value of y depends upon that of x, y is said to be a **function of x**, and the graph is often called the **graph of the function.**

As we give to x any value we choose in calculating the table of corresponding values of x and y from which we finally plot the graph, the quantity x is called the **independent variable,** whereas, since the value of y depends upon that chosen for x, the quantity y is called the **dependent variable.**

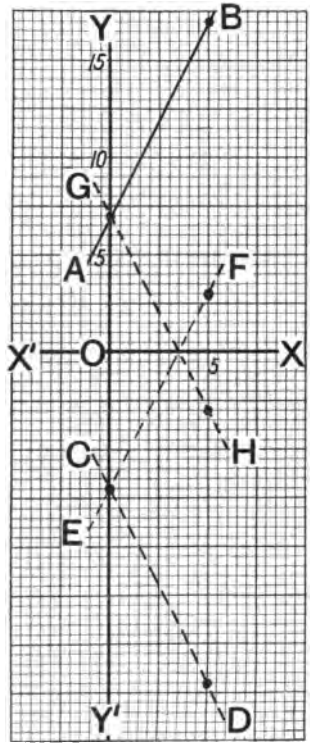


FIG. 296.

346. **General equation of a straight line.** Consider the equation $x - 5 = 3x + 2$, used in Art. 344. We transformed this equation to $0 = 2x + 7$, and then put $y = 2x + 7$. Now if we do this with a number of simple equations, we shall find that the coefficient of x (in this example 2) may be any number whatever, integral or fractional, and further it may be $+$ or $-$. Similarly, the number added (in this example 7) may have any magnitude and may be $+$ or $-$. These two numbers are called **constants,** and are denoted respectively by m and c. Hence, the equation in its most general form may be written $y = \pm mx \pm c$, which reads "y is equal to plus or minus mx plus or minus c," and means that y may equal (1) $+ mx + c$, or (2) $- mx + c$, or (3) $+ mx - c$, or

(4) $-mx - c$. This is the **general equation** of a straight line, and will represent *all* straight lines. The equation of *any particular* straight line is found when we know the numerical value and sign of both m and c.

Thus, when m and c are both positive we have $y = 2x + 7$.
" " " " " negative " $y = -2x - 7$.
" " m is positive and c negative we have $y = 2x - 7$.
" " m is negative and c positive " $y = -2x + 7$.

If we calculate a table for each line, as in Art. 344, and plot on squared paper, we get the four lines AB, CD, EF, GH, Fig. 296, and we observe the following facts:—

(1) When m is + (lines AB and EF) the line is inclined to OX at an angle less than 90°.

(2) When m is − (lines CD and GH) the inclination exceeds 90°.

(3) When c is + (lines AB and GH) the line cuts the axis YOY' above O, and when negative (lines CD and EF) it cuts below O.

Ex. 1. *Plot the graphs of the equations* (1) $y = x$; (2) $y = x + 3$; (3) $y = x - 3$.

The Tables are as follows:—

(1)

x	0	1	2	−2	−1
y	0	1	2	−2	−1

(2)

x	0	1	2	−2	−1
y	3	4	5	+1	+2

(3)

x	0	1	2	−2	−1
y	−3	−2	−1	−5	−4

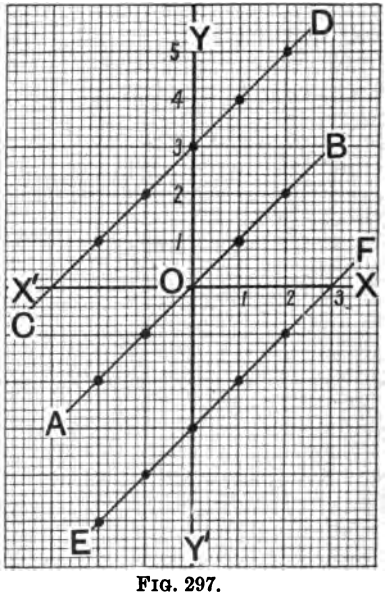


FIG. 297.

In Fig. 297, AB is graph (1), CD graph (2), and EF graph (3). The student will note from this example that when $m = +1$ and

$c = 0$, then $y = x$, and the graph passes through the origin and is inclined at 45°.

When $m = +1$ and $c = +3$, then $y = x + 3$, and the graph is still inclined at 45°, but it cuts the axis of y at the point $+3$.

When $m = +1$ and $c = -3$, then $y = x - 3$, and the graph is still inclined at 45°, but it cuts the axis of y at the point $y = -3$.

The distance from the origin to the point at which a line cuts either axis is called the **intercept** of the line on that axis.

Thus, the intercept of line AB on each axis is 0.
" " " CD on axis of y is $+3$.
" " " EF " y is -3.
" " " CD " x is -3.
" " " EF " x is $+3$.

Also note, the intercept on the axis of y is the value of y when $x = 0$, and the intercept on the axis of x is the value of x when $y = 0$.

Ex. 2. *Plot the graphs of* (1) $y = 1{\cdot}5x + 4$; (2) $y = -2x + 4$.

Tables are :—

(1)

x	0	4
y	4	10

(2)

x	0	2
y	4	0

The graphs are (1) AP, (2) PQ, Fig. 298.

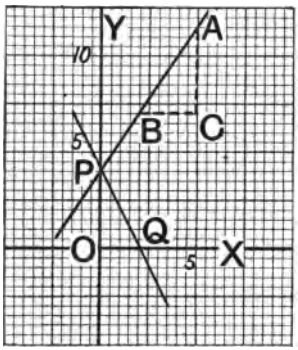


FIG. 298.

Note.—Since m is negative in the line PQ ($m = -2$), the line is inclined to the axis OX at an angle greater than 90°, and as m is positive in the line AP ($m = 1{\cdot}5$), the line is inclined at an angle less than 90°. When m is positive, y increases as x increases, and when m is negative, y decreases as x increases.

347. **Equation of a Straight Line in Various Forms.** The equation of a straight line in the form $y = \pm mx \pm c$ (Art. 346) is called the tangent form of the equation, for the constant m gives the slope of the line, and is equal to the tan. of the angle of inclination of the line to the axis of x, measured anti-clockwise, when the scales for y and x are equal.

The equation of a line may be obtained in other forms.

(1) *Equation of a line passing through two points whose co-ordinates are respectively* x_1, y_1 ; x_2, y_2. Fig. 299.

Let A be the point x_1, y_1,
B be the point x_2, y_2,
and C be any point having co-ordinates x, y.

The triangles ABF, ACE are similar.

$$\therefore \frac{CE}{BF} = \frac{AE}{AF}.$$

But $CE = y - y_1$, $AE = x - x_1$
$BF = y_2 - y_1$, $AF = x_2 - x_1$

$$\therefore \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the equation of the line passing through the given points.

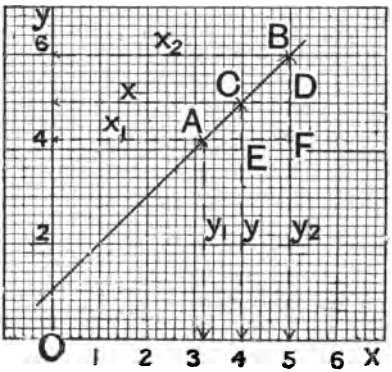


FIG. 299.

Ex. Let the co-ordinates of the points be A = 3, 4 ; B = 5, 7. Then $y_1 = 4$, $y_2 = 7$, $x_1 = 3$, $x_2 = 5$, and we have

$$\frac{y - 4}{7 - 4} = \frac{x - 3}{5 - 3},$$

$$\frac{y - 4}{3} = \frac{x - 3}{2}.$$

This can be transformed to the tangent form,

for $2(y - 4) = 3(x - 3)$
$2y - 8 = 3x - 9$
$2y = 3x - 1$
$y = 1{\cdot}5x - 0{\cdot}5.$

(2) *Equation of a line which cuts off intercepts c, b respectively from the axes of y and x.* Fig. 300.

Let the co-ordinates of any point D on the line be x, y.

Then $$\frac{DE}{AO} = \frac{EB}{BO}.$$

But DE $= y$, AO $= c$, EB $= (b - x)$, BO $= b$.

$\therefore \quad \frac{y}{c} = \frac{(b - x)}{b}$ is the required equation.

Simplifying, we have $$yb = bc - xc$$

Divide both sides by bc, and re-arrange $$\frac{x}{b} + \frac{y}{c} = 1 \quad . \quad . \quad (2)$$

(This equation can also be deduced from Equation (1) by substituting c, o for y_1, x_1 and o, b for y_2, x_2.)

The line AB (Fig. 300) is the line $\frac{x}{4} + \frac{y}{5} = 1$.

348. We have demonstrated that every simple equation may be written in the form $\mathbf{y = \pm mx \pm c}$, and also that every such equation corresponds to a graph which is a straight line. The converse of this is true, viz., **every graph which is a straight line corresponds to some simple or linear equation of the form $\mathbf{y = \pm mx \pm c}$**, and we shall now show *how the equation is to be found when the graph is given.*

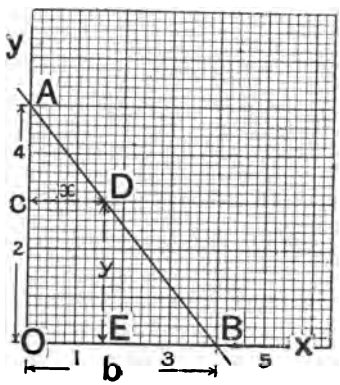


FIG. 300.

Consider first a particular case. In Ex. 2 of Art. 344, above, we plotted two graphs from their given equations. See Fig. 298. Graph (1) corresponds to the equation $y = 1{\cdot}5x + 4$.

The general equation for any straight-line graph is $y = \pm mx \pm c$. In this particular graph, m is $+ 1{\cdot}5$, and c is $+ 4$. If we read off the intercept, on the axis of y, we find that this is $+ 4$, and apparently corresponds to c in the general equation. Again, if we select any two points, as A and B, on the graph, and take the length AB as the hypotenuse of a right-angled triangle ABC, we find that BC = 3 divisions, and AC = 4·5 divisions, and hence that $\frac{AC}{BC} = \frac{4{\cdot}5}{3} = 1{\cdot}5$. This value apparently corresponds to m. If we repeat this investigation for graph (2) we again find the intercept on the axis of $y = 4$, and the value of m, taking the right-angled triangle OPQ formed by the axis OX and OY, is $\frac{OP}{OQ} = \frac{4}{2} = 2$, but as this graph is inclined at a greater angle than 90° to the axis XOX′, we must take the negative value for m, hence $m = -2$.

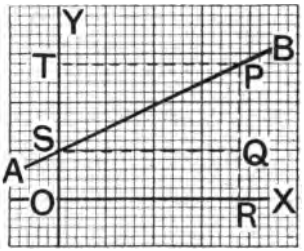


FIG. 301.

349. Now take a general case. We know that the co-ordinates of every point on the given line will satisfy the equation $y = \pm mx \pm c$. Suppose we take the co-ordinates of the point P, Fig. 301, as x and y; the ordinate y is the length PR, and the abscissa x is the length PT.

The intercept on the axis of Y is OS. We require to find the values of c and m in the general equation $y = \pm mx \pm c$, which represents any line, to make it into a particular equation for the particular line AB. As the line is inclined at less than 90° to XOX′, we take m as $+$. If we put $x = 0$ in the equation,

$y = mx \pm c$, we get
$y = 0 \pm c$
hence $y = \pm c$.

The value of y when $x = 0$ is the intercept OS; hence the

value of c is the intercept OS of the given line on the axis of Y, and for the line AB this is $+ c$.

Again, to find the value of m, we now have the equation

$$y = mx + c$$

$$\text{from which } m = \frac{y - c}{x}.$$

In Fig. 301, y = PR
c = OS = QR
hence $(y - c)$ = (PR − QR) = PQ.
Also x = TP = SQ
hence $\frac{y - c}{x} = \frac{PQ}{SQ}$, and this value is the required value of m.

As the ratios of the lengths of corresponding sides of similar triangles are equal, we can find the value of m by drawing any right-angled triangle having the given line as hypotenuse, and the two sides containing the right-angle parallel respectively to the axes of reference, and by calculating the value of the ratio $\frac{\text{length of side parallel to axis of Y}}{\text{length of side parallel to axis of X}}$. This value or ratio is sometimes called the "**slope of the line.**"

As an example, find the equation of the given line AB in Fig. 301. The equation is of the form

$$y = \pm mx \pm c.$$

The value of c is OS $= + 5$.

The value of m is the ratio $\frac{PQ}{QS} = \frac{9}{19} = 0{\cdot}47$, and it is positive, hence the equation of the line AB is

$$y = 0{\cdot}47x + 5.$$

This can be verified by plotting the graph of the equation $y = 0{\cdot}47\ x + 5$; it will be found to correspond to the given line AB. The student should also note that in all linear equations in which he has hitherto found the graphs that the coefficient of x corresponds to the value m, while the constant c corresponds to the intercept on the axis of y.

Ex. Find the equation of each of the given lines AL, CP, EU, *Fig.* 302.

(1) For line AL.

Form of equation is $y = \pm mx \pm c$.

c = intercept on axis OY = -3.

m = ratio $\frac{LM}{MN} = \frac{7}{3{\cdot}5} =$ 2, and is $+$.

Hence equation of AL is $y = 2x - 3$.

(2) For line CP.

Form of equation is $y = \pm mx \pm c$.

c = intercept on axis OY = 0.

m = ratio $\frac{PQ}{RQ} = \frac{3{\cdot}5}{5} =$ 0·7, and is $+$.

Hence equation of CP is $y = 0{\cdot}7x$.

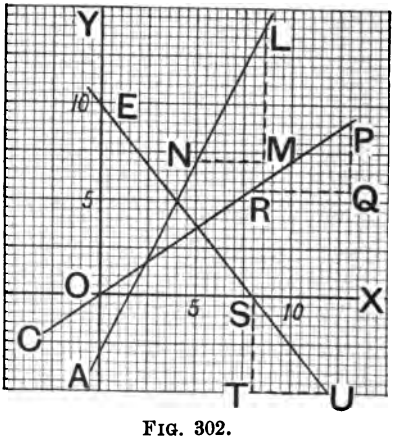


FIG. 302.

(3) For line EU.

Form of equation is $y = \pm mx \pm c$.

c = intercept on axis OY = $+10$.

m = ratio $\frac{ST}{TU} = \frac{5}{4} = 1{\cdot}25$, and is negative (since angle ESX between OX and the line EU exceeds 90°).

Hence equation of EU is $y = -1{\cdot}25\,x + 10$.

350. In the preceding examples we have plotted graphs in which the same scale is used for the ordinates and abscissæ. Now, in practice, the co-ordinates of points to be plotted may differ very considerably in magnitude, and hence we often adopt a different scale for ordinates from that used for abscissæ. As an example, consider a series of values of x and y as shown in the following Table :—

x	0·1	0·2	0·3	0·4	0·5	0·6
y	15	25	35	45	55	65

In this case, our greatest abscissa or x value is only 0·6, whereas the corresponding ordinate or y value is 65. In order to get all the points within an ordinary sheet of squared paper, we adopt the scales shown in Fig. 303, in

which for values of x each inch division is taken as 0·1, and for values of y each inch division is 20.

The equation to a graph in which different scales are used is, as before, of the form $y = \pm mx \pm c$, and also the value of c is the intercept on the axis OY, and the value of m is the ratio $\frac{LM}{MN}$, but we must observe that the value of c is the magnitude represented by the intercept on the scale adopted along OY,

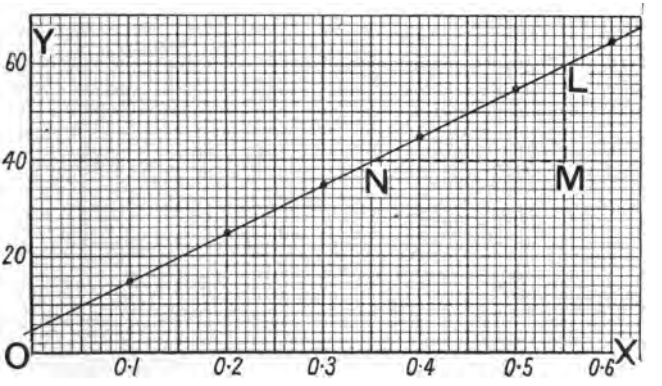


FIG. 303.

i.e., 5 (not 2·5, since each division = 2 units along OY). Similarly, in finding the value of m, the value of LM must be the magnitude represented by a length LM on the scale marked along OY, and the value of MN must be the magnitude represented by a length MN on the scale marked along OX.

Thus, LM = 10 divisions = 20 units on scale along OY.
MN = 20 „ = 0·2 „ „ „ OX.

Hence $\frac{LM}{MN} = \frac{20}{0·2} = 100$, and this value is the value of m.

The equation is thus $y = 100x + 5$.

351. When corresponding values of two variable quantities are obtained experimentally, the data obtained will not be free from error. When these corresponding

values are plotted on squared paper, the graph is obtained by drawing a line or curve to lie as evenly as possible among the points, passing through some, and leaving the others some on one side and some on the other side of the graph. With a little practice the student will readily determine which line or curve lies best within the plotted points. These graphs are extensively used in laboratory and workshop experiments.

By plotting on squared paper data obtained experi-

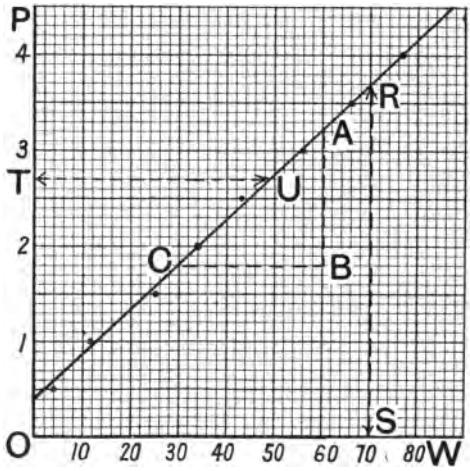


FIG. 304.

mentally, we not only obtain a graphical illustration of the way in which one quantity varies relatively to another, but we are also enabled (1) to correct errors in our experimental observations, (2) to ascertain approximately values within the range of our experiment which we have not actually observed, and (3) to predict approximately values which lie outside the range of our experiment. As an example, the force P in pounds necessary to lift a weight W in pounds with an experimental screw-jack is tabulated below, and the corresponding graph is shown in Fig. 304.

P lbs. .	0·5	1	1·5	2	2·5	3	3·5	4
W lbs. .	4	11·5	25	34	43	56	66	77

The values of P and W differ considerably in magnitude, so we adopt different scales. The vertical scale for P is 1 in. = 1 lb., hence each $\frac{1}{10}$ in. represents 0·1 lb.; whereas for W each inch = 20 lbs., hence each $\frac{1}{10}$ in. represents 2 lbs.

The values of P and W having been obtained experimentally, it is to be expected that the points will not all lie accurately upon a straight line; and it will be observed that where $W = 4$, 11·5, 25, and 43 the values of P are not on the graph, but the other values of P lie accurately along the graph.

By analogy to the equation $y = \pm mx \pm c$, the equation of this graph is $P = \pm m W \pm c$, and we must find numerical values for m and c from the graph.

The intercept c on the vertical axis is positive, and corresponds to 0·4 lb., and $m = \frac{AB}{BC} = \frac{1·42}{30} = 0·047$ and is positive. Hence the equation of the graph, or the **Law of the Machine**, is $P = 0·047\ W + 0·4$.

From this equation we can calculate values of P and W corresponding respectively to given values of W and P which lie outside the range of the graph—for example, we can find the force which will lift half a ton, or the load lifted by a force of 15 lbs.

Where $W = \frac{1}{2}$ ton or 1,120 lbs., substitute this value for W and we get

$$\begin{aligned} P &= 0·047\ W + 0·4 \\ &= (0·047 \times 1120) + 0·4 \\ &= 52·64 + 0·4 \\ &= 53·04 \text{ lbs.} \end{aligned}$$

This force lifts $\frac{1}{2}$ ton.

When $P = 15$ lbs., substitute this value for P and we get

$$15 = 0·047\ W + 0·4$$

$$\frac{15 - 0·4}{0·047} = W.$$

$310·6 = W$, hence a force of 15 lbs. lifts 310·6 lbs.

Values of P and W which do lie within the range of the graph may be obtained directly from the graph, or they

may be calculated from the equation ; thus, the force required to lift 70 lbs. is shown graphically by RS which = 3·69 lbs., and the load lifted by a force of 2·7 lbs. is shown graphically by TU which = 49 lbs. By calculation, if we put W = 90 lbs. in the equation we get $P = (0{\cdot}047 \times 70) + 0{\cdot}4$ = 3·69 lbs., and also by putting P = 2·7 we get $2{\cdot}7 = (0{\cdot}047 \times W) + 0{\cdot}4$ from which W = 49 lbs.

352. **Graphical solution of simultaneous equations.**

Ex. To solve graphically

$$6x + 9 = 3y \quad \text{(Eq. 1)}$$
$$4y = -12x + 40 \quad \text{(Eq. 2)}$$

we first arrange each equation to have y by itself on the left-hand side. Thus, $6x + 9 = 3y$ becomes $3y = 6x + 9$, and dividing by 3 becomes $y = 2x + 3$ (Eq. 3).

Also by dividing by 4, the equation $4y = -12x + 40$ becomes $y = -3x + 10$ (Eq. 4).

We now plot $y = 2x + 3$, and $y = -3x + 10$ on squared paper.

The Tables are :—

For $y = 2x + 3$ and graph AB.

x	0	3
y	3	9

For $y = -3x + 10$ and graph CD.

x	0	4
y	10	− 2

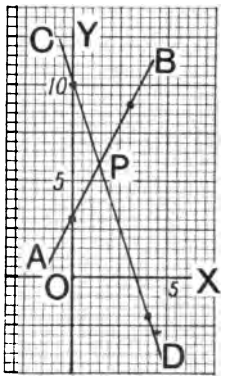


FIG. 305.

The two graphs intersect at P, and the co-ordinates of P are $x = 1{\cdot}4$, $y = 5{\cdot}8$.

Solving the two equations algebraically, we also find that $x = 1{\cdot}4$ and $y = 5{\cdot}8$; hence two simultaneous equations are solved graphically by plotting their graphs on squared paper, and reading from the graphs the co-ordinates of the point of

intersection. The x co-ordinate is the value of x, and the y co-ordinate is the value of y, which satisfies the two equations.

Exercises.

(1) Solve $3x + 4y = 18$, $5y - x = 13$.

(2) Solve $15x + 10y = 35$, $6x - 11y = -16$.

(3) Solve $\frac{1}{4}x + \frac{1}{2}y = 5$, $\frac{3}{8}x + \frac{7}{8}y = 8{\cdot}5$.

(4) Solve $2x + 10y = 3{\cdot}74$, $5x - y = 0{\cdot}25$.

(5) Two quantities are connected by a law $y = mx + c$. When $x = 1$, $y = 11$, and when $x = -1$, $y = 1$. Find the values of m and c, and the value of y when x is 30.

Answers.

(1) $x = 2$, $y = 3$.

(2) $x = 1$, $y = 2$.

(3) $x = 4$, $y = 8$.

(4) $x = 0{\cdot}12$, $y = 0{\cdot}35$.

(5) $m = 5$, $c = 6$; 156.

353. Two variable quantities may be related in such a manner that when corresponding values are plotted on squared paper, the graph obtained by joining the points is

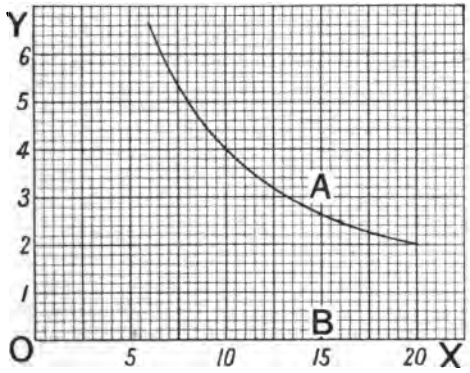


FIG. 306.

not a straight line. A value of one variable corresponding to a given value of the other variable can always be obtained (within the limits of the curve) by direct measurement from the graph.

As an example, plot the following values of x and y and find the value of y when $x = 15$:—

x	6	8	10	12	14	16	18	20
y	6·6	5	4	3·3	2·8	2·5	2·2	2

From curve, Fig. 306, it is seen that when $x = 15$, $y = AB = 2{\cdot}66$.

As an exercise, the student should plot a curve from which the cube roots of numbers can be read, taking the numbers as abscissæ and the cube roots as ordinates, as shown in the Table below :—

x	0	1	5	8	10	20	27	40	50	64	70	80	90	100
$y = \sqrt[3]{x}$	0	1	1·71	2	2·15	2·71	3	3·42	3·68	4	4·12	4·31	4·48	4·64

From the graph, read off the cube root of 2·7, 12·69, and 77·4. Note that the values of x are not equidistant.

As a practical example, the following corresponding values of the number of pounds of water passing over a weir, 2 in. wide, per minute, and the height of the surface of the water (head of water) above the weir were found experimentally :—

Head of water in mm. . . .	26·1	35·44	44·22	52·43	58·1	67·2	75·1
Lbs. of water discharged per min. .	52·9	79·33	109·4	140·5	161·25	199·2	235

In plotting these values on squared paper, we plot the *head* horizontally and the *discharge* vertically, as in Fig. 307, and we commence the vertical scale at 40, as our smallest ordinate is 52·9. Let each inch or 10 divisions horizontally represent a head of 20 mm., and each inch or 10 divisions vertically represent a discharge of 40 lbs. per minute. Each division horizontally then represents a head of 2 mm., and each division vertically represents a discharge of 4 lbs., per minute. Now draw an even curve through as many as

possible of the points. Any points lying off the curve are those in which errors have been made in observation or measurement during the experiment. From this curve we can obtain by measurement the discharge for a head of, say, 65 lbs., lying within the range of the experiment, by reading

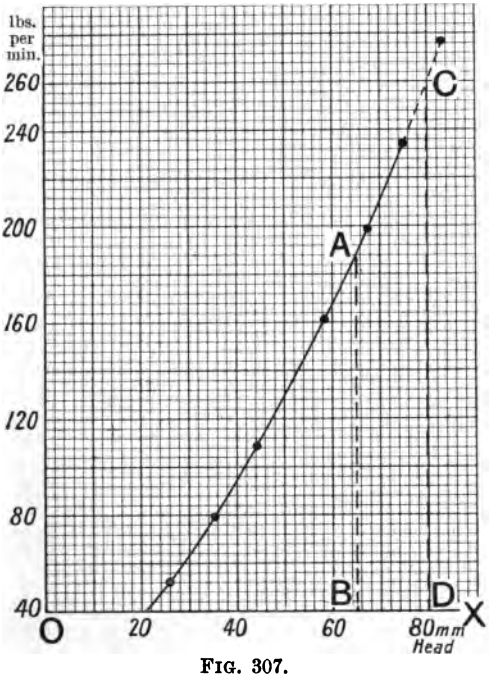


FIG. 307.

off the value AB, which is seen to be 188 lbs. per minute. We can also estimate the probable discharge for a head of, say, 80 mm., lying outside the range of the experiment, by producing the curve, as shown dotted, and reading off the value of CD, which is 260 lbs. per minute.

Exercise. The discharge in lbs. per minute through an orifice in the

bottom of a tank and the corresponding head of water in the tank are as follows (obtained experimentally) :—

Head in feet . . .	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0
Discharge per minute in lbs. . . .	12·1	18·4	22·5	25·5	28·0	30·5	33·0	35·0	36·2	39·0

Plot on squared paper, find from the curve the discharge for a head of 4·25 ft., and the probable discharge for a head of 6 ft.

Ans. 36; 42·2 lbs. per min.

In cases in which corresponding values of x and y, when plotted, do not give a straight line, it is frequently possible to obtain a straight line graph by plotting quantities obtained from x and y, thus when $y = a + bx^2$, we plot y and (x^2).*

Ex. The following corresponding values of x and y were measured. There may be errors of observation. Test if there is a probable law $y = a + bx^2$, and, if this is the case, what are the probable values of a and b. (B. E., 1911.)

x	1·00	1·50	2·00	2·30	2·50	2·70	2·80
y	0·77	1·05	1·50	1·77	2·03	2·25	2·42
x^2	1·00	2·25	4·00	5·29	6·25	7·29	7·84

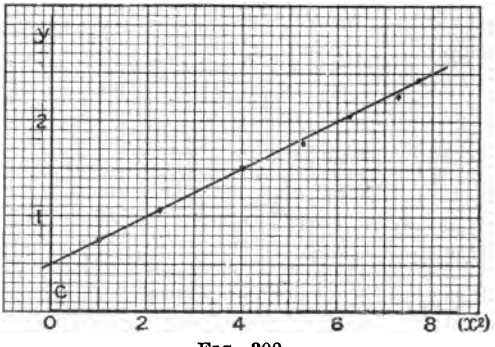


FIG. 308.

The equation is $y = m(x^2) + c$, (x^2) corresponds to x in the usual

* For a full discussion, see companion volume on Practical Mathematics.

linear equation, $m = \frac{2}{8} = 0{\cdot}25$; $c = 0{\cdot}5$, hence the law is $y = 0{\cdot}25\,x^2 + 0{\cdot}5$. Fig. 308.

The following suggestions may prove useful in suggesting quantities to give a straight line when plotted :—

(1) If $y = axy + b$, plot values of y and of (xy).
(2) If $y = \frac{a}{x} + b$ „ „ „ y and of $\frac{1}{x}$.
(3) If $y^2 = 4\,ax$ „ „ „ y^2 and x.
(4) If $y^2 = nx^2 + c$ „ „ „ y^2 and x^2.
(5) If $y = ax^n$ „ „ „ log y and log x.
(6) If $x = \frac{a}{y} + b$ „ „ „ $\frac{1}{y}$ and x.
(7) If $y = an^x$ „ „ „ log y and x.
(8) If $y^2 = b^2\left(1 \pm \frac{x^2}{a^2}\right)$ „ „ „ y^2 and x^2.

354. **Graphical solution of a quadratic equation.** In solving such equations graphically, we write all the terms on one side of the equation and put y on the other side, *e.g.*, $8x^2 - 2x = 15$ would be written $y = 8x^2 - 2x - 15$.

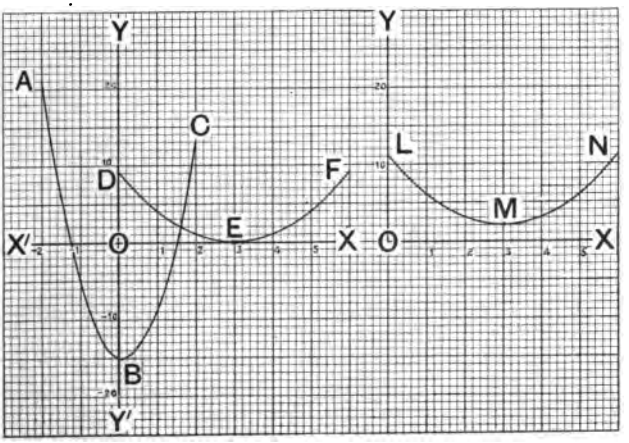


FIG. 309.

Now give to x a series of values both + and −, and calculate corresponding values of y, and then plot the x and y values on squared paper. The graph may—

(1) Cut the axis of x twice; *e.g.*, $8x^2 - 2x = 15$.
(2) Touch the axis of x; *e.g.*, $x^2 - 6x = -9$.
(3) Lie entirely above the axis; *e.g.*, $x^2 - 6x + 11 = 0$.

The first example gives two real roots $1\frac{1}{2}$ and $-1\frac{1}{4}$, and the x co-ordinates of the two points in which the curve crosses the axis give the two roots, just as in the case of the graphical solution of a simple equation, Art. 344.

The second case corresponds to two equal roots 3, the equal roots being given by the x co-ordinate of the point at which the curve touches the axis of x.

The third case corresponds to two imaginary roots, which cannot be found graphically.

Examples 1 and 2 above are solved graphically in Fig. 309, and the graph for Ex. 3 is also shown. The co-ordinates of the plotted points are shown in the Tables below :—

Graph ABC.

x	2	1·5	1	0·5	0	−0·5	−1	−1·5	−2
y	13	0	−9	−13·5	−15	−12	−5	6	21

Graph DEF.

x	5	4	3	2	1	0
y	4	1	0	1	4	9

Graph LMN.

x	0	1	2	3	4	5	6
y	11	6	3	2	3	6	11

355. A further, and in some cases a more convenient, graphical solution is as follows :—

The equation $8x^2 - 2x = 15$ may be written $8x^2 = 2x + 15$. If we plot a graph $y = 8x^2$ we have the parabola AOB (Fig. 310). If we plot the graph $y_1 = 2x + 15$, we have a straight line CD. At the points of intersection of the straight line and parabola, we have $y = y_1$, *i.e.*, the ordinate CE or DF for the line = the ordinate CE or DF for the parabola. Hence, $8x^2 = 2x + 15$ for the values of x corresponding to the points of intersection, and as the roots of the equation are the values of x which make

$8\ x^2 = 2\ x + 15$, the abscissæ, or x values, for the points C and D give the required roots of the equation. From Fig. 310, these values are seen to be $x = 1{\cdot}5$ or $-\ 1{\cdot}25$, which values agree with those found in Art. 354 and shown in Fig. 309.

If the roots are equal, as in the equation $x^2 - 6\ x + 9 = 0$, the line CD (Fig. 311), or $y = 6\ x - 9$, will be tangential to the parabola AOB, or $y = x^2$, at the point G at which

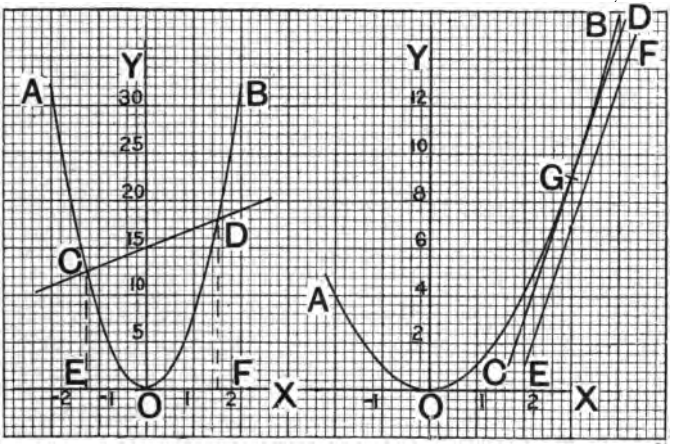


FIG. 310. FIG. 311.

$x = 3$, which value gives the root of the equation. Again, if the roots are imaginary, as in the equation $x^2 - 6\ x + 11 = 0$ of Art. 354, the line and parabola will not intersect. The line EF, or $y = 6\ x + 11$, is shown in Fig. 311, and it is seen that it does not touch the parabola AOB.

356. In cases in which two unknown quantities are to be found, we require two equations, called simultaneous equations. Such equations can be **solved graphically** by plotting the corresponding graphs, the points of intersection giving, by their co-ordinates, the values of x and y, which are the roots of the equations.

Ex. 1. *Solve*

$$x + y = 7.$$
$$xy = 10.$$

The graph $x + y = 7$ is a straight line. Written in the form $y = \pm mx \pm c$, it is $y = -x + 7$. In Fig. 312, the graph is AB.

The graph $xy = 10$ is the rectangular hyperbola CD. For negative values of x, we have a graph EF in the third quadrant, as corresponding values of y are negative. As the line AB does not enter the third quadrant, this part of the graph $xy = 10$ is not required. The co-ordinates of G and H are the roots of the equations.

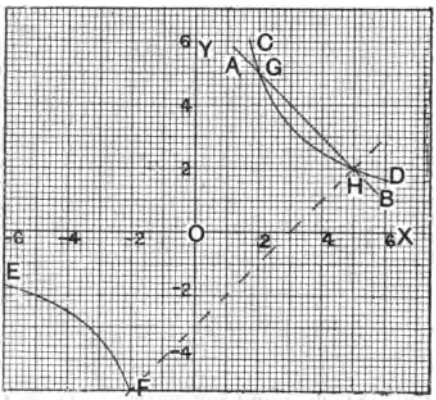


FIG. 312.

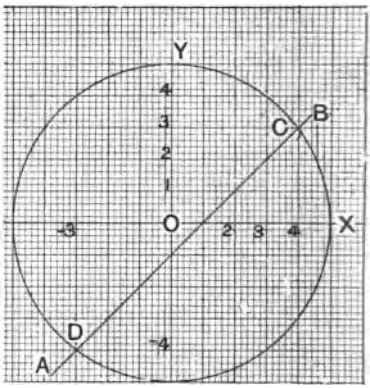


FIG. 313.

Exercise. Solve

$$x - y = 3$$
$$xy = 10.$$

Note.—Both parts CD and EF of the graph $xy = 10$ will be required. The dotted line FH (Fig. 312) is the graph $x - y = 3$. The roots of the equations are $x = 5$ or -2, $y = 2$ or -5. The points of intersection are F, H.

Ex. 2. *Solve*

$$x - y = 1.$$
$$x^2 + y^2 = 25.$$

The graph $x - y = 1$ is the straight line AB (Fig. 313).

The graph $x^2 + y^2 = 25$ is a circle, centre at O, and radius $= 5$. The co-ordinates of the points of intersection C, D of the line and circle give the roots of the equation. At C, $x = 4$, $y = 3$, and at D, $x = -3$, $y = -4$.

Exercise. Solve

$$x + y = 7$$
$$x^2 + y^2 = 25.$$

Note.—The line $x + y = 7$ intersects the circle $x^2 + y^2 = 25$ in points $x = 4, y = 3$ and $x = 3, y = 4$. It is not shown in Fig. 313.

Ex. 3. *Solve* $x^3 + y^3 = 28.$
$x + y = 4.$

To plot the graph $x^3 + y^3 = 28$, we have $y = \sqrt[3]{28 - x^3}$. From this we deduce the following facts, which the student should verify.

When $x = 0$, $y = \sqrt[3]{28} = 3{\cdot}036$
,, $x = 1$, $y = 3$
,, $x = 2$, $y = 2{\cdot}71$
,, $x = 3$, $y = 1$
,, $x = 3{\cdot}036$, $y = 0$.

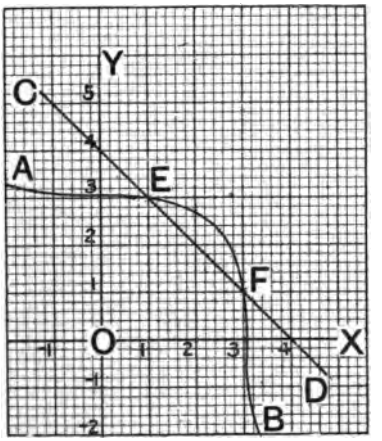


FIG. 314.

For negative values of x, y is + and is greater than 3·036, and increases as x increases in magnitude.

For positive values of x greater than 3·036, y is negative and increases as x increases. The graph is shown at A, B (Fig. 314). The graph $x + y = 4$ is the line GD. The co-ordinates of the points of intersection E, F are the roots of the equations.

Exercise. *Solve* $x^3 - y^3 = 26$
$x - y = 2.$

Note.—The graph $x^3 - y^3 = 26$ is similar to the graph AB, but inverted relatively to OX.

Ans. $x = 3$ or -1
$y = 1$ or -3.

Ex. 4. *Solve* $x^2 + y^2 = 25.$
$xy = 12.$

The graph $x^2 + y^2 = 25$ is a circle AB (Fig. 315), centre at origin O, and radius $= \sqrt{25} = 5.$

The graph $xy = 12$ is the rectangular hyperbola CD, or EF, for negative values of x give negative values of y.

The points of intersection are marked 1, 2, 3, 4.

The co-ordinates of (1) are
$x = 4, y = 3.$

The co-ordinates of (2) are
$x = -4, y = -3.$

The co-ordinates of (3) are
$x = 3, y = 4.$

The co-ordinates of (4) are
$x = -3, y = -4.$

Exercise. *Solve* $x^2 - y^2 = 16$
$xy = 15.$

Note.—The graph $x^2 - y^2 = 16$ comprises four branches, each similar in form to CD and EF and lying one in each quadrant.

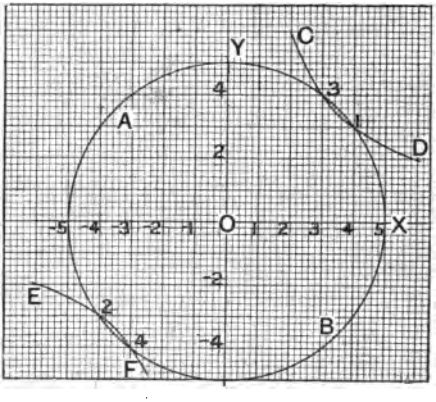


FIG. 315.

357. **Equation of the Circle.** Fig. 316. If we take the centre of a circle as the origin, and two diameters mutually perpendicular as axes of x and y, then if the co-ordinates of any point A on the circle are x and y, we have $OA^2 = x^2 + y^2$.

But OA is constant; call it a.

The equation of a circle is thus $x^2 + y^2 = a^2$.

Ex. $x^2 + y^2 = 16$ is the circle, Fig. 316, of radius $\sqrt{16}$ or 4.

If the centre of the circle is at a point whose co-ordinates are h, k (Fig. 317), and if the co-ordinates of any point A on the circle are x, y, then

$$\begin{aligned} AC &= AE - CE \\ &= AE - BD \\ &= y - k. \end{aligned}$$

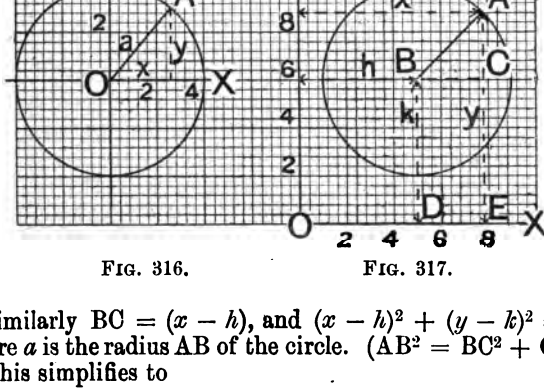


FIG. 316. FIG. 317.

Similarly BC $= (x - h)$, and $(x - h)^2 + (y - k)^2 = a^2$, where a is the radius AB of the circle. ($AB^2 = BC^2 + CA^2$.)

This simplifies to

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = a^2 \quad (1).$$

The general form of the equation to a circle is usually given as

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (2).$$

By comparing (1) with (2), it will be seen that $g = -h$, or $h = -g$. Similarly $k = -f$, and $c = h^2 + k^2 - a^2$.

The co-ordinates of the centre in the general equation are thus $-g$, $-f$, and the radius is $a = \sqrt{g^2 + f^2 - c}$.

Ex. The equation of a circle is $x^2 + y^2 - 6x - 4y - 3 = 0$. *Find its radius, and the co-ordinates of the centre.*

The general equation is $x^2 + y^2 + 2gx + 2fy + c = 0$.

Our equation is $x^2 + y^2 - 6x - 4y - 3 = 0$.

Hence $2g = -6, g = -3$
$2f = -4, f = -2$, and $c = -3$.
$a = \sqrt{g^2 + f^2 - c} = \sqrt{9 + 4 + 3}$
$= \sqrt{16}$
$= 4$.

∴ The co-ordinates of the centre are $-g$, $-f$, or 3, 2, and the radius is 4.

Examples of this kind may also be solved by making the left-hand side of the equation into the sum of two squares of the form $(x - h)^2$ and $(y - k)^2$, thus

$x^2 + y^2 - 6x - 4y - 3 = 0$ may be written
$(x^2 - 6x) + (y^2 - 4y) = 3$.

Make the expressions in brackets into complete squares.

$(x^2 - 6x + 9) + (y^2 - 4y + 4) = 3 + 9 + 4$
$(x - 3)^2 + (y - 2^2) = 16$

(9 + 4 is added to the right side to balance that added to left.)

By comparison with the equation $(x - h)^2 + (y - k)^2 = a^2$, it is readily seen that the co-ordinates of the centre are 3, 2, and the radius 4. This method is equivalent to shifting the axes of reference so that the origin is at the centre of the circle. All ordinates then become $(y - k)$, and abscissæ $(x - h)$, for the axis OY moves a distance k upwards, and OX moves a distance h to the right.

358. **Polar Co-ordinates.** Fig. 318. The polar co-ordinates of any point P are (1) *its distance* OP *from a fixed point* O, *called the pole or origin;* (2) *the angle* θ (XOP in Fig. 318) *which* OP *makes with a fixed line* OX (called the initial line) passing through the origin. The distance OP is called the **radius vector,** and is usually denoted by r. The angle θ is always measured counter-clockwise. Hence the polar co-ordinates of a point are often called the "$r\theta$ co-ordinates."

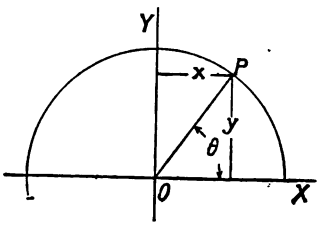


FIG. 318.

Any equation of a curve in rectangular co-ordinates may be readily transformed into one in polar co-ordinates, if

the initial line and axis of x coincide, by substituting $r \cdot \cos. \theta$ for x and $r \cdot \sin. \theta$ for y.

Thus, the equation of a circle in rectangular co-ordinates is $(x-h)^2 + (y-k)^2 = a^2$ when the radius is a, and the centre is at a point h, k. For a circle of radius a, with centre on the axis of y, we have $h = o$, and $k = a$. Its equation is thus $(x-o)^2 + (y-a)^2 = a^2$, in rectangular co-ordinates. By writing $x = r \cdot \cos. \theta$ and $y = r \cdot \sin. \theta$ we have the corresponding equation in polar co-ordinates, $(r \cdot \cos. \theta)^2 + (r \cdot \sin. \theta - a)^2 = a^2$. This simplifies to $r^2 \cdot (\cos.^2\theta + \sin.^2\theta) = 2r \cdot a \cdot \sin. \theta$.

$$r^2 = 2r \cdot a \cdot \sin. \theta$$
$$r = 2a \sin. \theta,$$

which is the polar equation of a circle of radius a.

Select any value for a, and calculate r for values of θ from 0° to 180°, and plot the resulting curve, thus satisfying yourself that it is a circle of radius a.

359. **Maximum and minimum values.** When a continuous graph is of such a shape that one value is greater

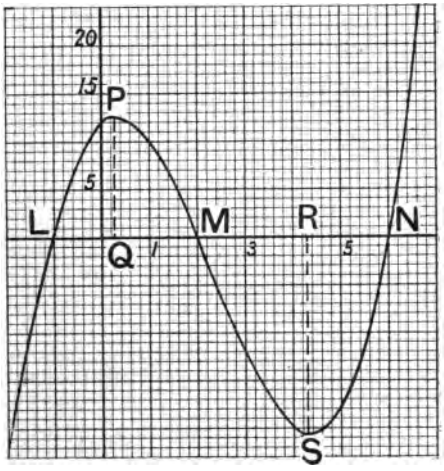


FIG. 319.

than any neighbouring values on either side of it, that value is said to be a *maximum* value, and a value which is less than any neighbouring value on either side of it is called a *minimum* value. In the graph, Fig. 319, the

ordinate PQ is a maximum, and the ordinate RS is a minimum, value. It must be observed that the above definition allows of more than one maximum or minimum value for any graph, and curves frequently occur in which there are several given values, *e.g.*, a sin. or cos. curve.

Exercises. (Arts. 354 and 355.)

Solve the following equations graphically :—

(1) $x^2 + x - 12 = 0$. *Ans.* -4 or $+3$.
(2) $x^2 - 6x + 9 = 0$. *Ans.* 3.
(3) $x^2 + 0{\cdot}5x - 3 = 0$. *Ans.* 1·5 or -2.
(4) $x^2 - 4 = 0$. *Ans.* 2 or -2.

Exercises (a). (Answers are on p. 332.)

Each of the following exercises is to be solved by plotting on squared paper.

(1) Plot points whose co-ordinates are respectively $(+2, +3), (+2, -3), (-2, -3), (-2, +3)$, and find the area of the figure obtained by joining the four points. Give the area in number of squares enclosed.

(2) Find the equation of the line joining the two points whose co-ordinates are $-2, 1$ and $6, 7$.

(3) By experiment and subsequent plotting upon squared paper, the law of a screw-jack is found to be $P = \frac{1}{75}W + 3{\cdot}5$; P being the lifting force and W the load lifted in lbs. Plot the probable graph represented by the above law. Find also the load lifted when $P = 20$ lbs.

(4) Plot the following values of P and W. They relate to a lifting appliance.

P (force in lbs.) . .	4	4·5	5	5·6	6·1	6·4	7·0	8·2	9·0
W (load lifted in lbs.) .	20	30	40	50	60	70	80	100	120

Find the value of P when W is 1 cwt.

(5) The following values of F and R were obtained experimentally. Plot a graph connecting F and R, and find its equation.

F	3·1	4·2	5	6·1	7·3	9	11·2
R	12	24	36	48	60	84	108

(6) The circumference c of a circle is proportional to the radius r. When $r = 3$, $c = 9{\cdot}42$, and when $r = 10$, $c = 31{\cdot}42$. Plot these two points. Join them by a straight line, and from this graph find the value of r when $c = 78{\cdot}54$.

(7) A spiral spring carries a marker moving over a graduated scale. When a load of 5 lbs. is on the spring the marker reads 5·3 in., and when the load is increased to 12 lbs. the marker reads 5·72 in. From this data, draw a straight-line graph which will represent the relation between the load and the scale reading. What does the marker read when there is no load on the spring, and when there is 25 lbs. ?

(8) The percentage efficiency of a Worthington steam pump at various speeds was determined experimentally as follows :—

Double strokes per minute .	10	20	30	40	50	60	70	80	90	100
Percentage efficiency .	25	40	48·5	55	59	63·5	66	68·5	71	72·5

Plot a graph, and find the probable efficiency when the number of double strokes per minute is 55.

(9) In a centrifugal pump experiment, the relation between the motor horse-power and the number of lbs. of water delivered per minute was found to be as follows :—

M.H.P. .	1·15	1·21	1·32	1·5	1·55	1·58
lbs. delivery	245	255	296	360	400	420

Find the probable delivery when the M.H.P. is 1·28.

(10) The relation between pressure P and temperature T of steam shown in the Table below was found experimentally. Find the value of T when P is 105, and the value of P when T is 300.

P lbs. per sq. inch .	5	10	15	20·5	27	31	36	44	50	60	70	80	90	100	110	120
T Fah. .	235	243	251	260	270	276	282	290	296	306	314	322	329	336	342	348

(11) The deflection D in inches at the centre of a beam, and the span L in inches are given below, the load being constant. Find the value of D for a span of 32 in.

L	15	20	25	30	35	40	44
D	0·025	0·075	0·165	0·275	0·43	0·65	0·84

(12) With an experimental beam, the deflection at the centre, and the load carried at the centre, are observed. Find the probable deflection for a load of 45 lbs.

Load in lbs.	0	4	8	12	16	20	24	28	32	36	40
Deflection in inches.	0	0·05	0·08	0·12	0·16	0·19	0·23	0·27	0·31	0·35	0·39

(13) A copper wire was loaded gradually until it broke, and the extension for each load was measured, with the following results. Find the probable extension for a load of 40·5 lbs.

Load in lbs.	0	5	10	15	20	25	30	32	35
Extension in inches.	0	·02	·03	·04	·06	·08	·29	·38	·82

Load in lbs.	37	40	41	42	43	44	45	46	47
Extension in inches.	1·32	2·43	2·97	3·64	4·24	4·95	6·05	7·7	wire broke

(14) With an experimental apparatus to illustrate the turning-effort on a crank-pin through 180° from the inner dead centre, the following readings were taken. Plot a curve to show this variation.

Degrees . .	10	20	30	40	50	60	70	80	90
Turning-effort .	2·5	4·25	6·75	8·5	9·75	10·5	11	10·55	10·5

Degrees . .	100	110	120	130	140	150	160	170
Turning-effort .	10·0	9·0	7·75	6·75	5·5	4·0	2·5	1·5

(15) The number N of coils in a spiral spring, and the number P of lbs. required to produce an extension of 1 in. are obtained experimentally. Find the probable value of P when N is 10 and also 13.

N	7	8	9	11	12
P	14·6	13·0	11·3	10·0	8·75

(16) Plot the following values, which were obtained experimentally, and find the twisting-moment required for a rod 0·275 in. in diameter :—

Diameter of rod in inches . . .	0·185	0·25	0·312	0·344	0·375
Twisting-moment to produce a twist of 1 radian . .	3·75	12·1	28·6	42·0	56·0

(17) The number N of double strokes per minute of a pump, and the number W of lbs. of water delivered by the pump are found experimentally.

N	26	34	58	72	83	92	100	104	154
W	6·5	10	20·25	25·25	29	33	35·5	37	43·25

Find the probable delivery when $N = 50$.

(18) Plot the graphs of $3y = 4{\cdot}8x + 0{\cdot}9$ and $y = 2{\cdot}24 - 0{\cdot}7x$. What are the co-ordinates of the point where they cross? Measure the angle each line makes with the axis of x and the angle between the graphs. (B.E. 1903.)

(19) The following values represent the areas of cross section of a body perpendicular to its axis :—

A sq. in. . .	250	292	310	273	215	180	135	120
x in. from end	0	22	41	70	84	102	130	145

Plot A and x on squared paper. What is the probable cross section 50 in. from the end? What is the average cross section and the whole volume? (B.E. 1903.)

(20) A series of soundings taken across a river channel is given by the following table, x feet being distance from one shore, and y feet the corresponding depth. Draw the section. Find its area. (B.E. 1904.)

x	0	10	16	23	30	38	43	50	55	60	70	75	80
y	5	10	13	14	15	16	14	12	8	6	4	3	0

(21) A feed pump of variable stroke driven by an electromotor at constant speed; the following experimental results were obtained :—

Electrical horse-power .	3·12	4·5	7·5	10·74
Power given to water .	1·19	2·21	4·26	6·44

Plot on squared paper, and state the probable electrical power when the power given to the water was 5. (B.E. 1905.)

(22) Mr. Scott Russell found that at the following speeds of a cana boat the tow-rope pull was as follows :—

Speed in miles per hour	6·19	7·57	8·52	9·04
Tow-rope pull in pounds	250	500	400	280

What was the probable pull when the speed was 8 miles per hour? There was reason to believe that the pull was at its maximum at 8 miles per hour, because this was the natural speed of a long wave in that canal. (B.E. 1905.)

(23) In the following table, A is the area in square feet of the horizontal section of a ship at the level of the surface of the water when the vertical draught of the ship is h ft. When the draught changes from 17·5 to 18·5 ft., what is the increased displacement of the vessel in cubic feet? (B.E. 1906.)

h	15	18	21
A	6020	6660	8250

(24) The speed of a ship in knots (nautical miles per hour) has been noted at the following times :—

Speed .	11·23	12·56	13·50	14·11	14·53
O'clock .	4	5	6	7	8

Plot on squared paper. What is the distance passed through during the hour after 6 o'clock? (B.E. 1906.)

(25) The following corresponding values of x and y are given in a Table :—

x	1·22	1·37	1·50
y	5·88	8·32	9·71

What is the probable value of x when y is 8? (B.E. 1906.)

(26) The following results were obtained by experiment of water discharged through an orifice :—

Duration of experiment.	15 min.	15	15	15	15	15	10	10
Actual discharge . .	576 lbs.	660	733	827	915	1011	737	738
Head of water . .	1·5 in.	2·0	2·5	3·27	4·01	5·0	6·0	7·0

Plot a curve showing the relation between discharge in lbs. per minute and head in inches. Determine also the discharge in gallons per hour when the head is $5\frac{1}{2}$ inches. (B.E. 1907.)

(27) The following values of x and y are thought probably to fulfil the law $y = a + bx$. Try if this is so. Find the most probable values of a and b. What is the probable error of each value of y? (B.E. 1907.)

x	10	20	30	40	50	60	70	80
y	180	303	492	603	807	919	1131	1200

(28) The area of the horizontal section of a reservoir is A sq. ft. at the height h ft. from its lowest point. When h is 30, what volume of water is in the reservoir? (B.E. 1907.)

h	0	2·5	5	7·5	10	12·5	15	17·5	20	22·5	25	27·5	30
A	0	2510	3860	4670	5160	5490	5810	6210	6890	7680	8270	8620	8780

(29) Plot the following values of x and y on squared paper. What is the average value of y between $x = 0$ and $x = 50$? (B.E. 1907.)

x	5	16	26	35	50
y	13	22	24	27	31

(30) Plot the following values of x and y on squared paper. They are thought to follow the law $y = a + bx$. If so, find a and b. What is the value of x when y is 50 and the value of y when x is 17·5?

x	3	6	9	12	15	18	21
y	10	16	22·5	28·5	34	40·5	46·5

(31) A straight line passes through points plotted on squared paper; it is $y = a + bx$.

If one of the points is $y = 2$, $x = 1$, and if another of the points is $y = 6$, $x = 2{\cdot}5$, find a and b. What is the value of y when $x = 1{\cdot}753$? What is the value of x when y is $5{\cdot}3$? (B.E. 1909.)

(32) In a table of values of x and y I find the following entries:—

x	10	11	12	13
y	·1115	·1128	·1150	·1202

What is the probable value of y when x is $11{\cdot}5$? What is the probable value of x when y is ·1165? (B.E. 1909.)

(33) The cross section of a tree (A sq. in.) at distance x in. from one end is as follows:—

x	10	30	50	70	90	110	130	150
A	120	123	129	129	131	135	142	156

What is the volume of the tree in cubic inches and in cubic feet, its total length being 160 in.? (B.E. 1909.)

(34) A man finds that if he uses x horses, his daily expenditure in pounds is

$$0{\cdot}3x + \frac{7{\cdot}3}{1 + x}.$$

Calculate this for various values of x and find what value of x makes his daily expenditure a minimum. (B.E. 1909.)

(35) x is distance in chains measured along a straight line AB from the point A, the values of y are offsets or distances in chains measured at right angles to AB to the border of a field. Draw the shape of this border. Find the area in square chains between the first and last offset and the straight line and border. Notice that the intervals in x are not equal. (B.E. 1908.)

x	0	1·50	3·00	5·00	7·50	9·00
y	0·53	0·27	0·46	0·42	0·35	0·52

Answers (a).

(1) 24.
(2) $y = 0{\cdot}75x + 2{\cdot}5$.
(3) 1237·5 lbs.
(4) 8·6 lbs.
(5) $F = \frac{1}{12}R + 2$.
(6) 25.
(7) 5; 6·5 in.
(8) 61·5.

(9) 274 lbs.
(10) 339 ; 55.
(11) 0·34 in.
(12) 0·43 in.
(13) 2·7 in.
(15) 10·5 ; 8·4.
(16) 16.
(17) 17.
(18) (0·8, 1·65) ; 58°, 145° ; 87°.
(19) 304 sq. in. ; 226·6 sq. in. ; 32,857 cub. in.
(20) 756 sq. ft.
(21) 8·5.
(22) 510.
(23) 6,660 cub. ft.
(24) 15·91 mls.
(25) 1·345.
(26) 423.
(27) 20 ; 15·4.
(28) 174,498 cub. ft.
(29) 22.
(30) 2 ; 4 ; 23 ; 39.
(31) $\frac{8}{3}$; − 0·6 ; 4·075 ; 2·27.
(32) 0·1138 ; 12·45.
(33) 20,448 cub. in.
(34) 4.
(35) 3·66 sq. chains.

Exercises (b).

(1) *A straight line passes through a point whose co-ordinates are* (4, 3), *and makes an angle of* 35° *with the axis of* x. *Calculate the value of* c *and find also the value of* m *in the equation* $y = \pm mx \pm c$ *for this line. Verify your result by plotting. Ans.* $c = 0{\cdot}2$; $m = 0{\cdot}7$.

(2) *A straight line passes through the points whose co-ordinates are, respectively,* (2, 7) *and* (6, 10). *Show that the value of* m *in the equation for this line is* 0·75, *and find the value of* c. *Ans.* 5·5.

(3) *The bending moment at any point distant* x *feet from the free end of a cantilever is* $\frac{w}{2}x^2$, *when* w *is the load in lbs. per foot run of the cantilever. Plot a curve from which the bending moment* M *at any point of a cantilever* 12 *ft. long can be determined, if* $w = 20$ *lbs.*

(4) *An arch across a roadway is shaped to the form of a semi-ellipse whose major axis is* 100 *ft.* (*the span of the arch*) *and minor axis is* 60 *ft.* (*the rise being thus* 30 *ft.*). *The equation for this ellipse is* $\frac{x^2}{2{,}500} + \frac{y^2}{900} = 1$, *when the origin is at the centre of the ellipse. Find by calculation, and by plotting, the height of the arch above a point* 20 *ft. from the centre of the road. Ans.* 27·48 *ft.*

CHAPTER XXI.

VECTORS.

360. In Mathematics, Mechanics, and Physics we have to deal with two distinct types of quantities. When we speak of 50 shillings, meaning a sum of money, we have fully specified the quantity we are speaking of, for we have specified the *kind of quantity*, viz., money, and we have specified the amount or magnitude of the quantity, viz., 50 shillings. If we move a body from any point to another point, say 50 ft. away, we say that the body has had a displacement of 50 ft. ; here we have stated the kind of quantity, viz., **displacement**, and also the amount, viz., 50 ft. ; but, before we can specify the point to which the body has been moved, we still require to state a further characteristic, viz., in what direction the displacement has been made. A quantity such as a displacement, then, is not fully specified until we know *quantity*, *kind*, and *direction*.

Quantities which are fully specified by kind and amount only and are independent of direction are called **Scalar Quantities**; examples are numbers, length, mass, time, temperature, etc.

Quantities which have a definite direction, and hence are fully specified only by kind, amount, and direction, are called **Vector Quantities***; examples are forces, displacements, velocities, accelerations, momentum, etc.

361. Both scalar and vector quantities can be represented geometrically by a line drawn to scale. The length of the line, in terms of the unit employed for the scale, represents in each case the amount of the quantity. If we are dealing with a scalar quantity, we may draw the line in any direction on the paper. If we are dealing with a vector quantity, we cannot draw the line in any direction we please, because the vector quantity has direction as well as magnitude, and

* Vector quantities are often referred to simply as vectors.

hence we must draw the line on the paper parallel to the given direction of the vector quantity.

Thus, a line 1 in. long drawn anywhere, *e.g.*, OA (Fig. 320), will represent 50 shillings money; if our scale is 1 in. = 50 shillings, but it will only represent, to the same scale, a displacement of 50 ft., provided the displacement is along the direction OA, *i.e.*, 45° to the horizontal reference line OX.

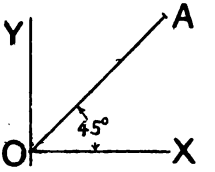


FIG. 320.

When a vector, *e.g.*, a displacement is represented by a line, either drawn on paper, or specified in space, we are still in doubt as to whether displacement is from O to A or from A to O, *i.e.*, we require to know which way along the given direction line the displacement takes place. This is usually called the **sense** of the vector quantity, and it is denoted by an arrow-head placed anywhere on the direction line. Other terms used for sense are **ort** and **clinure**.

Summarizing, we can represent fully a vector quantity by a line if

(1) The length represents to scale the amount of the quantity.

(2) The direction of the line corresponds to that of the vector quantity.

(3) The sense, ort, or clinure is shown on the line.

362. Some vector quantities require not only direction to be specified, but also *their actual position on the paper, or in space, e.g.*, a force is not completely specified until we know its magnitude, direction, sense, *and a point through which its direction passes.* A force is thus a particular kind of vector quantity, for it has a definite position as well as direction, and hence it is called a **localized vector** or a **rotor**.

363. A vector quantity may be specified numerically if we adopt a fixed line of reference and specify the angle, measured anti-clockwise, between the reference line and the direction of the vector quantity. The line of reference is usually taken as horizontal.

Let r units be the magnitude of the vector, and let θ be the angle its direction makes with the horizontal reference line OX (Fig. 321). Then the vector is specified numerically

as the vector r_θ, the symbol r denoting the magnitude, and the suffix θ the direction with the fixed reference line. The sense is understood as being away from O.

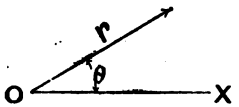


FIG. 321.

The displacement OA (Fig. 320), would be written as a displacement $50_{45°}$.

Similarly, to specify numerically a rotor or localized vector, we adopt a fixed line of reference, usually horizontal, and a fixed point O in that line (Fig. 322). Then, a rotor of magnitude r and inclination θ and passing through a point distant x from O measured along OX is written as ${}_x r_\theta$. The distance x is called an intercept, and is + or − according as it is to be measured to the right or left of O. As an example, a force of 50 lbs. whose direction is inclined at 30° to the horizontal, and which passes through a point 5 ft. from O, would be written as the force ${}_5{}^{50}{}_{30°}$.

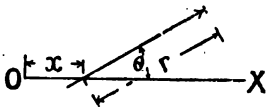
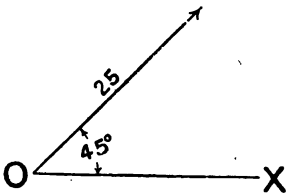
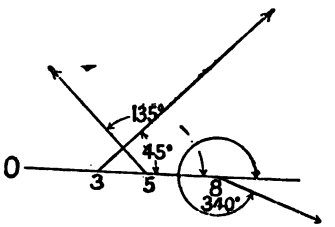


FIG. 322.



1″ = 20 f. s.
FIG. 323.



1″ = 40 lbs.
FIG. 324.

Ex. Represent graphically and numerically the following vectors—
(1) A velocity of 25 ft. per second in a direction North-east.
(2) Three forces (*a*) 50 lbs. acting North-east, (*b*) 30 lbs. acting North-west, and (*c*) 25 lbs. acting 20° South of East and passing through points 3, 5, and 8 ft. respectively from O.

Numerically.

(1) ${}^{25}{}_{45°}$ (2) ${}_3{}^{50}{}_{45°} + {}_5{}^{30}{}_{135°} + {}_8{}^{20}{}_{340°}$

Graphically, in Figs. 323 and 324.

364. **The sum of two vector quantities.** Consider a vector, say a displacement OA_3 (Fig. 325). If we divide it into three equal parts, OA_1, A_1A_2, A_2A_3, and consider each of these parts separately as a displacement, then the displacement OA_3 may be regarded as the sum of three separate displacements, $OA_1 + A_1A_2 + A_2A_3$.

Similarly, the sum of any number of parallel vectors of the same sense is a single vector, parallel to each of, and having the same sense as, the separate vectors, but having a magnitude equal to the sum of the magnitudes of the separate vectors. The vector sum is itself a vector joining the beginning of the first vector to the end of the last vector when all the vectors are placed end to end. If we regard

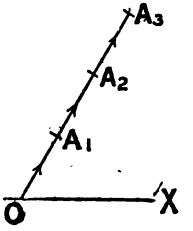


FIG. 325.

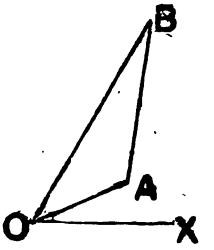


FIG. 326.

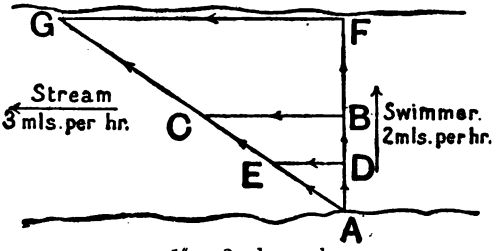
the separate vectors as a series of steps to be taken in order, then the vector sum is a single step equal in effect to the sum of the separate steps.

365. Now consider vectors which are not parallel, and consider each vector as a step. If we have two vectors, say displacements OA and AB (Fig. 326), then by taking the step OA we arrive at A, and by a further step AB we arrive at B; a single step OB, equivalent to the two separate steps, would be the step OB.

The vector OB is the **vector sum** of OA and AB, and may be defined as that vector which is obtained by placing the separate vectors end to end, with the beginning of the second in contact with the end of the first, and thenjoining the beginning of the first to the end of the second.

The vector OB = vector OA + vector AB.

This rule for the sum of two non-parallel vectors will be clear if we consider a swimmer crossing a wide river in which the stream is flowing at a certain velocity, say, 3 miles per hour. Let us assume that the swimmer can swim 2 miles per hour in still water, and that he heads straight across the stream. The swimmer's direction is AB (Fig. 327), and that of the stream is BC. Let AD be $\frac{1}{2}$ mile. Then in $\frac{1}{4}$ hour the swimmer would be at D, if swimming in still water. But, in the same time, the stream has carried him $\frac{3}{4}$ mile or a distance DE down the stream; hence, at the end of $\frac{1}{4}$ hour the swimmer is actually at E. If AB is 1 mile, the swimmer would arrive at B, in still water, in $\frac{1}{2}$ hour, but the stream carries him $1\frac{1}{2}$ miles, or a distance BC, hence his actual position is C. If we plot his position in this way for different intervals of time, and join up all



1″ = 2 mls. per hr.

FIG. 327.

points as E, C, etc., we find the actual path of the swimmer to be AG. Now find the path by the rule above for the sum of two vectors, one a displacement of 2 miles, and the second a displacement of 3 miles—perpendicular to the first. The vector sum has the direction AC.

Again, as the vector AF is the swimmer's velocity in still water in miles per hour, and as the vector FG is the velocity of the stream in miles per hour, the resultant velocity (Art. 367) is given in magnitude and direction by the vector AG.

Since AF = 2 and FG = 3, $\therefore$ AG = $\sqrt{4+9}$

= $\sqrt{13}$ miles per hour.

This result means that the actual velocity of the swimmer, relatively to the bed of the stream, or as seen by a person standing on the bank, is $\sqrt{13}$, or 3·6, miles per hour in the direction AG.

Similarly, the **sum of any number of vectors** is the vector obtained by placing the end of the first vector in contact with the beginning of the second, the end of the second in contact with the beginning of the third, and so on, and then joining the beginning of the first to the end of the last.

Thus, in Fig. 328

Vector OE = OA + AB + BC + CD + DE.

for OB = OA + AB.

OC = OB + BC = OA + AB + BC.

OD = OC + CD = OA + AB + BC + CD.

OE = OD + DE = OA + AB + BC + CD + DE.

366. It does not matter in what order we take the vectors, we obtain the same vector for the sum.

Thus, in the above example, Fig. 328 shows that

OE = OA + AB + BC + CD + DE,

and Fig. 329 shows that OE is also the sum, say, of

OA + BC + AB + DE + CD.

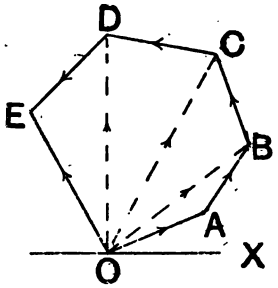


FIG. 328.

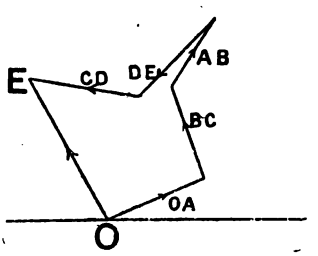


FIG. 329.

The above statement agrees with the corresponding algebraic law that

$$a + b + c + d + e = \text{say } (a + c + b + e + d),$$

or, the **order is immaterial in addition.**

367. The **vector sum** of a series of vectors is called the **resultant,** and the two or more vectors which collectively give the vector sum or resultant are called **components.**

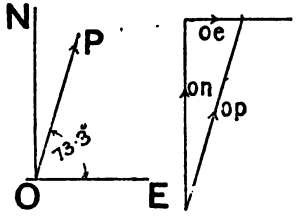
We may require to find the resultant of a given series of vectors, or we may be given a single vector and be required to find its components in any given directions.

The latter operation is called **resolving the given vector in the given directions.**

Ex. 1. *A ship steers due North at* 10 *miles per hour, and is moving in a current which travels East at* 3 *miles per hour. What is the actual velocity and direction of the ship?*

ON, Fig. 330, represents the direction in which the ship steers; OE the direction of the current. Then OP is the actual direction of

the ship. The direction OP is determined by drawing the vector diagram. The vector *on* is 10 units long and parallel to ON, the vector *oe* is placed at the end of the vector *on*, and is 3 units long and parallel to OE. The vector sum is then *op*, and by measurement this is 10·4 units long. The actual direction of the ship is OP parallel to *op*, and the actual velocity is 10·4 miles per hour.



1″ = 10 mls. per hr.
FIG. 330.

This result can be found by calculation, for *on* is ⊥r to *oe*, hence

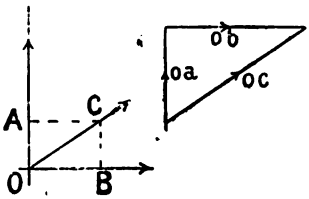
$$op = \sqrt{10^2 + 3^2} = \sqrt{109}$$
$$= 10{\cdot}44 \text{ miles per hour.}$$

The angle EOP is an angle whose tangent is $\frac{10}{3} = 3{\cdot}3333$. This angle is found from the Trig. Table to be 73·3 degrees.

Ex. II. *A swimmer wishes to cross a stream* 880 *yards wide, which is flowing at the rate of* 3 *miles per hour. He heads straight across the stream, and can swim* 2 *miles in* 1 *hour in still water. At what point will he land; what will be his actual velocity and path, and how long will he take to cross?*

Let OA, Fig. 331, be the direction across the stream, and OB the direction in which the stream flows. In the vector diagram, $oa = 2$ and is parallel to OA, $ob = 3$ and is parallel to OB; then *oc* gives the velocity and OC the direction of the swimmer.

$$oc = \sqrt{\tfrac{1}{4} + 9} = \sqrt{13}$$
$$= 3{\cdot}6 \text{ miles per hour.}$$



1″ = 4 mls. per hr.
FIG. 331.

The angle BOC = angle whose tangent is $\frac{2}{3}$ or 0·6666, *i.e.*, 33·6°.

If OA = $\frac{1}{4}$ mile (880 yards), then OB = $\frac{3}{4}$ mile, and OC = $\frac{\sqrt{13}}{4}$ miles = 0·9 miles. The swimmer thus lands $\frac{3}{4}$ mile down stream and relatively to the bed of the stream he swims 0·9 mile. He has to swim the equivalent of $\frac{1}{4}$ mile in still water, and he can swim 2 miles per hour. Hence the time taken is $\frac{1}{4}$ hour. We get the same result if we find the time taken to swim the path OC at the velocity *oc*, for

$$OC = \frac{\sqrt{13}}{4} \text{ miles.} \quad \text{Velocity } oc = \sqrt{13} \text{ miles per hour.}$$

$$\text{Hence time to swim OC} = \frac{\sqrt{13}}{4} \div \sqrt{13} = \tfrac{1}{4} \text{ hour}$$

Ex. III. *A cyclist travels due South at* 12 *miles per hour, and the wind actually blows from the South-west at* 5 *miles per hour. What is the apparent velocity and direction of the wind to the cyclist?*

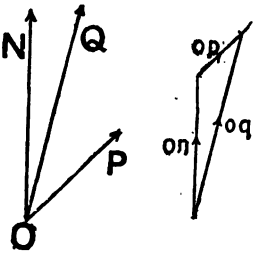
If a cyclist travels 12 miles per hour due South in a calm atmosphere, he will experience a wind resistance due to his own motion equivalent to that of a wind having a velocity of 12 miles per hour due North. Hence the cyclist in this example really experiences the resultant of two winds, one blowing due North at 12 miles per hour, and represented by ON, Fig. 332, and the other blowing from South-west to North-east at 3 miles per hour and represented by OP.

The vector diagram is thus $on = 12$ parallel to ON
$op = 5$ „ „ OP.

The vector sum is $oq = 15{\cdot}93$ mls. per hr.
The direction OQ is 13° East of North.
To calculate the magnitude and direction of OQ, we have

Angle between on and $op = 135°$.

$oq^2 = (on)^2 + (op)^2 - 2\,(on)\,(op)$ cos. 135°.
$= 12^2 + 5^2 - (2 \times 12 \times 5 \times -\cos. 45°)$.
$= 144 + 25 + (120 \times 0{\cdot}7071)$.
$= 144 + 25 + 84{\cdot}9$.
$= 253{\cdot}9$.
$oq = 15{\cdot}93$.



1″ = 16 mls. per hr.
FIG. 332.

The angle NOQ is given by

$$\frac{\text{sin. NOQ}}{\text{sin. 135}} = \frac{5}{15{\cdot}93}.$$

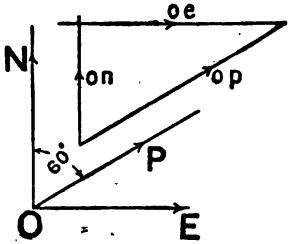
$\therefore$ Sin. NOQ = 0·2219,
and $\therefore$ angle NOQ = 13° (correct to the nearest degree).

Ex. IV. *An aeroplane is travelling* 30° *North of East at a velocity of* 50 *miles per hour. What are the components of this velocity in the directions North and East?*

Let on, oe, op, Fig. 333, be the vector diagram. We know that op, drawn parallel to OP must = 50. We require to find two vectors, on and oe parallel respectively to ON and OE, which will give op as their vector sum. Hence from the ends of op draw on, oe to intersect as shown. Then $on = 25$ is the northerly component and $oe = 43{\cdot}3$ is the easterly component.

To calculate on and oe we have

Angle NOP = 60°.
Hence $oe = op \times$ sin. 60°.
$= 50 \times 0{\cdot}866$.
$= 43{\cdot}3$ mls. per hr.
$on = op$ cos. 60.
$= 50 \times 0{\cdot}5$.
$= 25$ mls. per hr.



1″ = 40 mls. per hr.
FIG. 333.

368. To determine the **difference of two vector quantities.**

Let us consider, first, two parallel vectors, say, displacements. One body is moved 50 ft. from a given point O in the direction OA to the point A, Fig. 334. A second body is moved 30 ft. in the direction OA to B. The difference between the two displacements, or vectors, OA and OB is a displacement or vector BA, or that vector (BA) which must be added to the vector (OB) to be subtracted, in order to

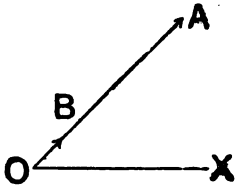


FIG. 334.

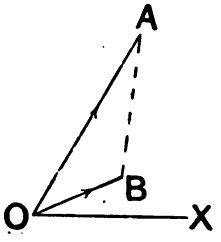


FIG. 335.

make the vector sum (OB + BA) equal to the given vector (OA).

Similarly, when OA and OB, Fig. 335, are not parallel. The difference between the vectors OA and OB is the vector BA, which is the vector to be added to the vector OB to make the sum = OA. Hence, to find the difference between any two vector quantities OA, OB, set out the two vectors from a common point O and join the ends B and A of the two vectors. This gives the vector BA, which must be added to the vector OB to give OA as the vector sum.

[The student should note that the vector BA = OA − OB, whereas the vector AB would = OB − OA for OB + BA = OA and OA + AB = OB.]

If we again consider the illustration of the swimmer in a stream (Art. 365), the difference between the vector AG (resultant velocity) and the vector AF (velocity of the swimmer) is the vector FG (or velocity of stream) necessary to make the path AG.

Ex. I. *Two ships at sea are travelling respectively* 25 *knots per hour North-east and* 20 *knots East, and their positions are such that they are approaching each other. What is their velocity of approach?*

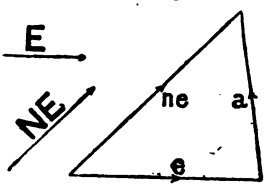
The directions of the ships are shown at E. and N.E., Fig. 336.

The vector diagram is

ne = 25 and parallel to N.E.

e = 20 and parallel to E.

The vector a = 17·8 is the vector difference of the two given velocities and hence gives the velocity with which the ships approach each other.

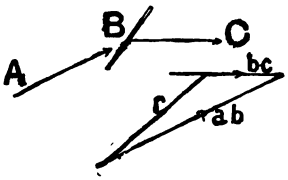


1″ = 20 knots.

FIG. 336.

Note.—The student will get a better idea of this kind of problem by assuming that the ships are travelling in the same direction at different speeds, say 25 and 20 knots. Their **velocity of approach** is 5 knots, or the vector which added to the velocity 20 gives the vector 25. The two steps 20 and 5 make up the single step 25. Similarly, in Fig. 336, the two steps e = 20 and a = 17·8 make up the single step ne = 25. Hence a is the velocity and direction of approach. If the ships do not ultimately collide, there is a certain time at which they are at a minimum distance apart; after that they separate with a velocity equal to their velocity of approach.

Ex. II. *A jet of water travels at a velocity of* 50 *ft. per second in a direction* AB, *Fig.* 337, *and impinges upon an inclined plate* B *which has a velocity of* 20 *ft. per second in the direction* BC. *Find the velocity of the water relatively to the plate, assuming that the jet impinges upon the plate without shock.*



1″ = 50 f.s.

FIG. 337.

The water is travelling faster than the plate and hence is approaching the plate and will ultimately impinge upon and run along it. The velocity of the water relatively to the plate is the velocity at which the water approaches the plate. The vector sum of the velocity of the water relative to the plate and the actual velocity of the plate must = the velocity ‖ AB.

Hence, find the vector difference of velocities ‖ AB and ‖ BC. This is r in vector diagram and = 33·5 ft. per second.

369. **Rotors.** A vector quantity which must occupy a definite position has already been referred to (Art. 362) as a localized vector or a rotor, and a force has been given as an example ofa rotor quantity.

The line representing a vector quantity other than a rotor may be moved anywhere parallel to itself. The line representing a rotor quantity must not be moved except along its own direction, *e.g.*, a displacement of 50 ft. in a direction West to East may be represented by the line AB (Fig. 338), or by CD, or by any line parallel to AB or CD, whereas a force of 50 lbs. acting from West to East and passing through O may only be represented by OF or FG or GH, or some segment along the line OX. We shall now confine our attention to those rotors which are forces, and we shall use the term "force" instead of the more general term "rotor quantity." It must be remembered, however, that the various rules for finding the vector sum and vector difference of a series of forces apply equally to other rotor quantities.

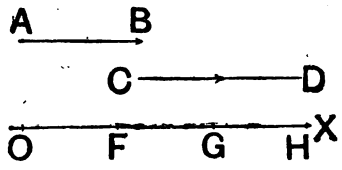


FIG. 338.

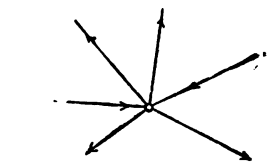


FIG. 339 (a).

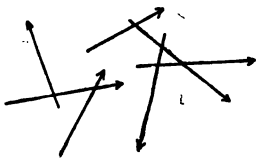


FIG. 339 (b).

370. A series of forces which act in the same plane are called **co-planar forces.**

A series of forces which act at the same point are called **concurrent forces,** whether they are co-planar or not.

We shall deal only with co-planar forces, and consider

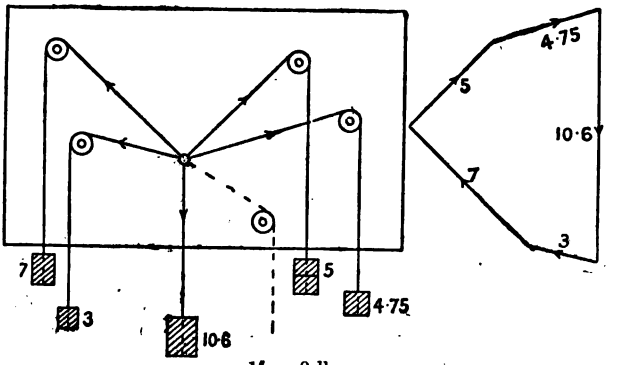
(1) Concurrent co-planar forces, Fig. 339 (a).

(2) Non-concurrent co-planar forces, Fig. 339 (b).

371. **Concurrent Forces.** If a number of forces act at a point, it can be shown experimentally that a single force,

acting at the same point, will produce equilibrium, *i.e.*, will prevent motion of the point due to the acting forces.

Take a drawing-board and four small pulleys, as shown in Fig. 340. To a small ring attach five strings, and pass four of the strings over the four pulleys, allowing the fifth string to hang freely. Now place weights, say of 3, 7, 5, and 4·75 lbs., on the strings, and by pulling on the fifth string in various directions, note that the small ring may be made to assume different positions. The magnitude of the force or pull in each string is equal to the weight attached to the end. The directions of the various forces exerted on the ring are indicated by the part of the string between the ring and the pulley over which the



1" = 8 lbs.

FIG. 340.

string passes, and each force or pull passes through the centre of the ring. The sense of each force or pull is away from the ring.

By pulling the fifth string vertically, move the ring to some definite position on the board, and secure it there by an ordinary pin. (If the pin is removed, the position of the ring will alter.) Now add to the fifth string a weight just sufficient to retain the ring in its position when the pin is removed. This weight or pull in the fifth string is a single force which balances the four forces 3, 4·75, 5, and 7 lbs. and it is called the "**equilibrant**" of the four forces. It also acts away from the ring.

We can determine the magnitude, direction, and sense of this equilibrant by means of a vector diagram, or as it is called in the case of forces, a **force polygon.**

From any point draw a vector equal and parallel to the force 3, from the end of the vector 3 draw a second vector equal and parallel to the force 7, and in turn vectors equal and parallel to the forces 5, 4·75. Now join the end of the vector 4·75 to beginning of vector 3.

This vector gives the magnitude and direction of the equilibrant, and in the present example it is vertical and equal to 10·6 lbs.

If the fifth string be now passed over a pulley, as shown in dotted lines, and a weight attached to its end, the position of the ring, and hence the directions of all pulls, will alter. Now draw a second force polygon as before, and note that the resultant vector is again equal to, and parallel to, the pull in the fifth string.

We conclude from the above experiment that, **when a series of forces act at a point and are in equilibrium (*i.e.*, the point does not move under the action of the forces) they can be represented in magnitude and direction by the sides of a closed polygon called the polygon of forces.**

We can regard any one of the series of forces as the equilibrant of the remainder of the series, since each force may be regarded as balancing the remainder. Again (as in a "tug of war," a pull along a rope in one direction will balance an exactly equal pull in the opposite direction), it will be seen that the four forces or pulls 3, 7, 5, 4·75 lbs. (Fig. 340) are equal in effect to a single vertical pull upwards of 10·6 lbs., for this pull would just balance the equilibrant pull of 10·6 lbs. downwards. The single force which is equal in effect to a series of forces is called the **resultant force**, and the separate forces of the series are called **component forces**. It will be also seen that the closing line of the force polygon gives the magnitude and direction of the resultant, while the sense of the resultant is opposite to that of the equilibrant.

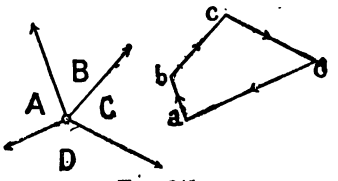


FIG. 341.

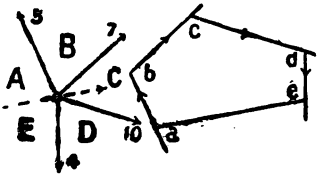
372. **Notation.**—It is usual in drawing force diagrams to denote each force by two letters placed on opposite sides of the force; thus the four forces shown in Fig. 341 would be called the forces AB, BC, CD, DA.

When we draw the force diagram, we letter the ends of the vector which is drawn parallel to the force AB as *ab*, that parallel to BC as *bc*, and so on, the capital letters denoting the forces in the one diagram, often called the **position**

diagram, and the corresponding small letters denoting the vectors in the vector diagram.

Ex. I. *Forces of* 5, 7, 10, *and* 4 *lbs. act at a point* O. *Find a single force* (*resultant*) *which has the same effect at the point* O, *and a force* (*equilibrant*) *to balance the four given forces.*

Let the forces 5, 7, 10, and 4 be denoted, as shown in the position diagram, Fig. 342, by letters AB, BC, CD, DE. Draw the vector diagram *abcde*. Then the vector *ae* gives the vector sum of the given forces, and hence is the resultant force. This resultant force is shown in dotted lines in the position diagram and it acts at O. If we reverse the sense, we have the vector *ea*, which is the equilibrant, and which would also act at O.



1″ = 16 lbs.

FIG. 342.

373. Fig. 343. If we have only three forces acting at a given point and they are in equilibrium, the vector diagram becomes a closed triangle, hence the following **theorem**, known as the **triangle of forces.**

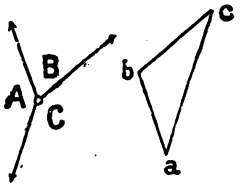


FIG. 343.

If three concurrent co-planar forces can be represented in magnitude and direction by the sides of a triangle, the forces are in equilibrium.

374. **Parallelogram of Forces.**—*The resultant or equilibrant of two concurrent forces is given in magnitude and direction by the diagonal of a parallelogram, the sides of which are equal and parallel respectively to the given forces.*

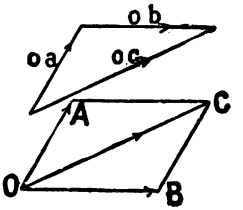


FIG. 344.

Thus if OA, OB, Fig. 344, represent to scale two forces acting at O, their resultant is given in magnitude and direction by the diagonal OC, and the equilibrant would be CO ; for if we draw the usual vector diagram to the same scale we have *oa* = OA *ob* = OB = AC, and hence *oc* = OC.

This construction is convenient, as it may be readily reversed to find the components of a given force. Thus, let OC, Fig. 345, be a given force. **To find components acting horizontally and vertically.**

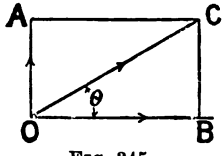


FIG. 345.

Set out OC to scale and draw OB and OA respectively horizontal and vertical.

Draw CA parallel to OB and CB parallel to OA.

Then OA and OB represent to scale two forces which have OC for their resultant and hence are the horizontal and vertical components of the force OC.

By calculation OB = OC cos. θ; OA = BC = OC sin. θ.

375. The resultant or equilibrant of *any number of forces acting at a point* can be found by repeated applications of

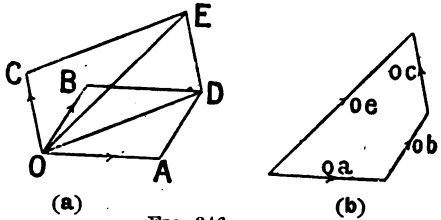


FIG. 346.

the parallelogram of forces. Fig. 346 (a) shows three forces, OA, OB, and OC.

$$OD = OA + OB$$
$$OE = OD + OC$$
$$= OA + OB + OC.$$

Fig. 346 (b) shows the vector diagram drawn to the same scale, and it will be seen that it gives the same result (oe = OE).

The resultant of two concurrent forces OA, OB, Fig. 347, can be calculated if the angle θ between the forces is known.

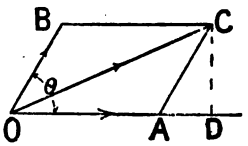


FIG. 347.

$$OC^2 = OD^2 + DC^2$$
$$OB = AC \text{ and angle } DAC = AOB = \theta$$
$$OD = OA + AD$$
$$= OA + AC \cdot \cos.\theta$$
$$= OA + OB \cdot \cos.\theta$$
$$\text{also, } DC = AC \sin.\theta = OB \sin.\theta$$
$$\therefore\ OC^2 = (OA + OB \cdot \cos.\theta)^2 + (OB \cdot \sin.\theta)^2$$
$$OC = \sqrt{(OA + OB \cdot \cos.\theta)^2 + (OB \cdot \sin.\theta)^2}.$$

376. The resultant or equilibrant of a number of forces acting at a point can be calculated by resolving all the forces

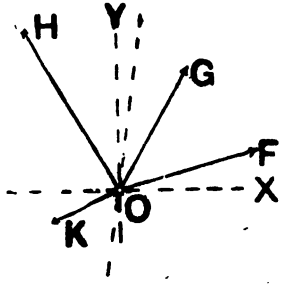
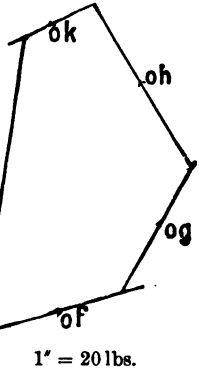


FIG. 348.

FIG. 349.



1″ = 20 lbs.

FIG. 350.

horizontally and vertically, finding a single horizontal force equal to all the horizontal components, a single vertical force equal to all the vertical components, and finally the resultant of the two forces thus found.

Ex. Let the given forces be OF, OG, OH, OK, = respectively to 15, 15, 20, and 8 lbs., *as shown in the diagram, Fig.* 348.

The horizontal components are

$$15 \cdot \cos. 15 + 15 \cos. 60 + 20 \cos. 120 + 8 \cos. 205.$$
$$= (15 \times 0{\cdot}966) + (15 \times 0{\cdot}5) + (20 \times -0{\cdot}5) + (8 \times -0{\cdot}9063).$$

The single horizontal force is thus

$$= 14{\cdot}49 + 7{\cdot}5 - 10 - 7{\cdot}25$$
$$= 4{\cdot}74 \text{ lbs.}$$ Call this OA.

The vertical components are

$$15 \sin. 15 + 15 \sin. 60 + 20 \sin. 120 + 8 \sin. 205$$
$$= (15 \times 0{\cdot}2588) + (15 \times 0{\cdot}866) + (20 \times 0{\cdot}866) + (8 \times -0{\cdot}4226).$$

The single vertical force is thus

$$3{\cdot}882 + 12{\cdot}99 + 17{\cdot}32 - 3{\cdot}38$$
$$= 30{\cdot}812 \text{ lbs.} \quad \text{Call this OB.}$$

The resultant of the two single forces is (Fig. 349)

$$OC = \sqrt{(OA)^2 + (OB)^2} = \sqrt{(4{\cdot}74)^2 + (30{\cdot}812)^2} = 31{\cdot}34 \text{ lbs.}$$

The direction θ which OC makes with OX is such that

$$\sin.\ \theta = \frac{AC}{OC} = \frac{OB}{OC} = \frac{30{\cdot}812}{31{\cdot}34}$$
$$= 0{\cdot}9832,$$
$$\text{hence } \theta = 79{\cdot}5 \text{ degrees.}$$

The result agrees with that found by the vector diagram shown in Fig. 350.

377. **Non-concurrent Forces.**—If a bar *oc*, Fig. 351, be pivoted to a pin at O, and weights P and Q be applied at the points *a* and *c* respectively, the bar would tend to rotate about O in a clockwise direction. This tendency to rotate can be exactly counteracted by an upward pull at any point of the bar to the right of O, the amount of the upward pull depending upon the distance from O at which it is applied. At some point *b* between *a* and *c* the upward pull would be found to be equal to P + Q, and experimentally it can be shown that the distance O*b* is such that $(P + Q) \times ob = (P \times oa) + (Q \times oc)$.

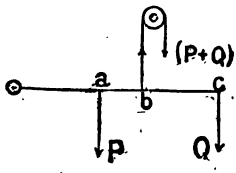


FIG. 351.

The product of any force and its perpendicular distance from any point is called its **moment about the point**; it gives a measure of the tendency of the force to produce rotation at the point. The above statement thus shows that the position of the equilibrant of a series of parallel forces is such that the moment of the equilibrant is equal to the sum of the moments of the separate forces about any point in the same plane as the forces.

It must be noted that the equilibrant is equal in magnitude to the vector sum of the separate forces, in this case (P + Q), but is opposite in sense. Although we could prevent rotation at O by a less or greater force than P + Q) if applied respectively to the right or left of *b*, we could not prevent motion of the bar *oc* parallel to itself, if the pivot

at O were free to move, unless the upward pull (P + Q) were just equal to the sum of the downward pulls P and Q.

A series of parallel forces such as P, Q, and (P + Q) will be in equilibrium if

(*a*) There is no tendency of the bar *oc* to move parallel to itself either upwards or downwards.

(*b*) There is no tendency of the bar *oc* to rotate about O, or any other point in the same plane as the forces.

378. The equilibrant or resultant of any series of parallel forces can be found graphically, and also the point at which it must act, as in the following example :—

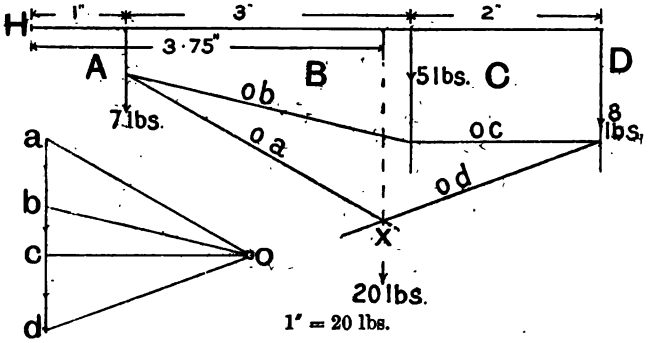


FIG. 352.

Take three forces, AB, BC, CD, Fig. 352, equal respectively to 7, 5, and 8 lbs. The force polygon is *abcda*. A single force *da* = 20 lbs. is the equilibrant.

We now require **to find a point through which the equilibrant passes.**

Choose any point O, called the **pole** of the vector diagram, and join *oa*, *ob*, *oc*, *od*. Between the forces AB, BC in the position diagram draw a line parallel to *ob*, which joins the pole to the point between the lines *ab*, *bc* in the vector diagram. Similarly, draw between forces BC and CD a line parallel to *oc*. From the end of *oc* draw *od* parallel to *od* in the vector diagram, and from the beginning of *ob* draw *oa* parallel to *oa*. The point X in which these lines intersect is a point through which the resultant or the equilibrant passes.

We can satisfy ourselves that this construction does give a point through which the resultant or the equilibrant passes, by choosing any

point, say H, and drawing through H a line perpendicular to the forces. Measure the distance from H to each force and ascertain whether the moment of the resultant is equal to the sum of the moments of the separate forces, thus—

Moment of AB = (7 × 1)
„ BC = (5 × 4)
„ CD = (8 × 6)
} Sum = 7 + 20 + 48 = 75

Moment of resultant = (20 × 3·75) = 75.

The diagram *abcda* is the force or vector polygon, and the diagram drawn between the forces AB, BC, CD, and the equilibrant in the position diagram is called the **link or funicular polygon.**

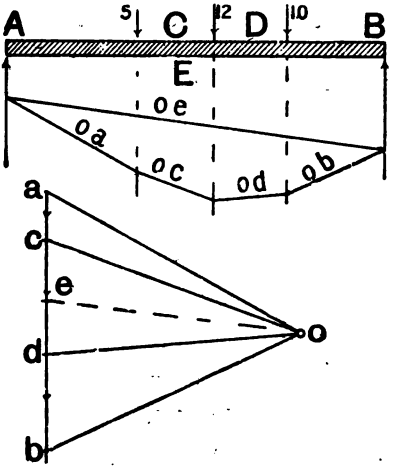


FIG. 353.

It will be noticed that the forces being in equilibrium, both these diagrams are closed. These are **the necessary conditions for the equilibrium of a system of co-planar non-concurrent forces.** They are utilized in solving various problems.

Ex. A beam 20 ft. long carries loads of 5, 12, *and* 10 *cwts. at points* 7, 11, *and* 15 *ft. from the left-hand end. Find the magnitude of each supporting force, assuming that they act vertically upwards.*

Let AC, CD, DB, Fig. 426, be the three loads, and BE and EA the supporting forces in the position diagram. The force polygon is *acdba*; we know that the two supporting forces together must be equal to (5 + 12 + 10) cwts.; and we have to find the magnitude of

each. Take any pole O, and draw *oa*, *oc*, *od*, *ob*. Commence from any point on the supporting force EA and draw between the forces in the position diagram lines parallel to *oa*, *oc*, *od*, *ob* of the vector diagram. We thus obtain four sides of a link polygon. As the whole system of forces must be in equilibrium (for the beam does not move) the link polygon must close. Hence draw *oe*, the **closing line** from the force EA to the force BE. From O, the pole, draw a line (shown dotted) parallel to this closing line *oe*. This line intersects the line *aodb* in *e*, and determines the magnitude of each of the supporting forces, for the force BE is given by *be*, and the force EA by *ea*. Thus BE = 15·85 cwts., EA = 11·15 cwts.

This result can be checked by calculation. The moment of the force EB about the end A must be equal to the sum of the moments of forces AC, CD, and DB.

Thus $EB \times 20 = (5 \times 7) + (12 \times 11) + (10 \times 15)$.

$$EB = \frac{35 + 132 + 150}{20}.$$

$$= \frac{317}{20}.$$

$$= 15{\cdot}85 \text{ cwts.}$$

Total load = 27 cwts., hence EA = (27 − 15·85)
= 11·15 cwts.

379. If a series of non-concurrent forces are not parallel, we can find their resultant or equilibrant in magnitude

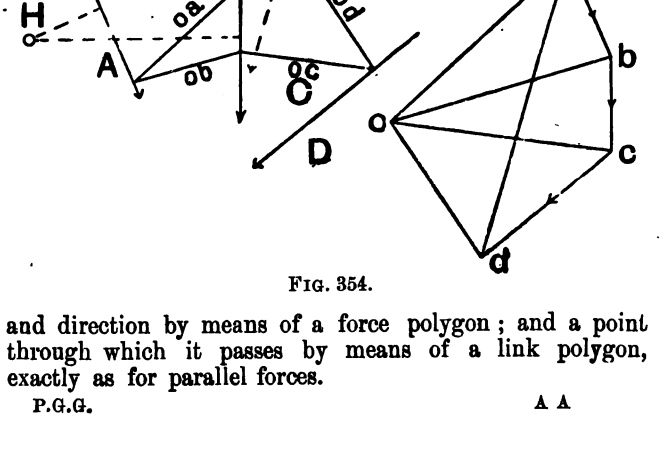


FIG. 354.

and direction by means of a force polygon; and a point through which it passes by means of a link polygon, exactly as for parallel forces.

In Fig. 354, the forces are AB, BC, CD. The force polygon is *abcda*. The resultant is *ad*, the equilibrant is *da*.

The link polygon is begun on the force AB. The two lines *ob*, *oc* are first drawn, then *oa*, *od* intersecting at X, which gives a point through which the resultant or equilibrant passes.

This construction can be shown to be correct by resolving each of the three forces AB, BC, CD, and the resultant, in any two directions perpendicular to each other, say horizontally and vertically; the component of the resultant in each direction would be found equal to the sum of the three components of the separate forces in that direction. Also, by taking moments about any point such as H, the moment of the resultant is found to be equal to the sum of the moments of the separate forces.

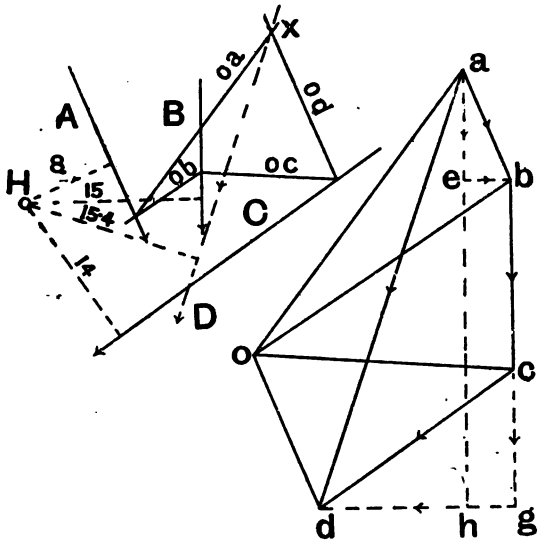


FIG. 355.

Ex. Forces of 5, 8, *and* 10 *lbs. act as shown at* AB, BC, CD, Fig. 355.

The resultant is 19·5 lbs. and acts through X, found by a force polygon and a link polygon, as shown.

The vertical component of AB = *ae*,
" " " of BC = *bc* (note force BC is vertical),
" " " of CD = *cg*,
and of resultant = *ah*,
and *ah* = *ae* + *cb* + *cg*.

Similarly, the horizontal component of the resultant is $-hd$ = horizontal components of AB, BC, and CD.

Note horizontal component of ab is $+eb$.

" " of bc is o as force BC is vertical.

" " of cd is $-gd$, as it acts to the left.

The algebraic sum is $-hd$.

Taking moments about H we note that each force tends to produce clockwise rotation about H, hence we can regard all moments as having the same sign. In measuring distances of each force from H, $\frac{1}{16}$ has been taken as unit.

The moment of force AB about H = (5 × 8) = 40.

" " BC " = (8 × 15) = 120.

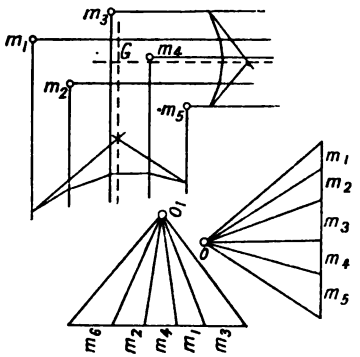
" " CD " = (10 × 14) = 140.

The sum of these is 300 units.

The moment of the resultant about H = (19·5 × 15·4)
= 300.

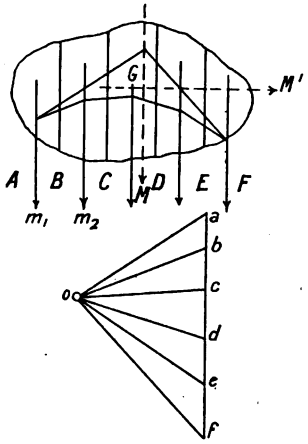
If we reverse the sense of the resultant, we have the equilibrant. The sum of the moments of the three forces and of the equilibrant about H is then zero, and the algebraic sum of the horizontal and vertical components of the forces and equilibrant is zero.

Centre of Gravity. Consider a series of isolated masses m_1, m_2, m_3, m_4, m_5. Treat each mass as a vertical force and find, by Art. 378, the line of action of the resultant force.



The funicular polygon is shown in the Figure. Now turn each force through 90° and repeat the process, thus finding the position of the resultant of the second system of forces. The point of intersection of the two resultants is the position of the centre of gravity of the masses.

To find the C.G. of an Irregular Area. Divide the area into a number of strips by vertical ordinates and consider each strip as a force m_1, &c., proportional to its area* and acting at the centre of the area. Find by a funicular polygon the position of the resultant M, and repeat the process for the area divided into horizontal strips, each strip being regarded as a horizontal force. The point G of intersection of the two resultants M, M_1 is the centre of gravity. If the area is symmetrical about one axis, divide into strips by ordinates perpendicular to the axis of symmetry, and the intersection of the line of action of the resultant with the axis of symmetry will be the required centre of gravity.



Exercises.

(1) An aeroplane steers due North with a velocity of 50 miles per hour, and is blown by the wind in a direction 25° North of East at 10 miles per hour. In what direction and at what actual speed does the aeroplane travel? *Ans.* 55 m.p.h.: 81° N. of E.

(2) A balloon rises vertically with a velocity of 40 miles per hour, and is blown due South at 15 miles per hour. What is the inclination to the horizontal of the path taken by the balloon? *Ans.* 70°.

(3) The wind is blowing due North with a velocity of 10 miles per hour, and an aviator, who can travel 50 miles per hour in a calm atmosphere, wishes to reach a town situated 20° North of East. In what direction should he steer? *Ans.* 9° N. of E.

(4) A ship steams North at a speed of 20 knots per hour, but the current is taking the ship East at the speed of 4 knots per hour. How far is the ship from its starting-point in $2\frac{1}{2}$ hours, and what is the direction? *Ans.* 51 sea mls.; 79° 41′ N. of E.

(5) A smooth board is inclined at 30° to the horizontal, and a metal roller rests on the board, being supported in position by a spring

* If the area be divided as explained in Art. 158 the strips will be equal in area and therefore of equal mass.

balance which exerts a pull in a direction inclined at 50° to the horizontal. If the balance indicates a pull of 20 lbs., what is the weight of the roller? *Ans.* 38 lbs.

(6) A picture weighing 50 lbs. is supported by a wire passing over a peg which is 5 ft. above the line of attachment of the wire to the picture. The points of attachment are 3 ft. apart. What is the tension in the picture wire? *Ans.* 26 lbs.

(7) In a steam engine, the piston is 20 in. diameter, and the mean steam pressure is 50 lbs. per square inch. The crosshead is guided between two parallel guide bars. The connecting-rod is 6 ft. long, and the crank is 1 ft. long. What is the stress in the connecting-rod, the pressure on the lower guide-bar, and the force on the crank-pin tangential to the crank-pin circle when the crank has rotated 35° clockwise from the inner dead centre? 15,900; 1,500; 10,450 lbs.

(8) Taking the data in the previous example, plot a curve showing how the turning-force on the crank-pin varies as the crank makes a complete revolution. What is the maximum turning-force? Show that the upper guide-bar for the crosshead has no pressure upon it during one half revolution of the crank from the inner dead centre, and that the lower guide-rod has no pressure upon it for the other half revolution.

(9) A ship is steering North-east with a speed of 15 knots, and a torpedo, travelling 30 knots, in a direction 10° East of North, is fired from a second ship. What is the velocity, in magnitude and direction, at which the torpedo approaches the first ship. If the torpedo hits the ship 3 minutes after it is fired, how far apart were the ships at the time of firing, and what was the bearing of the first ship relatively to the second? 19·5 knots; 17° W. of N.; 0·975 sea mls.; 17° W. of N.

(10) An aeroplane is in flight during a steady wind which blows at 20 miles per hour from the South. It is propelled relatively to the wind at a speed of 40 miles per hour. In what direction must the pilot *apparently* steer in order that his actual course shall be due West? At what speed will he travel West? (B.E. 1910.)

(11) A steamer is moving at 20 ft. per second towards the East; the passengers notice that the smoke from the funnel streams off apparently towards the South with a speed of 10 ft. per second. What is the real speed of the wind, and what is its direction? (B.E. 1910.) *Ans.* 22·5 f.s.; 27° W. of S.

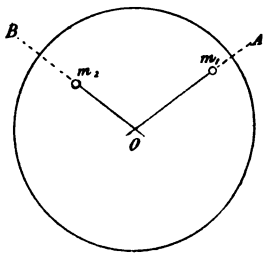
(12) There is a triangular roof truss ABC; AC is horizontal and 10 ft. long. The angle BCA is 25°, and BAC is 55°; there is a vertical load of 5 tons at B. What are the compressive forces in BA and BC? What are the vertical supporting forces at A and C? (B.E. 1910.) 4·75; 3·7; 3·375; 1·625 tons.

(13) The radial speed of the water in the wheel of a centrifugal pump is 6 ft. per second. The vanes are directed backwards at an angle of 35° to the rim. What is the real velocity of the water relatively to the vanes? What is the component of this which is tangential to the rim? (B.E. 1910.) 10·46; 8·6 f.s.

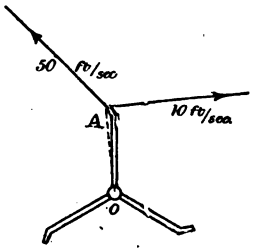
(14) The weight of a span of telegraph wire is 12·7 lbs. At one end the wire makes an angle of 5° and at the other an angle of 7°

with the horizontal. What are the pulling forces at these ends? (B.E. 1909.) 60·6 ; 60·9 lbs.

(15) The positions of two points A and B in a horizontal plane, referred to an origin O, are defined by the vectors $OA = 3''_{270°}$, $OB = 2''_{68°}$, angles being measured anti-clockwise from the East. Choose an origin O, mark the Eastward direction, and plot the points A and B. If C is the middle point of AB, verify that $\frac{1}{2}(3''_{270°} + 2''_{68°})$, the vector mean of OA and OB, is equal to the vector OC. Measure the length and direction of OC. (B.E. 1909.)

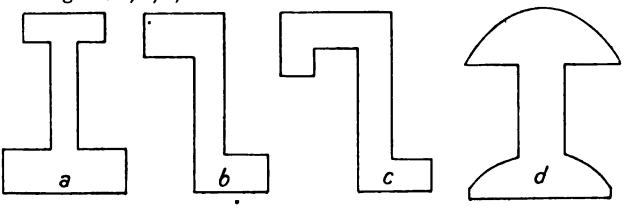


(16) A disc rotating about O has two masses m_1 and m_2 fixed to it as shown, which exert centrifugal forces on the bearing O of 5 lbs. and 8 lbs. along the radii OA and OB. Find the resultant centrifugal force. Measure its magnitude and the angle which its line makes with OA. (B. E., 1908.)



(17) You are given the plan of a lawn sprinkler which revolves about O. Water leaves at A with a speed through the nozzle of 50 ft. per second, as indicated. The rotation of the arms causes the nozzle A to move with a speed of 10 ft. per second in a direction perpendicular to the radius OA, as shown. Find the actual velocity of the jet, measuring its speed and showing its direction. (B. E., 1907.)

(18) Find graphically the position of the centre of gravity of each of the figures *a*, *b*, *c*, *d*.



CHAPTER XXII.

GRAPHIC DETERMINATION OF STRESSES IN BRACED STRUCTURES.

380. Any system of external forces acting on a structure tends to change the form of the structure. The combined effect of the disturbing forces may be (*a*) wholly **compressive**, *i.e.*, tending to press together the particles in the material of the structure, or wholly **tensile**, *i.e.*, tending to elongate or pull apart the particles in the material; or (*b*) the material may be subjected to compression in some parts and tension in other parts of the structure.

Material subjected to force of any kind is said to be **stressed**. A tensile stress is sometimes regarded as *positive*, and a compressive stress is then regarded as *negative*.

381. **Stresses in Braced Structures.** If a braced structure be subjected to a given system of one or more external forces which act at the joints of the frame, and the several members (or bars) of the frame be connected by pin-joints, the stress in any member will be either tensile or compressive, and the stress may be supposed to act along the axis of each member (or bar). Bars in tension are called **ties** and those in compression are called **struts**. The term "bar" or "member" is applied only to those parts of the frame which are capable of taking both tensile and compressive stresses. A member capable of taking tensile stress only, *e.g.*, a chain, or cord, is called a **half-member**.

382. A frame structure may have just sufficient bars to enable it to keep its shape under any system of loading (consistent with the strength of the frame), or it may have more, or less, than this number. A frame having *more* bars than is necessary for preservation of shape is said to

have **redundant** members, while one having *less* is called a **deformable frame.**

383. A frame comprising three bars 1, 2, 3, Fig. 356, has three joints, is not deformable, and has no redundant members. A frame having four bars 1, 2, 4, 5, is deformable, but if a fifth bar 3 be added, it is not deformable, and any additional bars, *e.g.*, a bar joining *a*, *b* would be redundant. A frame having seven bars, 1 to 7, Fig. 356, is not deformable, and has no redundant members. Now,

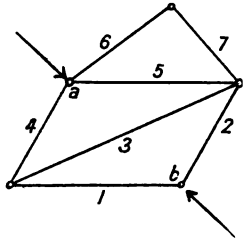


FIG. 356.

a non-deformable and non-redundant 3-joint frame has 3 bars;

a non-deformable and non-redundant 4-joint frame has 5 bars;

a non-deformable and non-redundant 5-joint frame has 7 bars;

and, generally, a non-deformable and non-redundant n-joint frame has $2\ (n-3)$ bars.

Hence, a frame having n joints must have $2\ (n-3)$ bars; if it has less it is deformable, and if more, it has redundant members.

384. If the member 3 in the frame 1 2 3 4 5, Fig. 356, were a *cord* and capable of taking only tensile stress, then forces applied as shown at the joints *a*, *b* would not deform the frame, but similar forces acting at the opposite corners would do so; and in order to prevent deformation of the frame under any loading it would be necessary to provide another *half-bar* (tensional member) joining *a*, *b*. A frame thus equipped is said to be **counter-braced.**

Redundant members should be avoided as much as possible, since they frequently produce initial stresses of unknown magnitude, and which, therefore, cannot be taken into account in investigating the stability of the structure.

385. Graphical methods of determining stresses in braced structures are based on the following principles:—If a braced structure is in equilibrium then—

(1) **The force polygon for every joint in the frame must close.**

(2) **The force polygon for the external forces must close.**

(3) **The funicular, or link, polygon for the external forces must close.**

386. **Triangular Frames**, or frames in which three bars only (or their equivalent) are used.

Ex. 1. *Fig.* 357. *A jointed frame* 1 2 3 *supports a weight* W. *Find the stress in each member.* The joint 3 to which W is attached is in equilibrium under the load W, and the forces acting along the bars CA, AB. We thus have three concurrent forces BC (or W), CA, and AB. Of these the magnitude of BC is known, and the others are found by drawing a triangle of forces *bca*, in which *bc* is made equal to W on a suitable scale, and *ca*, *ab* are drawn parallel respectively to CA and AB in the frame. The magnitude of the force in CA is *ca*, and that in AB is *ab* measured by the scale used for W. To ascertain the nature of the stress in CA and AB, we know that W acts downwards and is represented by *bc*, and to ensure equilibrium at joint **3**, the other forces CA, AB must balance BC, *i.e.*, $ca + ab = cb$; hence, the direction of the force in CA is *ca*, and the direction of the force in AB is *ab*. If we mark these directions on the frame diagram by arrow heads, we see that CA points toward the joint under consideration, and that AB points away from the joint, thus indicating that the force in CA is one pressing upon the joint, *i.e.*, CA is in compression, and that in AB pulls away from the joint, *i.e.*, AB is in tension.

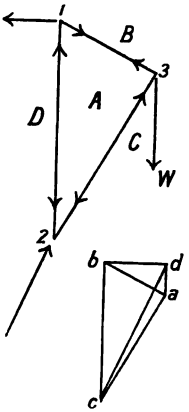


FIG. 357.

If we suppose the joint 1 to be a pin joint, and a horizontal member placed at 1, the force diagram for this joint is *bad*. As the pull in BA is away from the joint 1 (the member BA being in tension), the directions for the stresses

in AD, BD are *ad*, *db*, thus showing that the member AD is in compression and the member DB in tension. The forces at the joint 2 are given by the triangle *dac*. The forces in DA and AC are known to be in compression, hence the thrust at the joint 2 is a force *dc*, and the reaction at the joint, to balance this thrust, must be CD.

Ex. 2. *Fig.* 358. *A crane with vertical post* AD *and back stay* DE *carries a load* W *attached to a rope which*

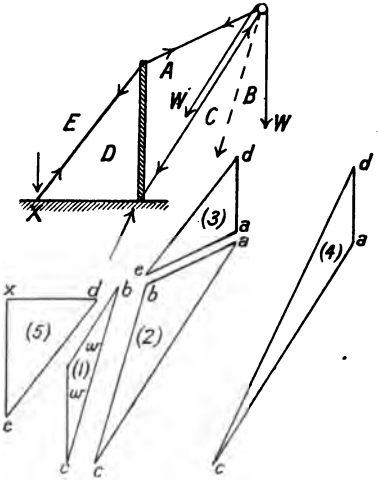


FIG. 358.

passes over a single pulley as shown. The direction of the lifting rope is parallel to the jib AC. *Find the forces acting in each member of the frame.*

(1) On attempting to determine the forces acting on the frame at the pulley wheel, we find four forces acting at that point, viz., weight of the load W; the pull in the cord on the other side of the pulley also equal to W; and the forces in the jib and tie of the frame. These four forces can be reduced to three by finding a single force BC, which is equivalent to the load W and the pull W. This is the

vector sum of W and W and is given in magnitude and direction by BC in the force diagram (1),

(2) Replacing W and W by force BC at the joint O, we now have three forces acting there, viz., BC and the forces in members CA and AB. The nature and magnitude of these forces are given by force diagram (2) determined as in Ex. 1.

(3) At the top of the crane post AD there are three forces, viz., the pull in the tie-bar, the pull in the back-stay DE, and the force in the post AD. The pull in AB having been already determined, we can draw the force diagram (3), first drawing $ea = ab$ reversed; then ad gives the compressive stress in the post and de the tensile stress in the back-stay. The vertical pull at X must be equal to the vertical component xe of the force de. The horizontal component is xd, shown in diagram (5).

(4) At the bottom of the crane post, usually a pivot, we have acting the compressive forces in DA and AC, and these are resisted by the pivot. To determine the magnitude and direction of this pressure, draw the diagram (4) in which da is taken from diagram (3) and ac from diagram (2); then the vector sum dc is the pressure on the pivot, and an equal and opposite force cd is the reaction at the pivot.

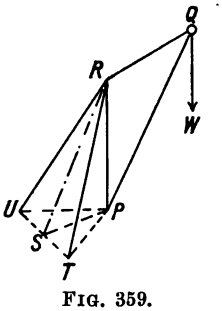


FIG. 359.

The five diagrams have been drawn separately for clearness. They could have been combined in one diagram.

Ex. 3. Fig. 359 shows a crane having two back-stays RU, RT.

In this example, we first replace RU, RT by a single stay RS in the same plane as PQR. Find the stress in RS as in the preceding example, and then find the components of the stress in RS in the two directions RU, RT.

Ex. 4. Fig. 360 shows a "shear legs" device AB, AC and stay AD.

First replace AB and AC by a

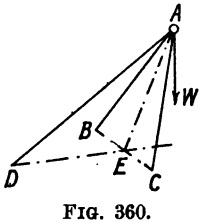


FIG. 360.

single member AE in the plane of AB and AC, and also in the plane of AD and AW. Then find the stress in AE and resolve it along AB and AC.

Ex. 5. Fig. 361. *A triangular frame* 1 2 3 *is supported at* 1 *and* 3 *and carries a load* W *at* 2. The supporting forces are DB at 1 and CD at 3. They may be determined as shown in Fig. 353, or they may be calculated by taking moments about each end 1 and 3 in turn, the perpendicular distance of W from 1 and 3 being known ; *e.g.*, if x is the perpendicular distance of W from 3, and l is the span, then

$$(\mathrm{DB})\ l = \mathrm{W}x \text{ or } (\mathrm{DB}) = \mathrm{W}x/l.$$

Similarly $(\mathrm{CD}) = \mathrm{W}(l-x)/l$. A further method of determining the supporting forces, and also the force in each member of the frame, is to draw a force polygon for the three forces acting at 2, viz., W and the forces in AB and CA. Begin with bc = W and draw ca and ab respectively parallel to CA and AB. Then the forces in CA and AB are given by ca and ab. Now draw ad parallel to AD, then dba is a force triangle for the joint 1, and db is the supporting force there. Similarly cda is a force triangle for the joint 3, and cd is the supporting force at 3. Note that $cd + db$ = W. Noting that the force W or BC acts in the direction bc, and following round the force diagram in "follow your neighbour" fashion, we ascertain that the forces in AB and AC each press on the joint 2, hence these members are in compression, and similarly the forces in AD pull on each joint 1 and 3, hence AD is in tension.

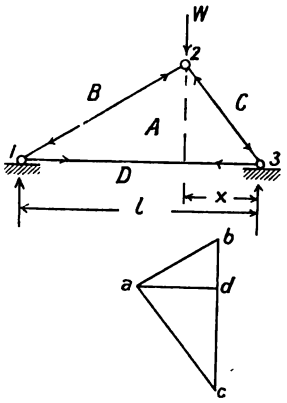
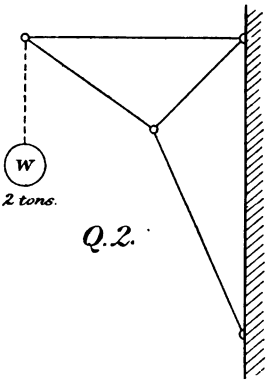


FIG. 361.

Exercises.

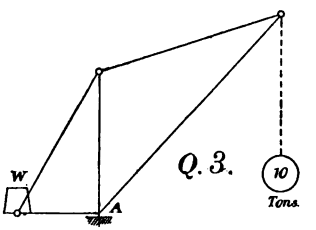
(1) A load of 2 tons hangs from a crane by a chain 16 ft. long. It is required to keep the load 3 ft. from the vertical by means of a horizontal rope attached to the bottom of the chain. What force in the rope will be necessary? (B. E., 1904.)

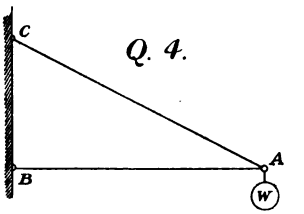
(2) Determine the forces in the four bars of the given crane when loaded with 2 tons as shown. Write the answers on the bars, using



the plus sign to signify a tension, and minus to denote compression. Adopt a scale of 1 in. to 1 ton. (B. E., 1910.)

(3) In the given crane, the weight W just balances about A the load of 10 tons. Find W. Determine the forces in the five bars of the frame, writing the answers on the bars themselves. Use + for a pull and − for a thrust. (B. E., 1912).





(4) A load W of 12 cwts. hangs from the point A. Determine the stresses in AC, AB, and also the thrust upon the wall at B, and the horizontal pull exerted at C.

(5) Determine W in the Fig. of Q. 4 if (1) the pull in the member AC is 15 cwts.; (2) the horizontal pull at C is 12 cwts.

387. **Incomplete Frames** are frames which assume a particular shape with a certain distribution of the load, and change shape when the load distribution is changed. Such frames can only be used in cases in which the greater part of the load is distributed in a definite way, and any variation to which the distribution is liable is allowed for by bracing or stiffening the joints.

Ex. 1. Fig. 362. A cord fixed at the ends and carrying intermediate loads will assume a shape which depends on the magnitude and position of the loads. Suppose, first, that we know the magnitude and position of each of the loads LM, MN, NP, PQ and the position of one end of the cord A.

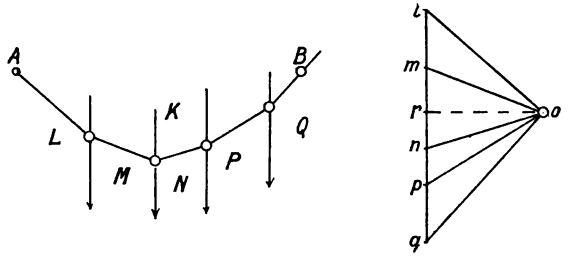


FIG. 362.

The form the cord will assume under this system of loading can be determined as follows :—Draw a load line *lmnpq*, making *lm* = load LM, *mn* = load MN, etc. Choose any pole O and join *ol*, *om*, *on*, *op*, *oq*. Draw a link polygon for this system, commencing such polygon at the definite point A. Thus draw a line parallel to *ol* from A to the line of action of load LM, thence draw a parallel to *om* to meet MN, and continue with KN ‖ *no*, KP ‖ *po*, finally drawing from PQ, to the right, a line parallel to *qo*. The second point of support B can be taken anywhere on the line parallel to *qo*. If it is required to be on the same level as A, it is located by drawing a horizontal line from A to meet the line parallel to *oq*.

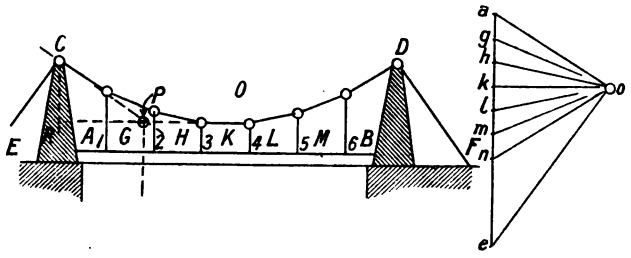
If, instead of the loads being given, we know *the form taken by the cord and also* **one** *of the loads*, say, LM, we can determine the other loads by drawing *lm* = load LM, and from

l and *m* drawing lines *lo*, *mo* intersecting in *o*, and parallel respectively to the two segments of the cord meeting on LM. Continue the load line *lm* downwards, and from O draw lines *on*, *op*, *oq* parallel, respectively, to the corresponding sections of the cord; then *mn*, *np*, *pq* give, respectively, the loads MN, NP, PQ.

A line *or* drawn through *o* perpendicular to the load line, gives the horizontal tension at any point of the cord.

388. A practical form of the above problem occurs in the suspension bridge, Fig. 363, in which a platform AB is carried by rods 1, 2, 3, 4, 5, 6 attached to a chain CD which passes over piers C, D and is secured to anchorage at E and F.

If we assume that the total load per foot run is constant,



(*a*) FIG. 363. (*b*)

and neglect the difference in weight of the rods 1, 2, 3 . ., and the links of the chain, each rod will have to carry an *equal* load, say W. If we know the **span** of the chain, *i.e.*, CD, and the **dip**, *i.e.*, the level of the lowest link below CD, and the magnitude of each load W, the shape taken by the chain may be found as follows:—

As the loading is symmetrical, the shape of the chain will be symmetrical about the centre, and as there are (in the Fig.) 7 links in the chain, the central link (between rods 3 and 4) will be horizontal.

Consider the half chain on the left; it is in equilibrium under 3 W, *i.e.*, the loads on the rods 1, 2, 3; the horizontal pull in the lowest link KO; and the pull in the link AO. These three forces must meet in a point P on the level of the central link KO, and vertically above the centre of force of the loads

3 W) *i.e.*, ¼ CD from the end. If we determine this point P, and join to the top of the pier C we have the direction of the link AO. To determine the direction of the links GO, HO: Set off a load line *ak* (Fig. *b*) and make $ag = gh = hk = W$. Draw *ao* parallel to link AO to meet a horizontal line from *k* in O. Join *go*, *ho* which give the direction of links G*o* H*o*.

The tension in each link is given by diagram (*b*); for *aog* is a triangle of forces for the joint of links AO, GO and rod 1; similarly *goh* is a triangle of forces for the next joint, and so on. The tensions in the links AO, GO, HO, etc., are thus given by *ao*, *go*, *ho*, etc. The line *ko* gives the horizontal tension in the central link or the *horizontal component* of the tension in any other link.

In order that there shall be no tendency to overturn the piers, the horizontal component of the tension in the anchor chain EC must equal that in link AO. Thus if CE is the direction of the anchor chain, and *aok* (Fig. *b*) is a force diagram for the half chain, draw *oe* parallel CE to meet *ak* produced in *e*, then *oe* gives the tension in the anchor chain.

The pressure on the pier C = *ae*, *i.e.*, the sum of the vertical components of AO and the anchor chain.

An expression connecting horizontal tension, total load, dip, and span may be obtained as follows:—

Let H = horizontal tension = *ko*, Fig. *b*.

w = load per foot run of bridge.

Then wl = total load, and $ak = \frac{1}{2} wl$, if l = span.

Let d = dip of chain below line CD.

Then considering triangles *aok* and CPR we have

$$\frac{ak}{ko} = \frac{\frac{1}{2} wl}{H} = \frac{CR}{RP} = \frac{d}{\frac{1}{4} l}$$

$$\therefore H = \frac{\frac{1}{4} l \cdot \frac{1}{2} wl}{d} = \frac{\frac{1}{8} (wl) \cdot l}{d} = \frac{\frac{1}{8} \text{ total load} \times \text{span}}{\text{dip.}}.$$

389. Fig. 364. If the number of rods is very great, the chain will take the form of a many-sided polygon which will not differ appreciably from a continuous curve. The horizontal tension H is then tangential to the centre of the curve, the pull at C is also tangential to the curve at C, and these tangents meet at a point P distant ½ l from one pier. Consider any portion ST of the chain (Fig. 365), T

being at the centre of the span; draw SU perpendicular to TU; bisect UT in V. Then the load on ST acts through V, and SVU is a

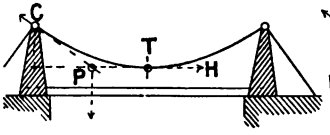


FIG. 364.

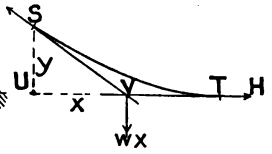


FIG. 365.

triangle of forces for the horizontal tension H, the pull at S, which is tangential, and the load. Let TU $= x$ and SU $= y$. Then if w = load per foot run, SU $= wx$, UV $= \frac{1}{2}x$, and as $H = \frac{1}{8}wl^2/d$ we have

$$\frac{SU}{UV} = \frac{y}{\frac{1}{2}x} = \frac{wx}{H} \quad \therefore x^2 = \frac{2yH}{w}$$

$$\text{But } H = \frac{1}{8}wl^2/d \quad \therefore x^2 = \frac{2y}{w} \cdot \frac{1}{8} \cdot \frac{wl^2}{d}$$

$$= \frac{l^2}{4d}y$$

and since $l^2/4d$ is constant, we have x^2 proportional to y, showing that the curve is a parabola.

390. Fig. 366. If the chain of Fig. 362 be inverted, as shown in Fig. 366, it will still be in equilibrium under the loads LM, MN, etc., provided the links are such as to stand compression. The force

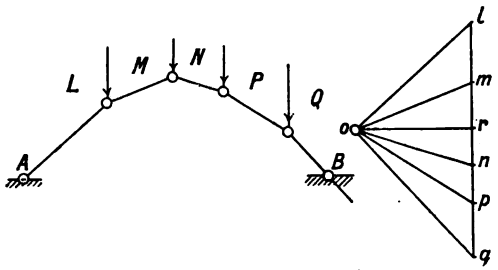


FIG. 366.

diagram is *olmnpq*. We thus get a kind of arch. As in the case of the suspension bridge, if a platform be connected by rods to the joints between the links, the platform being either above or below the links, we have a structure in equilibrium. When the rods are numerous, and the load per foot run is constant, the arch is parabolic, and we thus get either the parbolic arch of Fig. 367, or the Bowstring Girder of Fig. 368.

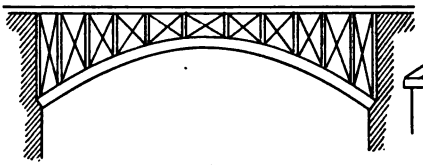


FIG. 367.

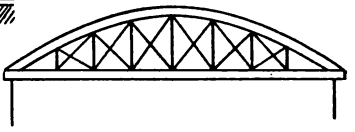


FIG. 368.

Exercises.

(1) Find the shape taken by a chain loaded as in Fig. 362, if the loads LM, MN, NP, PQ are respectively 4, 5, 9 and 3 cwts., and the distances between them are, respectively, 5, 7, and 4 feet. The supports A and B are to be on the same level, and A is 6 ft. from the load LM.

(2) If LM (Fig. 362) is 3 cwts., determine the loads MN, NP, PQ, and the horizontal tension in the chain.

(3) In the suspension bridge (Fig. 363 (*a*)) the span is 140 ft., the dip is 40 ft., and each rod 1, 2, 3, etc., supports a load of 10 tons. Determine the shape taken by the chain, and find the pressure in each pier. What is the horizontal tension in the chain? Draw a suitable anchor chain and determine the tension therein.

(4) In Fig. 366, if the links from A and B each have an inclination of 50° to the horizontal, and the loads LM, MN, NP, PQ are respectively 3, 5, 4, and 2 cwts., the loads being 10 ft. apart, determine the form of the chain.

Exercises on Articles 391 to 395.

(1) Copy the roof truss shown in Fig. 369 to double the scale shown in the Fig., and take each of the three loads as 5 tons. Draw a stress diagram, and tabulate the stresses in each member of the truss.

(2) Take the data of the previous exercise, and add a wind-pressure of 10 tons at joint 3, and 5 tons at each of the joints 2, 4. Assume the supporting force at 4 to be vertical. Find the supporting force at 0 and the stress in each member of the truss.

(3) Copy the truss shown in Fig. 370 to double the scale shown; assume loads of 5 tons at joints 1, 2, 3, 4, 5, loads of 2½ tons at joints 0, 6, and a *total* normal wind-pressure on the right of 18 tons. Determine the two supporting forces, that at 6 being vertical, and tabulate the stress in each member of the truss.

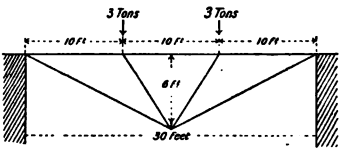
(4) Copy the truss shown in Fig. 371 to double the scale shown; take a load of 4 tons at each joint, and determine the stress in each member of the truss.

(5) The Warren Girder of Fig. 373 carries loads of 6 tons at each joint 3, 5, 7, 9, and each bar of the girder is 10 ft. long. Find the stress in each bar.

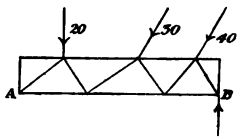
(6) Each load on the girder of Fig. 374 is 5 tons. Determine the stress in each bar of the girder.

(7) A girder of the form shown in Fig. 375 has 6 bays each 10 ft. square, and is loaded with a total load of 10 tons carried equally at joints 3, 5, 7, 9, 11. Find the stress in each member, and draw a stress diagram if the total load is carried as follows:—3 tons at 3, 5 tons at 7, and 2 tons at 9. What are the supporting forces in the latter case?

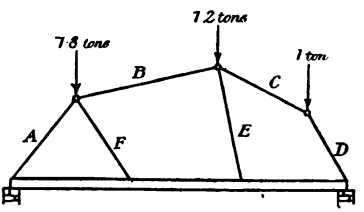
(8) Make a diagram of the stresses in the framing shown in the sketch, and figure *upon the frame diagram* the amount and nature of the stress in each member. (R.I.B.A., Inter., 1912.)



(9) The sketch represents a frame acted on by the given forces in its plane. Determine the magnitude of the forces at A and B that would just keep it in equilibrium.



Q. 9.



Q. 10.

(10) Determine and measure the thrusts in the members A, B, C, D, E, and F of the given truss when loaded in the manner shown. Let the scale be ⅛ inch to 0·1 ton. (B. E.)

391. **Compound Frames.** We now give a few examples of Roof Trusses, etc., showing how to determine the magnitude and nature of the stress in each member, and the reaction at points of support.

Ex. 1. Fig. 369 *is a line diagram of a roof truss. The*

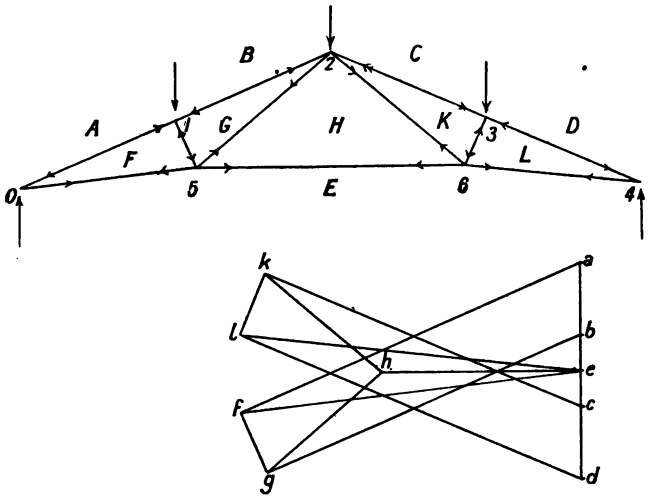


FIG. 369.

total load on the roof is equally divided between the joints 1, 2, 3 ; *the supporting forces at* 0, 4 *are vertical.*

The loads are AB, BC, CD, and as these are symmetrical about the centre, the supporting forces DE, EA, are equal to each other, and

$$DE + EA = AB + BC + CD.$$

The diagram for the external forces is *ab*, *bc*, *cd*, *de*, *ea*. To determine the stress in each member of the frame, we must **begin with a joint in which only two unknown quantities exist.** The joint O is in equilibrium under three forces, viz., EA, acting upwards, and the stresses in members AF, FE. The direction of each of these forces is known and the magnitude of EO is also known.

To determine the magnitudes of the stresses in AF, FE.—Draw *af* parallel to AF, *ef* parallel to EF, meeting at *f*: then *aef* is the force triangle for the joint O, and *af*, *fe* respectively represent the magnitude of the stress in bars AF, FE.

To determine the nature of the stress in AF *and* FE.—The force EA acts upwards, *i.e.*, in the direction *ea*, and going round the force triangle in the direction *ea*, *af*, *fe*, we find the force in AF is toward the joint and is compressive, whereas the force in FE is away from the joint and is tensile.

Proceeding to joint 1, we have four forces acting, viz., the load AB, and the stresses in FA, BG, and GF. Of the eight * quantities involved we know six, viz., the direction of each force and the magnitudes of FA and AB. The magnitudes of BG and GF are found by drawing the force polygon for the joint 1. We have *fa*, *ab*, already drawn ; draw from *b* a line *bg* parallel to BG, and from *f* a line *fg* parallel to FG. The closed figure *fabg* is the force polygon for the joint 1. The lengths of *bg* *gf* give the magnitudes of the stresses in the members BG, GF, and by going round the force diagram in the order *fabg*, we find that in each of the members BG, GF, the force acts towards the joint 1, and hence each is in compression. Proceeding to joint 2, we have five forces, viz., the load BC and the stresses in members CK, KH, HG, GB. There are thus ten quantities involved, of which only seven are known, *i.e.*, the direction of each force and the magnitudes of GB, and BC. We cannot draw the force polygon for the joint 2 until we know the magnitude of one other force. If we try the joint 5, we have four forces acting, viz., EF, FG, GH, HE, and of the eight quantities involved we know six., viz., the directions of each force, and the magnitudes of EF, FG. We can determine the magnitudes of GH and HE by drawing the force polygon *efghe* ; the lines *ef*, *fg* are already drawn ; draw *gh* parallel to GH and *eh* parallel to EH, intersecting in *h*. Then *gh*, *he* give respectively the magnitude of the forces GH and HE. There are now only two unknown quantities in the forces acting at joint 2, and the force diagram for that joint can be drawn. Three of the sides, *hg*, *gb*, *bc*, being already drawn, draw

* Four magnitudes and four directions.

from *c* a line *ck* parallel to CK, and from *h* a line *hk* parallel to HK, which completes the polygon *hgbckh*.

The remaining joints present no difficulty. The force polygon for joint 3 is *kcdlk*, the two unknown magnitudes being *dl* and *lk*; for joint 6 the polygon is *ehkle*, the only unknown magnitude being *le*; and for the joint 4 the polygon is *elde*, which will be found already drawn, the forces in EL and LD having been already determined. It should be noted that, as the roof is symmetrically loaded, the force diagram is symmetrical about the centre line *eh*, thus the stresses in corresponding numbers in each half of the roof are alike. Denoting a tensile stress as positive and a compressive stress as negative, we may arrange the result in tabular form thus :

Member . .	FA and LD	GB and KC	GH and HK	FG and LK	EF and EL	EH
Magnitude . .	{ length of *fa* or *ld*	*gb* or *kc*	*gh* or *hk*	*fg* or *lk*	*ef* or *el*	*eh*
Kind of stress .	−	−	+	−	+	+

Ex. 2. *Fig.* 370 *shows a roof truss subjected, in addition to the load of the roof, to a wind pressure which acts perpendicularly to the surface of the roof.*

We assume that the weight of the roof is distributed at the joints as follows :—The loads at joints 1, 2, 3, 4, 5 are equal, and at O, 6 the load is one-half the load at 1, 2, etc. The wind supposed to be blowing on to the right-hand side of the roof is uniformly distributed over that surface, hence the wind pressure at each of the joints 4 and 5 is $\frac{1}{8}$ P—where P represents total wind pressure—whereas the pressure at each of joints 3 and 6 is $\frac{1}{4}$ P, or half that acting at 4 or 5. The truss is supposed fixed at point O, and supported on a roller-bearing at 6.

We first compound, by the parallelogram of forces, the two forces (weight and wind pressure) acting on each of the joints 3, 4, 5, 6, and substitute these resultant forces for those originally given.

To determine the supporting forces at O *and* 6.—The supporting force at 6 will be vertical since the roller-bearing will supply no horizontal resistance. The magnitude of this

force and the direction and magnitude of the force at O may be found by means of a force polygon and link polygon for the external forces acting on the truss.

Draw the diagram of the forces known in magnitude and direction, as at *bcdefghk*. From *k* draw *ka* vertical, which

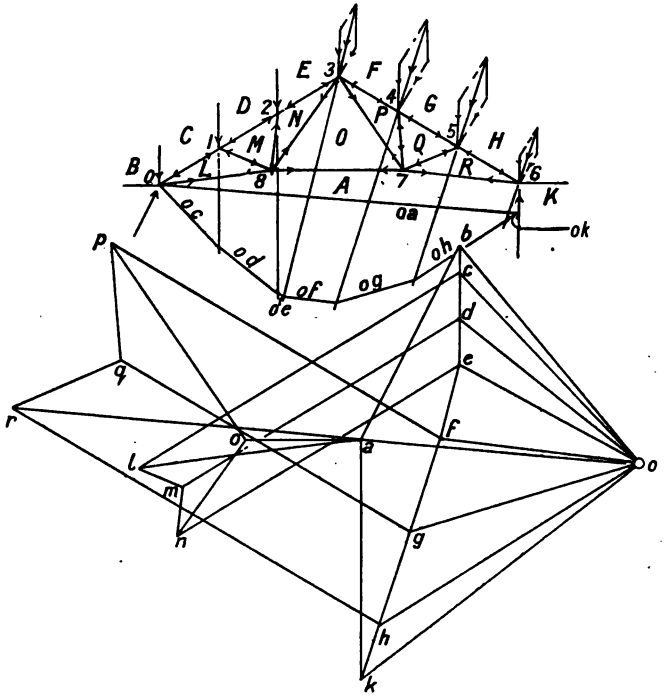


FIG. 370.

represents the direction of the supporting force KA acting at point 6. The magnitude of KA can be determined by drawing a link polygon for the system of forces (Art. 379). This polygon must be commenced at the point O, since that is the only point at present known to lie on the line of action of the supporting force at O. Thus commence at O

on the frame diagram and draw a line *oc* parallel to *oc* of the force diagram to meet the line of action of force CD. Similarly draw lines *od*, *of*, etc., respectively parallel to *od*, *oe*, *of*, *og*, *oh*, *ok*, in the force diagram, and draw the closing line *oa* of the link polygon.* Draw from the pole O a parallel to *oa* meeting the vertical from *k* in *a*. Then *ka* gives the magnitude of the force KA acting at 6, and *ab* gives the magnitude *and direction* of the supporting force at O. Having now determined all the external forces acting on the structure both in magnitude and direction, the force polygon for each joint may be drawn as already explained. The joints are dealt with in the order 0, 1, 2, 8, 3, 4, 5, 7, 6, and the stresses in the members are as in the appended table. The force diagram is *not* a symmetrical one.

Member . .	CL, DM, EN, FP, GQ, HR, LM, MN, PQ, QR,	AL, AO, AR, ON, OP
Magnitude .	Length of *cl*, *dm*, *en*, *fp*, *gq*, *hr*, *lm*, *mn*, *pq*, *qr*	Length of *al*, *ao*, *ar*, *on*, *op*
Kind of stress .	−	+

Ex. 3. In a roof truss of the form shown in Fig. 371, a difficulty arises after the force diagram has been drawn for the joints 1, 2, 3; at this stage, the stresses in members CN, NA, DO, ON, OP, and PA are known. Proceeding to joint 4 we have three unknown forces, viz., ER, RQ, QP, hence we cannot draw the force polygon for this joint. If we go to joint 5 we also have three unknown forces, viz., PQ, QT, TA. To overcome this difficulty, the members QR, RS are supposed removed, and a member R′ S′ connecting joints 5, 6 is substituted for them as shown in the small Fig. The diagram for the joint 4 is then *poder′p*, and that for the joint 6 is *r′efs′r′*, and for the joint 5 we now have *apr′s′ta*.

By this means we determine the position of the line *s′t*

* It must be noted that as forces AB, BC are concurrent and the link polygon begins at their point of action, the link between them reduces to a point, hence we have no line in the link polygon parallel to *ob*.

which gives the stress in the member joining joints 5 and 8. We also determine *ta*. If we now return to the actual truss for the joint 4, we have already determined fully the lines *po*, *od*, *de*, and we know that *pq* must end on the line *st*, hence we can determine *q* by producing *pr'* to meeet *st* in *q*. From *q* we now draw *qr* parallel to QR to meet *er* drawn parallel to ER in the point *r*. The force diagram for joint 4

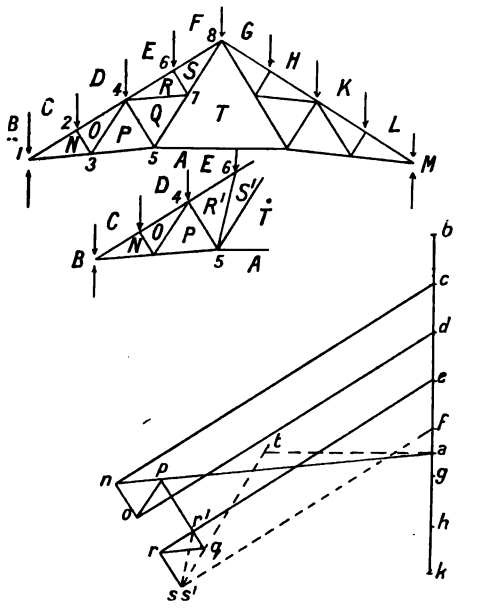


FIG. 371.

is now *poderqp*, for joint 5 the diagram is *apqta*; for joint 6 the diagram is *refsr*; for joint 7 the diagram is *tqrst*. Joint 8 presents no difficulty. The whole diagram is symmetrical about the line *at*, and the student should find no difficulty in dealing with the right-hand half of the roof.

If the loading is not symmetrical, the above construction must be carried out separately for each half of the roof-truss.

Ex. 1, 2, 3, 4, 5, *Fig.* 371 (a), *is the section of an open shed. The projecting roof* 2, 3, 4 *rests on supports* 1 2, 4 5, *fixed at* 1 *and* 5; 3 4 *is equally divided, and* HK *is perpendicular to* 3 4. *The loads are vertical and represented in lbs. Find and write down the stresses on*

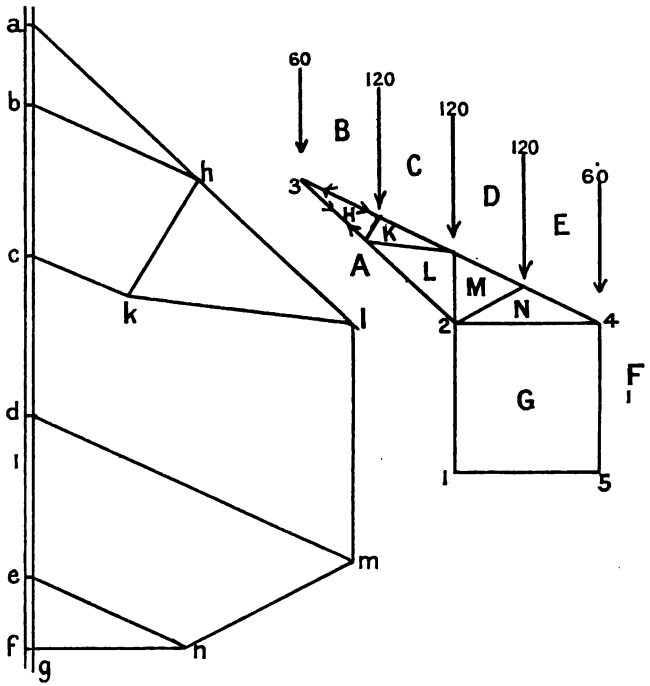


FIG. 371 (*a*).

the members of which the roof is composed; and indicate the nature of the stresses, whether tensile or compressive. In constructing the polygon of stresses, employ by preference for unit 1 *in. to* 100 *lbs.*

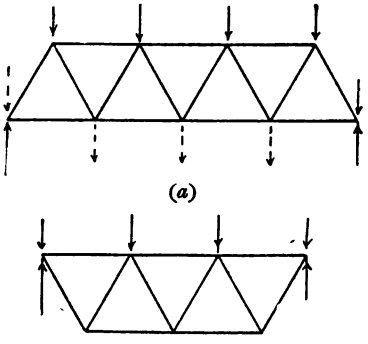
(B. E. (Hons.), 1893.)

The load line is *abcdef*. Note that there is no stress in the support F G, while that in G A is equal to the whole load. The Fig. shows the force diagram drawn in the usual manner. The stresses are as shown in the following table :—

Member .	AH	BH	HK	KC	KL	LA	LM	MD	MN	NE	NG	GF	GA
Stress in lbs. .	170	135	100	80	170	330	180	260	130	140	130	0	480
Kind of stress .	−	+	−	+	+	+	−	+	−	+	−	0	+

392. **Girders,** or beams for carrying platforms of bridges, are commonly built up of bars and plates connected by pin joints or rivets. Figs. 372, 375, are illustrations of common types. Each form is a triangulated framework; the continuous top and bottom members are called the booms or flanges: the cross bars constitute the web of the girder.

Ex. 1. The Warren girder shown in Fig. 372 consists of a number of equilateral triangles. The girder may be supported as shown at Fig. (*a*) or (*b*), and the load may be applied to either top or bottom (or both) booms. Fig. 373 shows a Warren girder supported by piers at the points 1 and 11 and carrying loads LM, MN, NP, PQ each = W attached at the joints 3, 5, 7, 9 in the bottom boom.



(*a*)

(*b*)

FIG. 372.

The load diagram is *qpnml*, and as the loading is symmetrical the supporting forces are vertical and each equal to $\frac{1}{2}$ *lq*.

In drawing the force diagram, commence at joint 1, and we have for joint 1 the triangle *lka*, *ka* being the stress in the diagonal member 1 2 or KA, and *al* that in the section AL of the lower flange. For joint 2, triangle *akb*. For joint 3, the closed figure *labcml*. For joint 4, the closed figure *cbkdc*, and we may note that the stress in the part 24 or BK of the upper flange is less than that in the part 46 or DK by the amount *bd*. For joint 5 we have the closed figure *nmcden*. (Note that the points *d*, *e*, being coincident, there is no stress in the diagonal member DE, or, in other words, the compressive stress due to certain loads is exactly balanced by tensile stress due to other loads.) For the joint 6 the closed figure is *edkfe*, the point *f* coincides with *d*, *e*, hence there is no stress in the member FE.

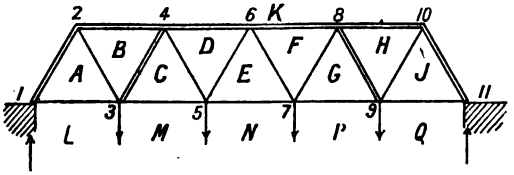
For the joint 7 the closed figure is *pmefgp*.
" " 8 " " " *qfkhg*.
" " 9 " " " *qpghjq*.
" " 10 " " " *jhkj*.
" " 11 " " " *qjkq*.

The kind of stress in each member may be ascertained by the method of Art. 386, and the results arranged in tabular form, as follows :—

Member .	BK, DK, FK, HK	AL, CM, EN, GP, JP	KA	AB
Kind of stress	all negative	all positive	−	+

Member .	BC	CD	DE	EF	FG	GH	HJ	JK
Kind of stress	−	+	0	0	+	−	+	−

It will be noted that all the members in the upper flange are in compression, all those in the lower flange are in tension, and of the



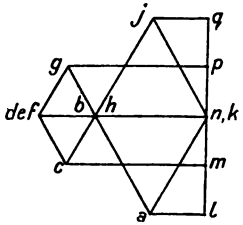


FIG 373.

diagonals, the two central ones have no stress, while those leaning towards the centre of gravity of the load (the vertical through 6) are in compression, and those leaning away are in tension.

In Fig. 373, members in compression (or *struts*) are represented by a double line.

393. A convenient method of deciding whether a member is in tension or compression is to consider the way in which the structure will distort if the member

is removed, *e.g.*, if the member 46 is removed, the girder will distort about 5 as a hinge, the points 4 and 6 will approach each other, and hence the member 46 must be in compression. Again, if 79 is removed, the hinge is 8, and distortion separates the points 7 and 9, hence member 79 is in tension. Similarly, removal of 34 produces a distortion in which points 3 and 4 approach each other, whereas removal of 45 produces distortion in which points 4 and 5 separate.

394. Fig. 374. If a great load is to be placed on a Warren girder, it is usual to distribute it over all the joints by means of tie connexions as shown in Fig. (*a*). The force diagram then takes the form Fig. (*c*). For the external forces we have the loads acting at joints 2, 3, 4, 5, 6, 7, 8, and the supporting forces TU, XP, each equal to half the total load. The force diagram for the external forces is *pq*, *qr*, *rs*, *st* (downwards), *tu* (upwards), *uv*, *vw*, *wx* (downwards) and *xp* (upwards).

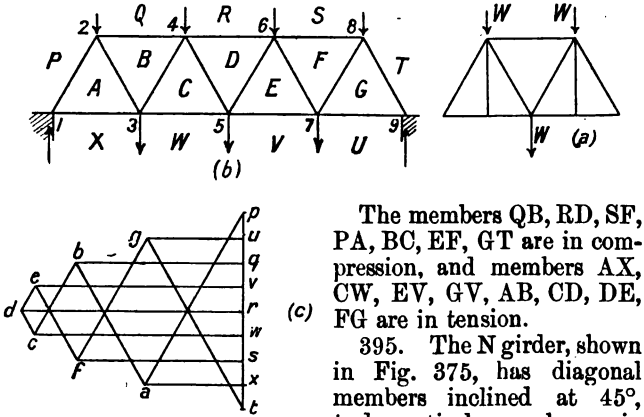
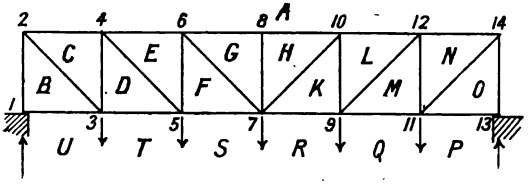


FIG. 374.

The members QB, RD, SF, PA, BC, EF, GT are in compression, and members AX, CW, EV, GV, AB, CD, DE, FG are in tension.

395. The N girder, shown in Fig. 375, has diagonal members inclined at 45°, and vertical members, in addition to the top and bottom flanges. The diagonals are inclined outwards from the centre when the number of bays is even, and when odd the centre bay is counterbraced. The force diagram is drawn as in the Warren girder. The external forces are the supporting forces UA, AP, and the loads PQ, QR, RS, ST, TU. By drawing the diagram for each joint, in the order 1, 2, 3, &c., . . . 14, we find for joint 1

a figure *uab*, and the points *u* and *b* coincide. Hence there is no stress in the member BU, and moreover, the supporting force at 1 is transmitted directly to the joint 2. For joint 2 we have a closed figure *bacb*, for 3 a figure *tubcdt*, for 4 a figure *dcaed*, for 5 a figure *stdefs*, for 6 a figure *feagf*, for 7 a



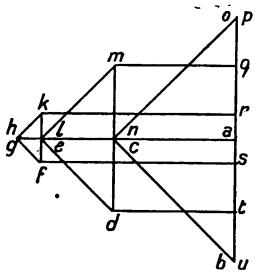


FIG. 375.

figure *rsfghkr* (note that *g*, *h* are coincident, hence there is no stress in the member GH), for 8 the figure is *hgah*. As the loading is symmetrical, the force diagram will be symmetrical about the central line *ahg*. There is no stress in the member OP, and the supporting force *ap* is transmitted directly by the member OA to the joint 14.

The function of the members GH, OP, and BU, in which there is no stress, is to preserve the rigidity of the structure under shock. Theoretically, the pin joint 8 should transmit either tension or compression, but any lateral shock on the members GA, HA would cause the collapse of the structure if the member GH were omitted. Similarly for the members AB, OB; any lateral force would move them from the vertical, if the members BU, OP were absent.

The kinds of stress are as follows :—

Compression: AB, AC, CD, AE, EF, AG, AH, KL, AL, MN, AN, AO;

Tension: BC, DT, ED, FS, FG, HK, KR, LM, MQ, NO, OP;

or, all members in the upper flange are in compression, all in the lower flange (except BU, OP) are in tension, all vertical members are in compression (except GH), and all diagonal members are in tension.

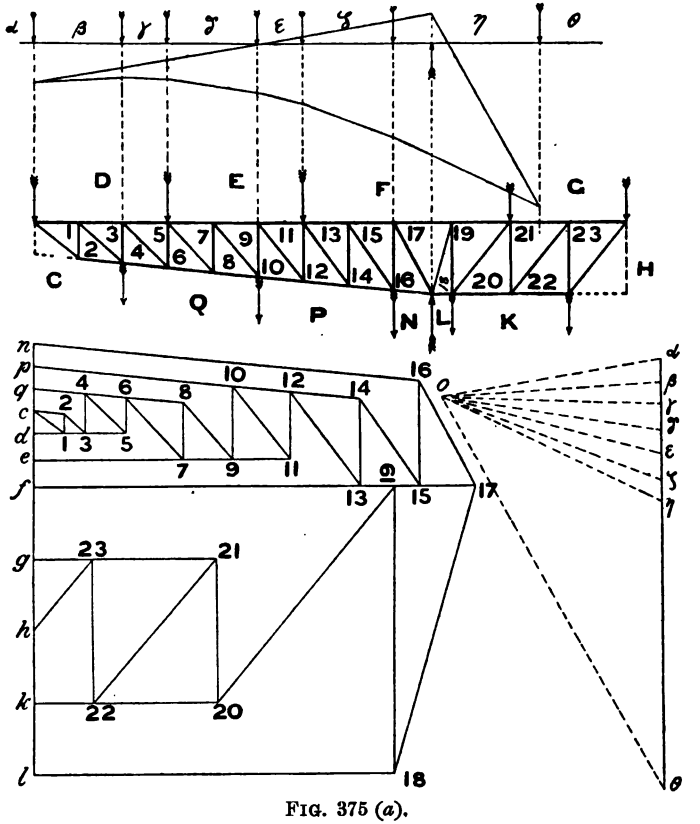


FIG. 375 (*a*).

Ex. The diagram (*Fig.* 375 (*a*)) *represents a swivel bridge, its pivot placed at* NL. *The loads, in tons, are placed, as indicated below, at* CQ, QP, *and* PN, *above at* CD, DE, *and* EF. *Determine the equal*

loads on the counterpoise to the right of NL, *placed above at* FG *and* GH, *below at* HK *and* KL. *For the construction of the funicular polygon, take* 1 *ton* = 0·4 *in. Find and write down the stresses in the members of the bridge.* (B. E. (Hons.), 1895.)

The truss is shown in thick lines in the Fig. The supporting force is NL. The loads are NP, PQ, QC, and CD, each 1 ton; DE and EF each 1½ tons. The four equal, but unknown loads on the counterpoise, are FG, GH, HK, KL.

The dotted members are useless and take no stress. To determine the four loads on the counterpoise, regard them first as a single load acting at their centre of gravity. Draw a funicular polygon, as shown, the loads being denoted as $\alpha\beta$, $\beta\gamma$, etc., and transferred to a horizontal line above the truss, to prevent confusion. Having drawn links parallel to $o\alpha$, -------- $o\eta$, we draw the closing line of the polygon from the line of force NL to line of resultant of the four loads on the counterpoise. From pole o draw $o\theta$ parallel to the closing line, thus determining $\eta\theta$ the sum or resultant of the four loads required. The stress diagram is now drawn, a different load line npq -------- l, being used to prevent confusion. This diagram should be begun at the point c. The stresses are as in the following table, and the four loads on the counterpoise are each 3·2 tons:—

Member .	D 1	D 3	D 5	E 7	E 9	E 11	F 13	F 15	F 17	F 19
Stress in tons	1·4	2·3	4·2	6·7	8·9	11·5	14·6	17·2	19·7	16 2
Member .	G 21	G 23	H 23	K 22	K 20	L 18	N 16	P 14	P 12	P 10
Stress in tons	8·2	2·6	4·1	2·6	8·2	16·1	17·3	14·7	11·5	9·0
Member .	Q 8	Q 6	Q 4	C 2	C 1	1, 2	2, 3	3, 4	4, 5	5, 6
Stress in tons	6·8	4·2	2·4	1·4	1·7	0·8	1·2	1·7	2·6	1·5
Member .	6, 7	7, 8	8, 9	9, 10	10, 11	11, 12	12, 13	13, 14	14, 15	15, 16
Stress in tons	3·7	2·5	3·4	3·2	4·2	3·0	5·2	3·9	4·7	4·7
Member .	16, 17	17, 18	18, 19	19, 20	20, 21	21, 22	22, 23			
Stress in tons	5·3	13·3	12·8	12·5	6·4	8·4	6·4			

Note.—Exercises are on p. 371.

CHAPTER XXIII.

MOMENTS.

396. **Moments.** Suppose a bar OA (Fig. 376) pivoted at O to be retained in a horizontal position by friction on the pivot O, the amount of the friction being adjustable. Let a force P applied at a distance x from the pivot, be just sufficient to cause the bar to rotate. If the friction on the pivot be increased, the force P will need to be increased say to P′ in order to produce rotation. Or if the original value P be retained, the distance x must be increased to x', at which distance P will produce rotation. From this experiment it will be evident that the tendency of a force to

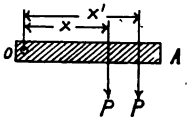


FIG. 376.

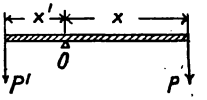


FIG. 377.

produce rotation about a point depends upon (1) the magnitude of the force, (2) the perpendicular distance of the point from the line in which the force acts. The tendency to produce rotation about O may thus be measured by the product of the force P and the perpendicular distance x from O on to the line of action of the force, *i.e.*, by $P \times x$, and this product is called the **moment** of the force P about O. If the rotation produced is clockwise, it is usual to call the moment a **positive** one, and, if counterclockwise, to call it **negative**. Thus, if P and P′ (Fig. 377) are forces acting at distances x and x' respectively from O, the moment of P is $+ Px$, and the moment of P′ is $- P'x'$. If $Px = P\,x'$, it is evident that there will be no rotation about O, and this may be expressed as $Px - P'x' = 0$, or, *the* **algebraic sum** *of the moments is zero when the body acted upon is in equilibrium.*

397. *If a number of forces acting upon a body tend to produce rotation, the net turning effect is obtained by finding the algebraic sum of all the moments. The body will be in equilibrium if this sum is zero.*

An apparatus for demonstrating this fact consists of a number of connected pulleys of radius OA, OB, OC, OD (Fig. 378), to which cords are attached, the cords passing over small pulleys at 1, 2, 3, 4, and supporting weights P_1, P_2, P_3, P_4. The cords are always tangential to the pulleys, hence each force P_1, etc., acts at a *perpendicular* distance from O equal to the radius of the corresponding pulley. If the weights P_1, etc., be adjusted so that the pulleys are in equilibrium, it will be be found that

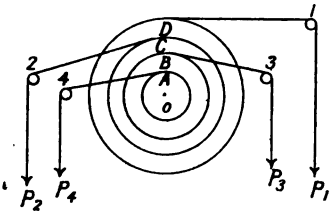


FIG. 378.

$$(P_1 \times OD) + (P_3 \times OB) = (P_2 \times OC) + (P_4 \times OA).$$

398. The moment of a force P about a given point K may be found graphically as follows :—

Fig. 379. Draw a line *ab* to represent the force P (or AB, using Bow's rotation), to any suitable scale, say 1 inch = 5 lbs. Select any point *o* as pole.* Join *oa*, *ob*, and from any point V in AB draw lines VR, VS parallel, respectively, to *ob*, *oa*. Through K draw a line parallel to AB, meeting the *links* VR, VS in points R, S. Draw VN ⊥ RS.

The moment of P about K is $= ab \times NV = RS \times oH$,

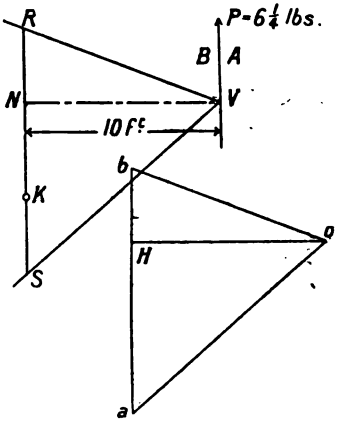


FIG. 379.

* For convenience, select O so that OH represents an integral number of lbs., cwts., &c.

ab being measured on a force scale, and NV on a distance scale. For, the triangles *aob*, SVR are similar; hence

$$\frac{ab}{RS} = \frac{oH}{VN}, \text{ or } ab \times VN = RS \times oH$$

$\therefore$ RS $\times$ *o*H gives the required moment.

The *length* RS will give the required moment, if measured on a particular scale called a *moment scale.*

To determine a Moment Scale. An example will illustrate the method. Suppose we require moment in lbs.-ft. units. In Fig. 379 we have used a linear scale of 1 in. = 10 ft. for the position diagram. The polar distance OH, being 1 in., represents 5 lbs. on the *force* scale. The moment scale will be 1 in. = 10 ft. $\times$ 5 lbs., or 1 in. = 50 lbs.-ft. The rule is—*Multiply the linear scale of the position diagram by the value of the polar distance measured on the force scale.*

The line RS measures $1\frac{1}{4}$ ins., thus the required moment of P about K is $(50 \times 1\frac{1}{4}) = 62{\cdot}5$ lbs.-ft.

399. Fig. 380. *To find the sum of the moments of two forces acting in the same direction about K*, we first draw a force diagram *abc*, and choose a pole *o* so that OH is an integral number of lbs. on the force scale. Then from any point V in AB draw two links VS, VR parallel respectively to *oa*, *ob*, and by drawing through K a line parallel to AB, we find RS, which gives, to a certain scale, the moment of AB about K. To find the moment of BC about K, produce RV backwards to meet the force BC in W, and draw links from this point parallel to *ob* and *oc* respectively. The link WR already drawn is parallel to *ob*; draw WT parallel to *oc*, then RT represents the moment of BC about K to the same scale that SR represents the moment of AB. Hence ST represents the sum of the

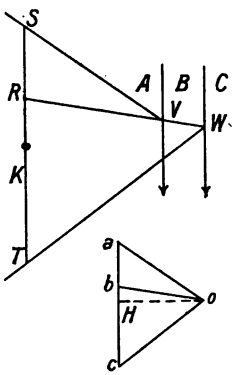


FIG. 380.

moments of AB and BC about K. This process can be extended to any number of forces.

400. If forces act in opposite directions, the algebraic sum of the moments about a point K is determined by a method similar to that of Art. 399. In this case a sign + or − must be given to each length SR, RT, etc., according as the force tends to produce clockwise rotation or the contrary.

Ex. Fig. 381. *Forces* AB, BC *act downwards, and* CD *upwards. To determine the algebraic sum of the moments of these three forces about* K.

Draw first a force diagram *abcd*, and choose a suitable pole *o*. From V in AB draw VR, VS parallel to *oa*, *ob*, to meet the line through K, then RS (+) is the moment of AB about K. Similarly, produce SV to meet BC in W, and from W draw WT parallel to OC, then ST (+) is the moment of BC about K, and RT (+) is the sum of the moments of AB and CD. Now produce TW to meet CD in X, and from X draw XU parallel to *od*. Then TU (−) is the moment of CD about K. All these moments are to the same scale. Since TU exceeds RS + ST by an amount RU, the algebraic sum is a moment given by RU, and it is negative, for RU is measured upwards.

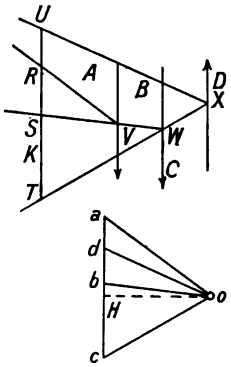


FIG. 381.

401. **Bending moment.** *If one or more external forces act on a beam, the sum of the moments of all these forces about any point in the beam is called the bending moment at that point.*

A funicular polygon will give the bending moment at any point of a beam or structure if the forces acting upon it are parallel.

Consider a beam supported at each end (Fig. 382), and carrying loads AB, BC, CD. The reactions EA, DE may be determined by a funicular polygon *m*XWV*n* (Art. 378), care being taken to make the polar distance *o*H suitable for a

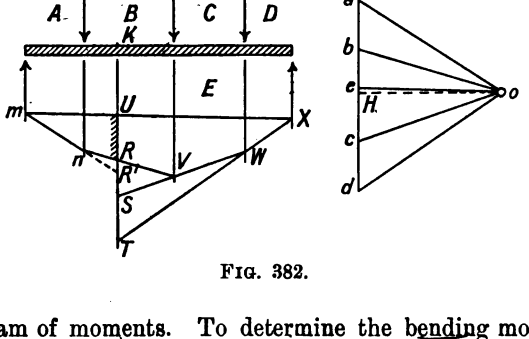


FIG. 382.

diagram of moments. To determine the bending moment at any point K on the beam, we may employ the construction of the preceding Art. to find the algebraic sum of the moments of BC, CD, DE which are the forces to the right of K. Thus draw the force diagram *bcde*, choose a suitable pole *o*, and draw from V in BC links VR, VS parallel to *ob*, *oc* ; produce VS back to meet CD in W, and from W draw links WS, WT parallel to *oc*, *od* ; produce TW to meet DE in X, and from X draw links XT, XU parallel to *od*, *oe*. Then RU, which is negative, gives the algebraic sum of the moments of BC, CD, DE about K.

In finding this value, it will be observed that we use the sides *n*V, VW, WX, X*m*, of the funicular polygon for the forces, and further, that the line RU, representing the algebraic sum of the moments, is the intercept of a line through K cut off by the funicular polygon. Hence, *the funicular polygon gives the bending moment at any point to a scale similar to that for finding moments, i.e.*, if the scale for the beam is $\frac{1}{10}'' = 1$ ft., and if the polar distance *o*H is, say, 5 lbs. on the force scale, then the bending moment scale is $\frac{1}{10}'' = (12 \times 5) = 60$ lbs. ins. or $1'' = 600$ lbs. ins. RU evaluated on this scale gives the bending moment at K.

It will be readily seen, from the diagram, that the same result would be obtained if we took the forces to the left of K, for R'U would give the moment of force EA, and R'R the moment of force AB, hence the difference RU is the actual bending moment at K.

402. **Couples.** Two parallel forces P (Fig. 383), equal in magnitude, but opposite in direction, and not acting in the same straight line, constitute a *couple*. The moment of

a couple is equal to the product of either force and the perpendicular distance between the forces. The moment of a couple about any point in the same plane as the couple is constant and equal to the moment of the couple. This is readily seen from Fig. 383, for considering any point *o*, the moment of P at A is positive and = P × OA, and the moment of P at B is negative and = P × (OB). The algebraic sum is P (OA − OB) = − P × AB = − P*x*, which is the moment of the couple.

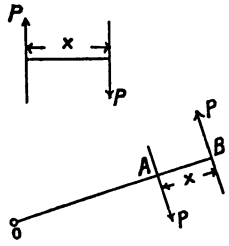


FIG. 383.

403. **Graphical determination of the moment of a couple.** Fig. 384. Let the two forces be AB, BC equal in magnitude but opposite in direction. The force diagram is *ab*, *bc*. Choose a suitable pole *o*, and join *o* to points *b* and *ac*. To find the moment of force AB about K, take any point V in AB and draw links VR, VS, parallel, respectively, to *oa*, *ob*. Then the moment of AB is given by RS. To find the moment of BC, produce RV to W, and choose W as the point in BC from which links WR, WT are drawn parallel, respectively, to *oc*, *ob*. Then TR is the moment of BC about K, and TS (which is TR − RS) is the moment of the couple about K. Since VS and WT are parallel, each being parallel to *ob*, it will be seen that the moment of the couple is constant whatever be the position of the point K.

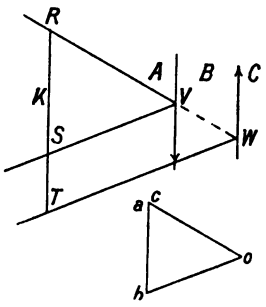


FIG. 384.

Exercises.

(1) Find graphically the moment of a force AB of 12 lbs. about a point K distant 8 ft., and verify your result by calculation.

(2) Two parallel forces of 3 and 9 lbs. respectively and 2 ft. apart act about a point K distant 5 ft. from the nearer force of 3 lbs. Find

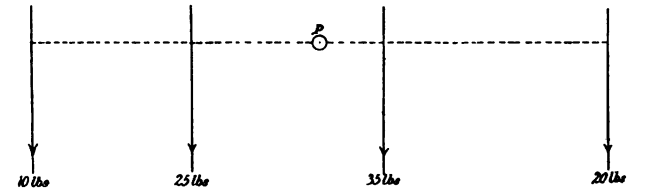
the resultant moment graphically when the forces have (a) the same sense, (b) opposite senses.

(3) Take the data of Exercise 2 (a) and add a third force of 7 lbs. distant 6 ft. from K, and on the opposite side to the forces of 3 and 9 lbs. The three forces are parallel and have the same sense. Find the resultant moment about K.

(4) A beam 12 ft. long is supported at each end and carries loads of 5, 12, and 8 cwts. distant 3, 7, and 9 ft. respectively from the left-hand end. Find graphically the bending moment at the centre, and draw a diagram from which the bending moment at any point of the beam can be determined.

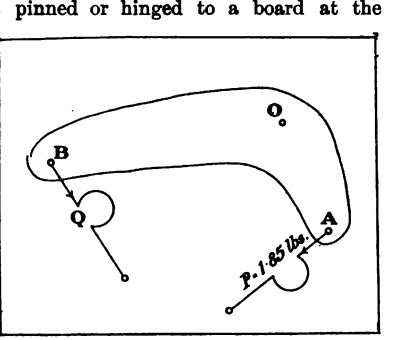
(5) Find graphically the moment of a couple consisting of two forces of 3·7 lbs. distant 1·8 ft.

(6) Four parallel forces act round a point P. Find graphically, and write down in foot-pounds, their resultant moment. Also draw



a scale of moments. Take, by preference, moment base or polar distance = 40 ft. Scale of length, 15 ft. = 1 in. Scale of forces, 15 lbs. = 1 in. (B. E., 1900).

(7) A cardboard lever is pinned or hinged to a board at the point O. Forces P and Q are applied to the lever as shown. The linear scale of the drawing being $\frac{1}{4}$, and the magnitude of P being 1·85 lbs., what is the moment of P about O? Find the magnitude of Q if the lever is balanced. Show a force which, being applied to the lever near O, shall relieve the pin from all pressure, and allow of its being removed without disturbing the equilibrium.



(B. E.)

CHAPTER XXIV.

KINEMATICAL DIAGRAMS; MOMENTUM; VECTOR PRODUCTS.

404. **Displacement.** If an object move from a position A to a position B, either along a straight line or curve, the path of the object is a displacement diagram. If the path of the object is a straight line, the direction of motion is constant. If the path is a curve, the direction of motion is variable, but at any given instant and for an indefinitely small period of time the direction of motion *is along the tangent* of the curve. Thus if a point P move from A to B along the curve APB (Fig. 385), the direction of the point at the instant it occupies the position P is along the tangent PT.

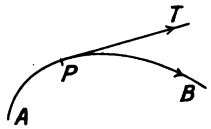


FIG. 385.

405. **Linear velocity** is the rate of motion of an object along its path. Thus velocity involves both *space* and *time*, and is measured in terms of compound units of space and time, *e.g.*, feet per second, miles per hour, etc.

406. **Uniform and uniformly variable velocity.** If the rate of motion of an object is constant during successive unit periods of time, however small, its velocity is constant or uniform; if the rate of motion increases or decreases by equal amounts with successive unit periods of time, the velocity is uniformly variable.

407. **Average or mean velocity.** The velocity of an object may vary considerably during a definite period of time. The space traversed divided by the time occupied between the extreme limits of motion gives the average or mean velocity; *e.g.*, a train running from Paddington to

Plymouth, a distance of 226½ miles, in 4 hrs. 7 mins. has an average velocity of 55·02 miles per hour, yet this may not be the actual velocity of the train for any finite period of the run.

408. **Acceleration** and **retardation** are rates of change of velocity, acceleration being an increase, and retardation a decrease, in rate.

409. **Rotation.** Consider a thin plate (Fig. 386) fixed to a flat table by a pin passed through a point O, and let the plate rotate about the pin O as centre ; any point in the plate, or rigidly connected with it, will move in a circle about O. The plate is said to have a motion of *simple rotation.* The point O is called the **permanent centre.**

410. **Angular velocity.** As the plate, Fig. 386, rotates about the centre O, a point A moves in a circle, and during one complete revolution or cycle of the movement, a line OA will sweep through an angle of 360°, or 2π radians, and if t seconds represents the period of time occupied in one revolution, then, dividing either 360° or 2π radians by t, we obtain the velocity of the point A in degrees or radians per second of time, *i.e.,* the *angular velocity of the point.* Angular velocity may, or may not, be constant. If the velocity varies, the average velocity between given limits is found by dividing the angle traversed by the time, as in the case of linear velocity.

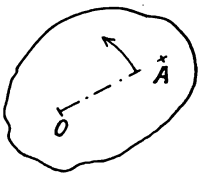


FIG. 386.

411. All points in the plate (Fig. 386) have equal angular velocity about the centre O, but the circumferential or linear velocity of points varies as the length of their paths, *i.e.,* directly as the distances of the points from the centre O. Thus for any point A we have

Linear velocity = angular velocity × radius OA.

From this it follows that when the angular velocity of any point in a body which has simple rotation is known, the angular velocity of every point in the body is given, and the linear velocity is easily determinable.

If V = linear velocity, θ = angular velocity, and R = radius, the above is expressed, $V = \theta R$.

If R = 1, we have $V = \theta$, or, in words, *the angular velocity of a point is numerically equal to the linear velocity of a point in the body at unit distance from the centre of rotation.*

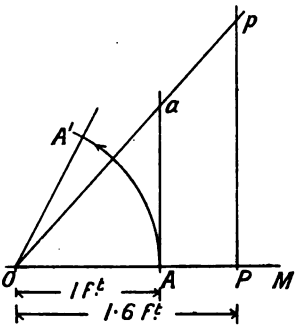


FIG. 387.

Ex. Fig. 387. *A fly-wheel of radius 1·6 ft. rotates with a uniform angular velocity of α° per sec. Represent the linear velocity of a point on the outer rim of the wheel.*

On a straight line OM make OA = 1 in. (or any convenient unit), and describe an arc AA'. Draw $Aa \perp$ OM, and draw OA', making $\alpha°$ with OM and intersecting AA' in A'. Make Aa = arc AA'. Then Aa measured to the scale used in the drawing gives the linear velocity of a point A at unit distance from O. Make OP on OM = 1·6 in. Draw $Pp \perp$ OM, meeting Oa in p, then Pp gives the linear velocity of the point P on the outer rim, for $\frac{Pp}{OP} = \frac{Aa}{OA}$.

412. Consider a link AB, Fig. 388, constrained to move in a plane POQ in such a manner that the end A moves along PO and the end B along OQ. In moving to a new position, the link has a rotary movement, since any new position the link may occupy is not parallel to the position AB. Each small element of rotation of the link takes place about some point as centre in the plane in which the link is moving, and for every position of the link this centre can be found. It is called the **instantaneous centre of rotation** or **virtual centre.**

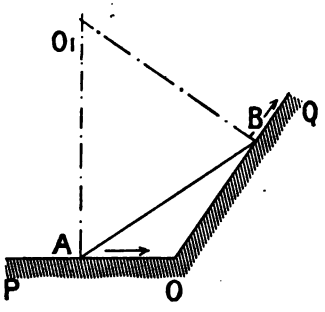


FIG. 388.

To find the instantaneous centre corresponding to the position AB, we may regard the point A as moving, at the instant, along a circle whose centre is somewhere along the perpendicular AO_1 to PO. (*See* Art. 128.) Similarly, we may regard the point B as moving, at the same instant, along a circle whose centre is somewhere along the perpendicular BO_1 to OQ. The intersection O_1 of the two perpendiculars is the required instantaneous centre for the position AB of the link, and it can always be found, whatever the motion of the link may be, provided we know the paths of two points in the link. It will be observed that, for each position of the link AB, we have a different point O_1 as the centre about which rotation takes place; hence the term *instantaneous* centre, as the link only rotates about this centre for an indefinitely small period of time.

All the relations that exist between the parts of a body which has simple rotation and the permanent centre also exist between a body which has complex motion and its instantaneous centre.

All points rigidly connected with AB have the same angular velocity about the point O_1 and linear velocities *proportional* to their distances from the centre O_1. This important principle enables many complex motions to be reduced to an equivalent simple rotation, and it is extensively used in dealing with problems involving the velocity of particular parts of mechanisms.

413. **Graphical representation of velocity.** *Ex.* 1. Let the link MN in the four-link chain shown in Fig. 389 be fixed. We require the direction of motion of the point Q in the link DR.

If the chain move within the limits of constraint, the point R moves in a circle about centre N. Similarly D moves in a circle about centre M. We now desire to know the position of the point about which the link DR is turning when it occupies a given position. For the position shown in the diagram, the direction of point R is perpendicular to the radius NR; the direction of D is perpendicular to MD; and the point O_1 in which DM, RN intersect is the point about which the link DR is turning at the instant. The

direction at the instant of any point, as Q, in the chain is perpendicular to O_1Q.

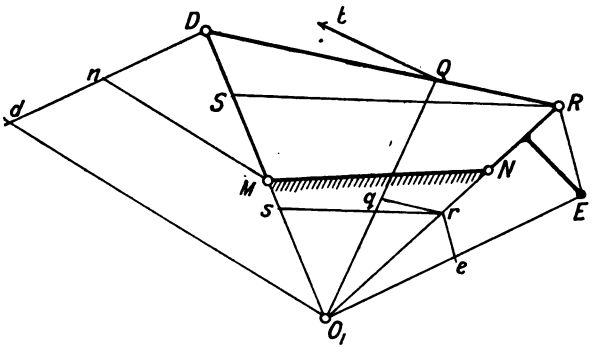


FIG 389.

The student will find it interesting and instructive to make a drawing of the chain showing the link DR at different periods of the motion, and by the methods of Chap. XVI. draw the locus of the instantaneous centre for one complete cycle. (In working this exercise make the link MN short relative to MD and NR.)

The linear velocity of any point in the chain shown in Fig. 389 is easily determined when the linear velocity of one point is known. Let the velocity of the point R be given. Determine O_1 the instantaneous centre corresponding to the given position of the chain. On RO_1 make Rr any convenient length to represent the velocity of the point R.

For the velocity of the point Q, join QO_1 and draw $rq \parallel RQ$, meeting O_1Q in q. Lhen Qq measured to the scale used for Rr gives the velocity of Q, and $Qt = Qq$ is a vectorial representation of the instantaneous velocity of the point Q.* Similarly for the point S. Join RS and

* It must be distinctly understood that when we say Qt represents the linear velocity of Q at the instant, we do not mean that Q moves from Q to t in one unit of time, since the velocity of the point may be rapidly varying and be greater or less immediately before—and after the instant under consideration, but rather Qt represents the direction in which Q is moving at the instant under consideration, and is the distance Q would move through if it continued at the same velocity through a unit period of time.

draw *rs* ‖ RS. Then S*s* gives the velocity of point S. The velocity of any point as E, not in either link but rigidly connected to a link, is found with equal facility. Thus join EO_1. Draw *re* ‖ RE. Then E*e* gives the velocity of the point E.

The **angular velocity** of any link in the chain is determined by similar constructions. Thus let the angular

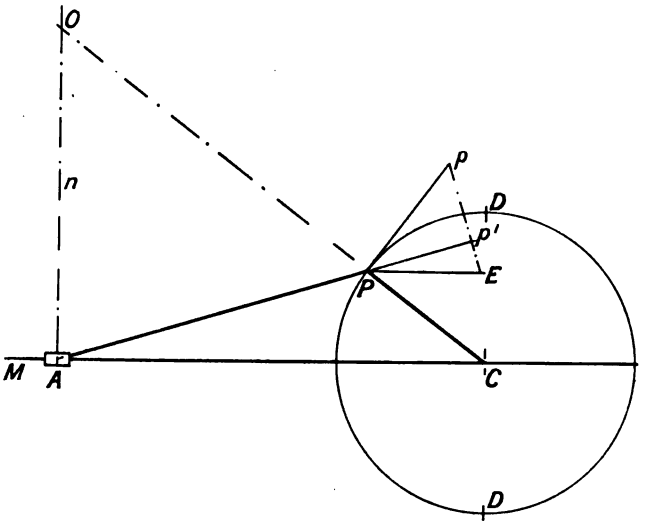


FIG. 390.

velocity of the link MD be given. Draw D*d* in any direction and of any length to represent the numerical value of the angular velocity of link MD. Join dO_1 and draw M*n* ‖ O_1d. Then D*n* measured to the scale used for D*d* gives the angular velocity of the link DR.

Ex. 2. In the mechanism shown in Fig. 390, the point A moves in the straight line MC, and the point P moves in a circle about C; therefore the point O in which CP intersects A*n* ⊥r AC is the instantaneous centre of the link AP corresponding to the position shown in the diagram. The

direction of motion of any point in the link AP is perpendicular to the radial from O to the point.

Ex. Determine the locus of the instantaneous centre O of the link CP for one complete cycle.

The linear velocity of points attached to the chain may be found as in the preceding example. The velocity of the slider A is, however, more conveniently found as follows :—

Set off the velocity of P as the vector Pp ; projecting this vector on to AP we obtain Pp', the component of Pp in the direction of the link PA.

Since every point in PA must have an equal component

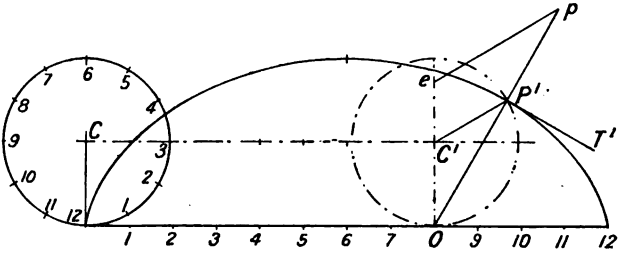


FIG. 391.

along the link, the velocity of A must be such that when projected on AP the component is Pp' ; hence we draw PE ∥ AC, meeting the perpendicular from p on to PA in E. Then PE gives the velocity of A.

If we investigate the velocity of A (Fig. 390) for different positions of the link AP, we see that the length of PE (the velocity of A) will depend upon the length of Pp' and also upon the angle EPp'. Now Pp' is a maximum when CP is ⊥r PA, for Pp' then = Pp ; but the angle EPp' is a maximum when CP is ⊥r CA. It can be shown that the velocity of A has its maximum value when the crank CP is approximately ⊥r to the link PA.* When P falls on AC, or AC produced, the projection Pp' is zero, hence the velocity of A is zero at each end of its stroke.

If the link AP is infinitely long, we get the mechanism of Fig. 394, and AP is parallel, in every position of P, to AC. The maximum

* The difference is a small angle, the sine of which is $\frac{1}{n(n^2+2)}$ nearly, where n is ratio PA/CP.

velocity of A then occurs when CP is vertical or $\perp^r$ CA, and the zero velocity at each end of the stroke.

Ex. 3. *The instantaneous centre of a rolling curve is the point of contact of the curve and its track.* Thus in Fig. 391, O is the instantaneous centre corresponding to the point P′ on a rolling circle which describes the cycloid shown, and P′T′ perpendicular to P′O is tangent to the cycloid, and gives the direction of motion at that point. To determine the velocity of a point, as P′, on the rolling circle when the velocity of a point is known, let the linear velocity of the centre C′ be given. Join C′P′, P′O. Make C′*e* any length to represent the given velocity of C′. Draw *ep* ∥ C′P′ meeting OP′ in *p*. Then P′*p* measured to scale gives the linear velocity of P′.

414. **Simple harmonic motion.** Fig. 392. If a point P moves in a circular path with a constant velocity, its projection M on a diameter of the circle moves with a motion which is called *simple harmonic motion*. Let a counter-clockwise rotation of P be regarded as positive and an opposite rotation as negative. Let the initial position of P coincide with C; the point M then occupies its mid position. As P rotates in a positive direction, M will, in the first quarter revolution, be displaced to the right from O to A; in the second quarter revolution M moves back to O; in the third it moves to the left from O to B, and in the fourth finally moves back from B to O. Displacement of M to the right of the mid position is regarded as positive, and to the left negative, the maximum displacement OA or OB being called the "**amplitude.**"

A graphical construction for the displacement of M from the mid position for any angular position of OP is shown in Fig. 392. Set out along OC (produced downwards) any suitable length KX to represent 2π radians (*i.e.*, one complete revolution). Divide the path of P into 12 equal parts, commencing from C, and divide KX into 12 equal parts, and number as shown. Displacements will be measured perpendicular to KX. When P is at 1, the distance of M from O is obtained by the intersection I of 1I, 1I respectively parallel and perpendicular to KX. Similarly, lines through corresponding points 2, 2, etc.; on the line KX

and circle CPB give other points, and a fair curve drawn through the twelve points so obtained is a curve of displacements for the point M from its mid position. This curve is a **sine curve**, for in any position of OP, the displacement OM = OP sin. COP.

Let the angle turned through per second by OP measured in radians be q (*i.e.*, the *angular* velocity of OP is q), and suppose that OP has been rotating for t seconds. It has thus rotated through qt radians from its initial position. If OP coincided with OC initially, then in t seconds the position of OP is inclined at an angle qt radians to OC, but if OP were already at an angle COP or α radians from OC initially, at the end of t seconds the angle between OC and OP would be $(qt + \alpha)$ radians. If a = amplitude, x = the displacement at time t, we have $x = a \sin (qt + \alpha)$. This expression becomes $x = a \sin (\theta + \alpha)$ if $\theta = qt$, the angle which OP has turned through from its starting position. The angle α between OC and OP at the instant from which time is reckoned (known as the **epoch**) is called the **angle of advance**,* and the position of OP when $t = o$, is called the **representative crank**.

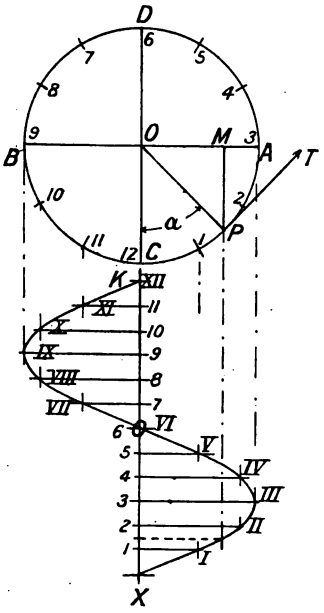


FIG. 392.

It should be noted that if OP starts from OC, the angle of advance $\alpha = o$, and the initial displacement of M is zero.

The ratio of the angle COP to a complete revolution, *i.e.*,

* This angle is frequently called the angle of epoch, or simply the epoch.

the ratio $\frac{(qt + a)}{2\pi}$ (since there are 2π radians in one revolution) is called the **phase** at the time t, and this constitutes a further method of specifying the position of M at any instant. If two simple harmonic motions are under consideration (*e.g.*, in connection with alternating electric currents) the angle between the two representative cranks is called the **lead** when measured forwards or in the positive direction of rotation, and **lag** when measured backwards or in the negative direction.

It will be readily seen that if T seconds is the time taken by OP for one complete revolution, and that if q is the angular velocity, then $T = \frac{2\pi}{q}$ seconds; T is called the **periodic time**, or the time of one *complete* vibration of M. Further, if f is the number of complete vibrations per second, called the **frequency**, then $f = \frac{1}{T} = \frac{1}{\frac{2\pi}{q}} = \frac{q}{2\pi}$. The expression $x = a \,.\, \text{sin.}\, (qt + \alpha)$ for displacement of M, may thus be written—

$$(1)\quad x = a \,.\, \text{sin.} \left(\frac{2\pi t}{T} + \alpha\right) \quad \ldots \ldots \left\{ \text{for } q = \frac{2\pi}{T} \right\}$$

or $$(2)\quad x = a \,.\, \text{sin.}\, (2\pi f t + \alpha) \quad \ldots \ldots \left\{ \text{for } q = 2\pi f \right\}$$

415. Consider now the velocity of M at any instant. As PM is perpendicular to AB, the velocity of M is equal to the component velocity of P along AB. Let PT represent the constant velocity of P, then its projection on AB will give the velocity of M at any instant. Now when P is at either A or B the projection of PT becomes a point, whereas when P is at either C or D it projects its true length on AB; hence the velocity of M is zero at each extreme position, and a maximum when passing through the mid position.

The point M has a retardation from O to A, an acceleration from A to O, a retardation from O to B and an

acceleration from B to O. It can be shown * that if a is the amplitude, α the angle of advance, and qt the angle turned through by OP from OC, then as $x = a \,.\, \text{sin.}\, (qt + \alpha)$, velocity $v = a.q.\, \text{cos.}\, (qt + \alpha)$, and acceleration

$$v' = -\, a.q.^2 \,\text{sin.}\, (qt + \alpha) = -\, q^2 x.$$

By giving to t various values, and knowing a, q, and α, we can calculate values of x, v, and v' corresponding to values of t, and plot three curves on a time base which will give respectively displacement, velocity, and acceleration of M. As we have already seen, the curve for x is a

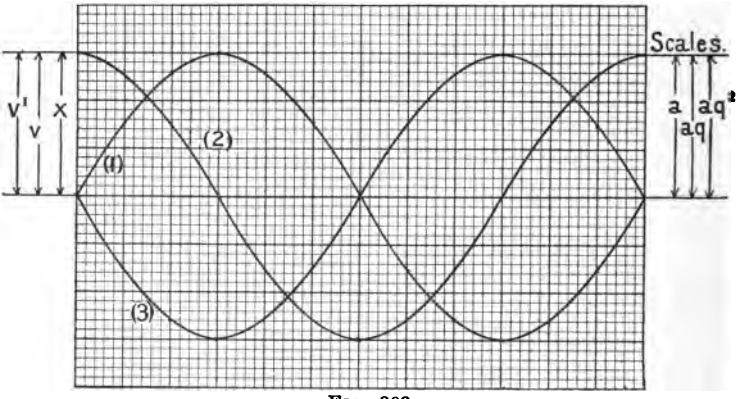


FIG. 393.

sine curve, and by plotting it is seen that the curve for v is a cosine curve, and that the curve for v' is also a sine curve having a phase $= \frac{1}{2}$ relatively to the curve for x.

The three curves are shown at (1), (2), and (3), Fig. 393, the scales for ordinates being in the ratio a, aq, and aq^2.

* For students acquainted with the calculus,

Let x = displacement

Then $\frac{dx}{dt}$ = velocity and $\frac{d^2x}{dt^2}$ = acceleration.

Hence $x = a \,.\, \text{sin.}\, (qt + \alpha)$

Velocity $\frac{dx}{dt} = aq\, \text{cos.}\, (qt + \alpha)$

Acceleration $\frac{d^2x}{dt^2} = -\, aq^2 \,\text{sin.}\, (qt + \alpha).$

$= -\, q^2 x$

416. A practical example of a mechanism in which one part moves with simple harmonic motion occurs in a donkey pump, or a small slotting or shaping machine shown diagrammatically in Fig. 394. The crank r carries a block which works in a slotted link A, guided so as to reciprocate parallel to OB as the crank rotates. The cutting-tool or the pump piston is carried by, or connected to, the slotted link. If θ is the angle AOB, then $r.\cos.\theta$ gives the displacement of the tool, etc., from the mean position, and $r\ (1 - \cos.\theta)$ gives the displacement from either extreme position.

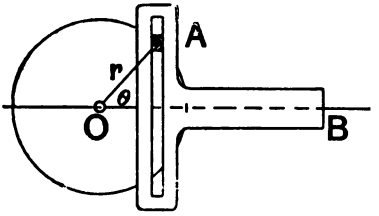


FIG. 394.

417. In a steam engine mechanism, the piston and crosshead have a motion which approximates to a simple harmonic motion, and approaches more nearly to a true simple harmonic motion as the length of the connecting-rod increases, hence a curve showing the displacement of the piston for angles turned through by the crank will *resemble* a sine curve in form. The mechanism shown in Fig. 394 is equivalent to a connecting-rod of infinite length.

418. Fig. 395. To draw a curve giving the displacement of the piston or crosshead in a steam engine, from its extreme, or from its mean position, for any angle turned through by the crank.

Let OP be the crank in its initial position, PC the connecting-rod, and C the crosshead. Let the crank rotate counter-clockwise. Divide the crank-pin circle P 3, 6, 9, into 12 equal parts. With points 1, 2, 3, etc., as centres and radius PC, determine points C_1, C_2, etc., and then with centres C_1, C_2, etc., and the same radius PC, draw the arcs

1*a*, 2*b*, etc. Then P*a*, P*b*, etc., give the crosshead or piston displacement from the *extreme* position C, when the crank-pin has made $\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{4}$, etc., part of a complete revolution, *i.e.*, when the crank has turned through 30°, 60°, 90°, etc., from the position OP. Draw through P a line AB

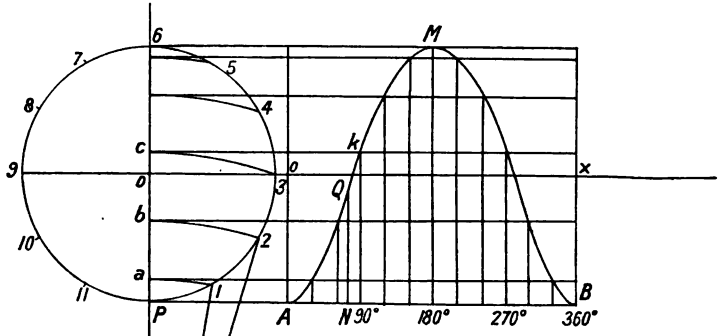
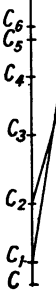


FIG. 395.



perpendicular to CP, and from A mark off equal divisions to represent angles turned through by the crank, and draw perpendiculars as shown, and then drawing through *a*, *b*, etc., lines parallel to AB intersecting these perpendiculars we obtain points on the **displacement curve**, thus lines through corresponding points *c*, and 90° intersect in a point *k* on the required curve. The points so obtained for a complete revolution of the crank are joined by a fair curve AMB, and from this curve the piston displacement for any angle turned through by the crank can be obtained. This displacement multiplied by the sectional area of the cylinder gives the volume of the steam in the cylinder for the corresponding position of the crank.

Ex. Let the stroke P6 of the piston be 2 ft. What is the displacement when the crank has turned through 75°? If the

cylinder is 10″ diameter, what is the volume of the steam for this position of the crank.

QN = piston displacement = 0·8 ft.

$$\text{Volume of steam} = QN \times \frac{\pi \times 5 \times 5}{144}$$

$$= 0{\cdot}44 \text{ c. ft.}$$

(2) The curve of piston displacements from the mid position of the piston is obtained by taking the line *ox* parallel to AB through *o* as the base line. The curve AMB is then the required curve, for when P*b*, for example, is the displacement from one extreme position, *ob* must be the displacement from the mid position. Ordinates below the base line then represent displacements to one side of the mid position, and ordinates above represent displacements to the other side. It should be noted that the mid position of the piston *does not correspond* to a rotation of 90° or 270° of the crank from its initial position.

419. **Steam engine valve mechanisms.** In the common form of steam engine, the passage of steam to and from the cylinder C is controlled by a slide valve V, which

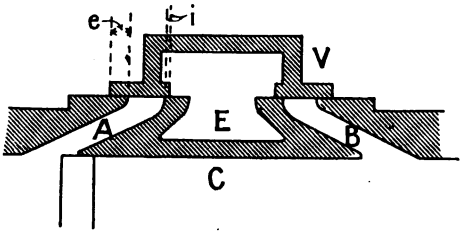


FIG. 396.

is shown in Fig. 396 in its mid position covering completely both ports A and B. If the valve moves to the right, as shown in Fig. 397, steam can pass through the port A to the left of the piston ; the piston is thus moved to the right, and steam already to the right of the piston can pass by the port B to the exhaust port E. If the valve V moves to the left of the mid position, steam is admitted to the right of the piston, and the left is opened to exhaust. The motion of the piston is transmitted by the crosshead, connecting-

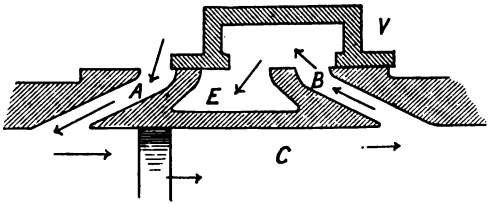


FIG. 397.

rod, and crank to the crank shaft, and on the crank shaft is an eccentric (equivalent to a very short crank) connected by a relatively long rod to the valve V. As the eccentric rod is long relatively to the crank equivalent of the eccentric, (*i.e*, the distance between the eccentric centre and that of the shaft upon which the eccentric disk is keyed), the motion of the slide valve can, without any sensible error, be regarded as simple harmonic ; we cannot, however, do this for the piston, since the connecting-rod is frequently not more than four or five times the length of the crank.

From a consideration of Fig. 398, in which the eccentric E is shown, for convenience, above the crank C*r*, it will be seen that the travel or stroke of the piston P is equal to twice the length of the crank C*r*, or = the diameter *ab* of the crank-pin circle, and also that the travel of the valve V is equal to the diameter of the eccentric-centre circle. Further, the piston occupies its extreme positions when the piston-rod, connecting-rod, and crank lie in the same line, and similarly for the valve when the eccentric rod ER and eccentric are in line. These positions are called "*dead points*," or the crank-pin is said to be on a "*dead centre*." The mid position of the valve occurs when the eccentric is perpendicular to the line of travel of the valve, but the mid

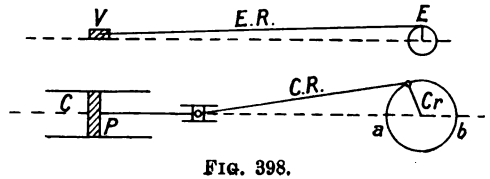


FIG. 398.

position of the piston does not occur when the crank is perpendicular to the line of travel of the piston. (See Art. 418.)

Since the steam must be admitted to, and exhausted from, the cylinder when the piston occupies definite positions in the cylinder, it will be seen that the relative positions in which the eccentric and crank are secured to the crank shaft is a matter of great importance. If a circle be drawn to represent the path of the eccentric centre, the points *a*, *b* (Fig. 399) correspond to the extreme positions of the valve, and the points *c*, *d* to the mid position shown in Fig. 396. The diameter *ab* is equal to the travel of the

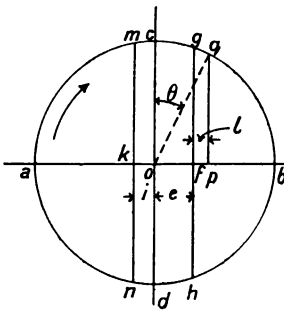


FIG. 399.

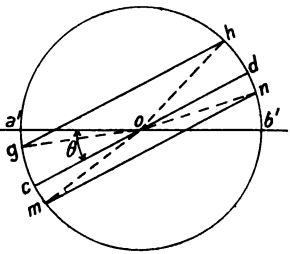


FIG. 400.

valve. In the mid position, the outer edge of the valve overlaps the port A by an amount *e*, called the *outside lap*; hence the valve must move to the left by an amount *e* before steam is admitted to the cylinder, and obviously steam will be cut off when the valve, in returning, is distant *e* from the mid position. Hence, if we mark off $of = e$, and draw $gh \perp^r ab$ through *f*, the points *g*, *h* represent, respectively, the position of the eccentric centre at "*admission*" and "*cut-off*"; *fb* will also represent the greatest opening of the valve for admission of steam, and the arc *gbh* gives the part of the revolution of the eccentric during which the admission of steam takes place.

The valve, in its mid-position, overlaps the inner edge of the port A by an amount *i*, called the *inside lap*; hence

the valve must move to the left of its mid position by an amount i before the port A is opened to exhaust, and exhaust will be cut off when the valve has returned to a distance i from its mid-position; hence, if we set off $ok = i$ to the left of o, and draw $mkn \perp ab$, the points n, m represent, respectively, the position of the eccentric centre at "*release*" and "*compression*," and ka represents the greatest opening of the valve to exhaust, and the arc nam is the part of a revolution during which exhaust takes place.

If the valve is required to admit steam exactly at the moment the piston arrives at the end of its stroke, *i.e.*, when the crank occupies a position parallel to oa, then the angle aog is the angle which must exist between the crank and the centre line of the eccentric. It is common, however, in practice to allow the valve to open slightly before the piston reaches the end of its stroke, say, by an amount l, called the *lead*. Set out $fp = l$, then q will give the position of the eccentric centre when the crank occupies the position oa, and the angle aoq is the angle which the eccentric must occupy relatively to the crank. The angle coq, by which the angle between the crank and eccentric exceeds 90°, is called the *angle of advance*. If r represents half the travel of the valve, it is seen that $r = oa = oq$, and that $op = (of + fp) = r \sin. \theta$, *i.e.*, outside lap + lead $= r \sin. \theta$.

As the crank occupies the position oa (Fig. 399) when the eccentric occupies the position oq, and is always an angle aoq behind the eccentric, it will be seen that if the diagram (Fig. 399) be rotated counterclockwise through an angle $(90° + \theta)$, so that the diameter dc now occupies the position shown in Fig. 400, then og, oh, on, om correspond respectively to the positions of the crank when admission, cut-off, release, and compression take place, and the corresponding positions of the piston can readily be found by using g, h, n, and m as centres, the length of the connecting-rod as radius, and cutting the line of dead-centres. Repeating this from centres a', b' will give the extreme positions of the piston, from which the displacements corresponding to cut-off, etc., may be measured.

420. **Valve diagrams** are ingenious graphical constructions by which investigations relating to the slide valve are made. One of the most useful is that of **Zeuner**, which depends upon the following :—

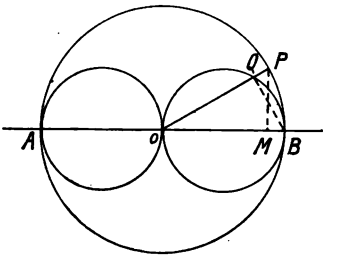


FIG. 401.

Let AB (Fig. 401) represent the valve travel, and the circle thereon the path of the eccentric centre. Upon AO and OB describe circles, and draw OP to represent any position of the eccentric. Then OM will be the displacement of the valve from the mid position ; and since OQB is a right-angle and OB = OP, we have $\frac{OM}{OP} = \frac{OQ}{OB}$; hence OQ = OM and also represents the displacement of the valve from the mid-position when the eccentric occupies the position OP. Now turn the diameter AB and the two

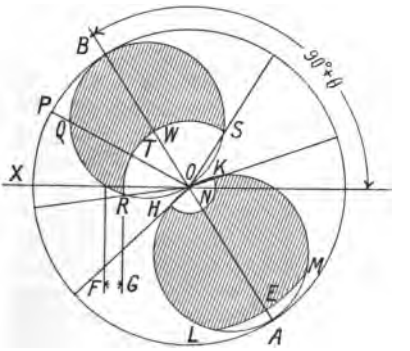


FIG. 402.

circles thereon backwards through an angle $(90° + \theta)$, so that the radials of the circle APB will correspond to

positions of the *crank*. (See Fig. 402.) The valve displacement from the mid position corresponding *to any position of the crank* is then obtained by drawing a radial to represent the crank, and measuring the intercept OQ on the valve circle to the scale upon which OB or OA was made equal to the half travel of the valve. The difference between the displacement from the mid position and the outside lap gives the valve opening to steam ; hence, if we draw an arc RS with centre O and radius = outside lap, the shaded area will give the opening to steam for any position of the crank, *e.g.*, QT is the valve opening when the crank has rotated an angle XOP from the inner dead centre. BW is the maximum valve opening. FG is the valve opening when the piston is beginning the stroke ; thus FG is the lead. OR is the position of the crank at admission, and OS at cut-off. With centre O and radius equal to the inside lap, draw an arc HK and make NE = width of cylinder port. Then the shaded area gives the opening to exhaust ; *e.g.*, NE is the maximum opening, OK is the position of the crank at release, and OH at compression.

421. **Oval diagram.** A useful diagram is Fig. 403, which represents valve displacements from the mid position in relation to piston displacements from one extreme position.

Let the curve ABC be drawn on a base AC representing a complete revolution of the crank, and let the ordinates be displacements of the piston from one extreme position, so that BD is the piston stroke, as in Art. 418. Draw a base line EF parallel to BD, and produce it to the right. Anywhere along EF produced draw a curve of valve displacements from the mid position corresponding to angles turned through by the crank from one dead position. (This curve is a sine curve having an amplitude = half travel of valve and an angle of epoch = θ, the angle of advance of the eccentric.) Points on the required oval curve are obtained by drawing through corresponding points on each curve ; thus, at 90° along ABC draw 90°—H parallel to DE, to intersect a line parallel to EF drawn through K on the valve displacement curve, the point K being that corresponding to 90° on that

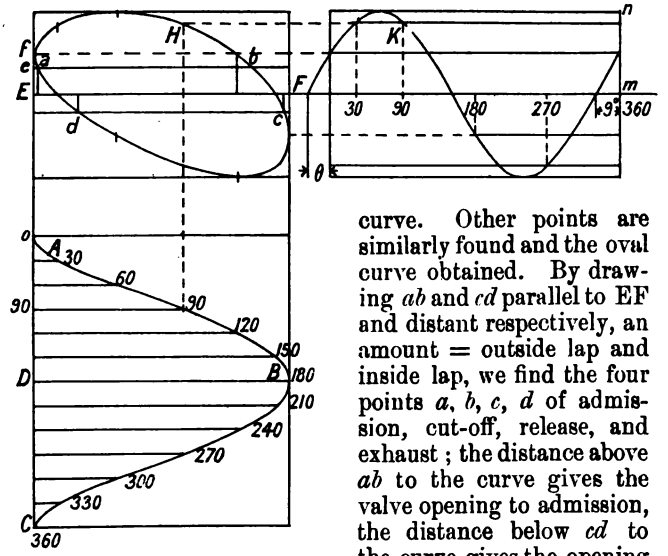


FIG. 403.

curve. Other points are similarly found and the oval curve obtained. By drawing *ab* and *cd* parallel to EF and distant respectively, an amount = outside lap and inside lap, we find the four points *a*, *b*, *c*, *d* of admission, cut-off, release, and exhaust; the distance above *ab* to the curve gives the valve opening to admission, the distance below *cd* to the curve gives the opening to exhaust, and *ef* is the lead.

Note.—The above method is applicable for finding the path of a point which has two simple harmonic motions simultaneously impressed upon it; *e.g.*, it may have a S.H. motion of amplitude *mn* and a certain frequency, and a second motion, approximately S.H., perpendicular to the first of amplitude BD and half the frequency. The resulting path would be a closed curve of figure 8 form, each half similar to the oval curve, and would be similarly obtained.

422. A diagram known as the **Harmonic Diagram** (Fig. 404) (due to Professor Osborne Reynolds) is extremely useful for slide valve investigations. Two curves are drawn, representing respectively the displacement of the piston and valve from their *mid* positions, on a base line 0—360, corresponding to a complete revolution of the crank from one dead centre.

The displacement curve ABCDE for the piston approximates to a S.H. curve, and the curve FGHK for the valve is a true simple harmonic curve, the phase of which is

(90° + θ) in advance of the piston curve, due to the angle of advance of the eccentric. The scale for the valve displacements must be exaggerated. The distance of *a b* from the base line is equal to the outside lap, and the distance of *c d* is equal to the inside lap. Then admission and cut-off occur at *a* and *b*, and by projecting upon the piston displacement curve, the piston displacement from the mid position is *e f*, or from one extreme position is *h f*. Scales may be drawn along *k l*, and *m n* to read percentages of stroke from extreme position if desired. The distance below *ab* to the curve FG gives valve opening to steam. The

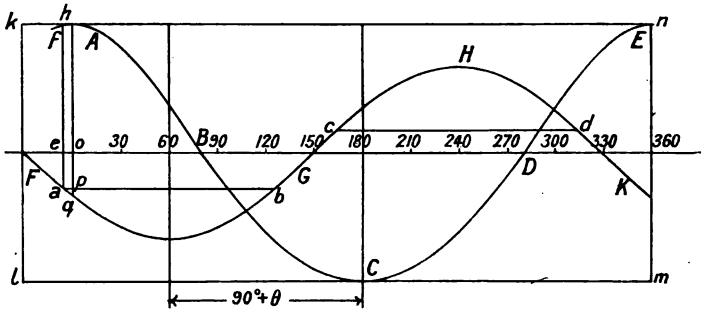


FIG. 404.

points *c* and *d* give release and compression, and they may also be projected on the piston displacement curve. Distances above *cd* to GHK give valve opening to exhaust. The angle turned through by the crank is also readily ascertained for cut-off, &c., from this diagram. *pq* is the lead, being the valve opening at the commencement of the piston stroke. To study the effect of varying the angle of advance, one of the two curves should be drawn on tracing-paper and moved about over the other curve.

423. **Momentum.** The momentum of a body is the product of the mass and the velocity of the body, the mass and velocity being expressed in suitable units, usually lbs. and ft. per second respectively. The unit of momentum is the momentum possessed by unit mass (usually a mass of 1 lb.) moving with unit velocity (usually a velocity of 1 ft.

per second). As velocity is a vector quantity (having sense as well as magnitude), it follows that momentum is a vector quantity, and hence it can be completely specified by a vector.

Now a body of mass m lbs. travelling in a given direction with a velocity of v ft. per second has mv units of momentum (each unit being of the order lb. $\times$ ft. $\div$ sec.), and it will continue to travel in the given direction with a velocity v, unless acted upon by some external force. The force may act in the direction of v, in which case the velocity, and consequently the momentum, are increased; or the force may act so as directly to oppose the velocity v, then the velocity and momentum are decreased; or the force may act so as to change the direction of motion only, leaving the velocity still $= v$. Any such force is said to be impressed upon the body.

The connection between force and momentum is obtained as follows:—

It can be shown, experimentally, that—

Force F in poundals = mass m (lbs.) $\times$ acceleration a (ft. per sec. per sec.);

or $F = ma$

If F is required in lbs., we have $F = \frac{ma}{g}$, g being the acceleration due to gravity.

Now acceleration a = change of velocity per sec. = $\frac{v-u}{t}$, if v = final velocity, u = initial velocity, t = time in seconds.

$$\therefore F = ma = m \cdot \frac{v-u}{t} = \frac{mv-mu}{t}$$

or $Ft = mv - mu$,

i.e., Force (poundals) $\times$ time (secs.) = change of momentum;

or Force (lbs.) $\times$ time = change of momentum $\div g$.

We can thus find the force necessary to produce a given change of momentum, or *vice versâ*. It is readily seen from the above that force in lbs. = change of momentum *per sec.* $\div g$, and it is in this form that the problem usually occurs in practice.

Ex. Fig. 405. A mass of M lbs. of water impinges per

second upon a fixed vane AB with a velocity v in the direction AC, and the vane changes the direction only of the water to BD, which is perpendicular to AC. Find the magnitude and direction of the resultant pressure on the vane.

The vane must do two things: (1) reduce the velocity v in the direction AC to zero and thus destroy per second

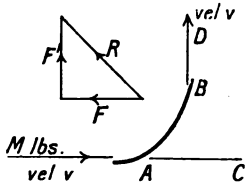


FIG. 405.

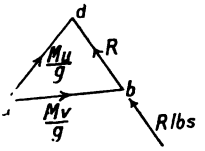


FIG. 406.

momentum Mv in the direction AC; (2) impart a velocity v in the direction BD and thus impress upon the water momentum Mv per second in the direction BD. The force to produce (1) is $F = \frac{Mv}{g}$ lbs. in the direction CA, and to produce (2) the force is $F' = \frac{Mv}{g}$ lbs. in the direction BD. The resultant pressure is the vector sum of F and F' given by R lbs. in Fig. 405, and it changes the momentum in one sec. from Mv units in the direction AC to Mv units in the direction BD.

Since F, F', and R, in Fig. 405, are proportional and parallel respectively to initial, final, and change of velocity, it follows that *change of momentum is proportional to the impressed force and takes place in the direction in which the force acts.* We may thus ascertain the final direction of motion, momentum, and velocity of a body possessing Mv units of momentum in a direction *ab* if acted upon for one sec. by a force R lbs. in the direction *bd*, by drawing (Fig. 406) $ab = \frac{Mv}{g}$ units, bd = R units (lbs.) and joining *ad*. The final momentum is then given by *ad* in magnitude and direction, and the velocity $u = \frac{ad \text{ (momentum units)} \times g}{M \text{ (lbs.)}}$ ft. per sec.

Exercise.

If a jet of water strikes a fixed vane and is deflected, the force F acting on the vane, being equal to the loss of momentum per second, is given by the vector equation

$$F = \frac{m}{t}(v_1 - v_2).$$

Find F for the case in which $\frac{m}{t} = 4$, the mass of water flowing per second; $v_1 = 50_{15}°$ ft./sec., the velocity of the water as it enters; $v_2 = 7_{80}°$ ft./sec., the velocity of the water as it leaves. (B. E., 1912.)

424. **Products of two vectors.** If two lines of length a, b respectively be two sides of a rectangle, the area of the rectangle is the product ab. If the two lines are lines representing vector quantities, they have sense as well as magnitude, and in this case it is useful, in many problems, to consider two separate products which can be derived from the lines.

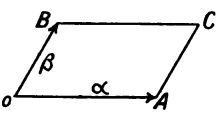


FIG. 407.

Consider two vectors α, β (Fig. 407). Set them out from a common point o. If we move the vector β parallel to itself along the vector α, we describe a parallelogram OACB, and the area of this parallelogram is called the **vector product** of the two vectors α and β, and is written as $[\alpha\beta]$, the square brackets denoting a vector product as distinguished from a scalar product, to be described later. If we keep the vector β fixed and move the vector α along β we get the same area OBCA, but the vector product is now written as $[\beta\alpha]$. The product $[\alpha\beta]$ may be considered as an area described by going round the figure OACB in the direction OA and may be regarded as positive, while the product $[\beta\alpha]$ may be regarded as described by going round the figure in the order OBCA, and hence may be regarded as negative. Adopting this convention, we see that $[\alpha\beta] = -[\beta\alpha]$. As the product described above has sense as well as magnitude, it is itself a vector quantity. A vector product has many useful applications in problems and theorems connected with moments of forces.

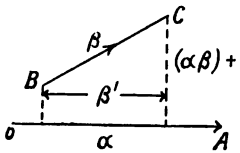


FIG. 408.

Consider next two vectors, α β (Fig. 408), and project the vector β

upon a. Let the projection be β'. Then the product of a and β' is a rectangle having sides respectively equal to a and β', and the product $a\beta'$ is written as $(a\beta)$, and is called the **scalar product** of two vectors, the curved brackets distinguishing this product from a *vector* product above described. As the vector β', when placed $\perp^r$ to a, may be placed in any one of the infinite number of planes which contain the vector a, it is seen that the scalar product has no particular sense; hence it is not a vector quantity, but a scalar quantity.

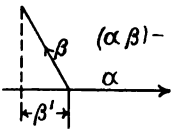


FIG. 409.

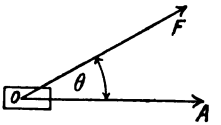


FIG. 410.

The scalar product is regarded as + if β' has the same sense as a (as in Fig. 408), and negative if β' has the opposite sense to a (as in Fig. 409). Further, if θ is the angle between a and β, we have $(a\beta) = \text{OA. (BC cos. } \theta)$

$$= \text{BC. (OA. cos. } \theta)$$
$$= (\beta a).$$

A useful application of a scalar product of two vectors is found in estimating the work done by a force acting upon a body which it displaces in a direction inclined to that of the force; thus let a force F (Fig. 410) act upon a body at O, the force always keeping the direction OF, and suppose it displaces the body by an amount OA.

Let β be the vector OF (force)
„ a „ „ OA (displacement).
Then work done = (βa) (scalar product).
$= \beta' \times a$, where $\beta' = \text{F. cos. } \theta$.

Also, as $(\beta a) = (a\beta)$ we have work = $(a' \times \beta)$
where a' = projection of a on β = OA. cos θ,
i.e., work = component of force in direction of displacement × displacement,
or = component of displacement in direction of force × force.

Exercises.

(1) If AP (Fig. 390) is = 5 CP, and CP makes 100 revolutions per minute, and is 10 inches long, plot a diagram from which the linear velocity of A may be found corresponding to angles 0° to 180° turned through by CP from the line CA.

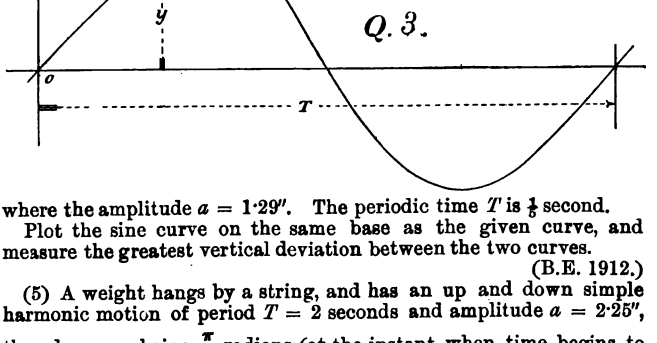
(2) Draw a curve showing the displacement of A (Fig. 390) from its mid-position for a complete revolution of CP. (See Fig. 395.)

(3) If the crank BC of Q. 12, p. 220, rotates with a uniform angular velocity of 5 radians per second, what are the angular, and linear, velocities of points D, E for the position shown in the diagram?

(4) A piece A of a certain machine has an oscillating motion. For successive positions of A the displacements y at the times t are noted, and a graph of the results is given.

The shape of the curve indicates that the motion is nearly simple harmonic, and analysis gives as a first approximation the sine curve

$$y = a \text{ sin. } \frac{t}{T} 360°$$



where the amplitude $a = 1{\cdot}29''$. The periodic time T is $\frac{1}{5}$ second.

Plot the sine curve on the same base as the given curve, and measure the greatest vertical deviation between the two curves.

(B.E. 1912.)

(5) A weight hangs by a string, and has an up and down simple harmonic motion of period $T = 2$ seconds and amplitude $a = 2{\cdot}25''$, the advance a being $\frac{\pi}{2}$ radians (at the instant when time begins to be reckoned). The displacement y from mean position at any time t is thus given by the equation

$$y = a \text{ sin. } \left(\frac{2\pi}{T} t + a \right) = 2{\cdot}25'' \text{ sin. } \left(\pi t + \frac{\pi}{2} \right)$$

Draw a curve showing the relation between y and t at any time from $t = 0$ to $t = T = 2$ seconds.

Adopt as the horizontal scale for time 3 inches to 1 second, and take the vertical scale for y full size. Read off the displacement y when $t = 0{\cdot}25$ second. (B.E. 1904.)

CHAPTER XXV.

DESCRIPTIVE GEOMETRY.

425. All the figures we have made drawings of hitherto are *plane* figures, which we have been able to place wholly in the plane of the drawing-paper. We have now to show how objects which have length, breadth, and thickness, *i.e.*, three dimensions, and lines occupying fixed positions in space, may be represented by drawings which supply precise information on all points relating to magnitude and position.

426. In Fig. 411, an ordinary 60° set-square is shown supported on three hat-pins placed *vertically* under the three corners A, B, C of the set-square and piercing the *level plane* surface marked HP in *a*, *b*, *c*. If the points *a*, *b*, *c* are joined by straight lines, a triangle is formed which is called *the* **projection** *of the set-square on the level plane* HP.

A second plane surface marked VP is shown occupying a vertical position to the left of the set-square and meeting the plane HP in the line XY. Suppose long knitting-needles placed at each corner of the set-square and *perpendicular to the plane* marked VP, meeting same in *a'*, *b'*, *c'*. If the points *a'*, *b'*, *c'* be joined by straight lines, the triangle so formed is called the *projection of the set-square on the* vertical plane VP.

The two planes HP and VP, upon which the projections are made are called **planes of projection.** The former occupies a horizontal position ; it is therefore a **horizontal plane of projection.** The latter occupies a vertical position, and is therefore a **vertical plane of projection.** The abbreviations HP and VP are often written for *horizontal plane* and *vertical plane* respectively.

A horizontal plane is sometimes called the *ground*, and the line XY in which the HP and a VP intersect is called the **ground line.**

A projection on a horizontal plane is called a **plan**; a projection on a vertical plane is called an **elevation.** Thus, *a b c* is the plan of the set-square ABC, whilst *a' b' c'* is its elevation on the given VP.

The set-square ABC placed as in Fig. 411 represents a triangle occupying a definite position in space; *abc* is

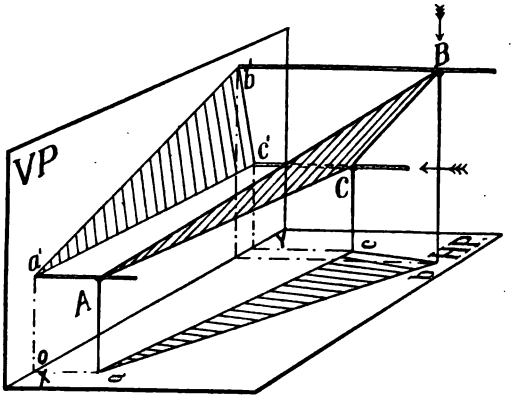


FIG. 411.

the plan of the triangle; *a'b'c'* is the elevation on the given VP.

Each edge of the set-square, as AB or BC, represents a *straight line in space*; thus *ab* is the plan of a straight line AB occupying a definite position; *a'b'* is the elevation of AB on the given VP. Similarly *bc*, *b'c'* are *plan* and *elevation* of *the straight line* BC.

Each corner of the set-square ABC represents a *geometrical point*, occupying a definite position, at a definite height above the HP, and at a definite distance in front of the given VP. Consequently, *a* and *a'* are respectively the

plan and *elevation of the given point* A; similarly, b, b', and c, c' are the *plan* and *elevation* of *points* B and C respectively.

If from the elevation a' of the corner A of the set-square, a perpendicular $a'o$ be dropped on to the horizontal plane, this perpendicular will be in the VP and will be perpendicular to XY; it will also be parallel to the vertical pin Aa. If from the plan a of the corner A of the set-square, a perpendicular ao be drawn on to the VP this line will be in the HP and will meet the vertical line from a' in the point o on XY. The line ao will be parallel to the needle Aa'. Thus Aa oa' is a rectangle, and Aa' = ao; Aa = $a'o$.

The points discussed in this article may be summarized thus:—

(*a*) *The projection of a given point in space is at the foot of the perpendicular drawn from the point on to the plane of projection.*

(*b*) *The projection of a given* **straight line** *is the straight line joining the projections of the ends of the line.*

(*c*) *The projection of a given* **triangle** *or* **any plane rectilinear** *figure, is the figure formed by straight lines joining up the projections of the corners of the triangle, or other figure.*

(*d*) *The plan of a point is the same distance in front of* XY *as the point is in front of the* VP.

(*e*) *The elevation of a point is the same distance above* XY *as the point is above the* HP.

(*f*) *If a line is parallel to* $\frac{VP}{HP}$ *its* $\frac{plan}{elevation}$ *is parallel to XY.**

(*g*) *If a line is in* $\frac{VP}{HF}$ *its* $\frac{plan}{elevation}$ *is in XY.*

427. In Fig. 411, which for convenience is reproduced in Fig. 412 (*a*), the planes upon which the plan and elevation of the set-square are made are shown in the positions which the planes of projection are always supposed to occupy, *viz.*, the one horizontal and the other vertical. Suppose the horizontal plane HP to rotate on the line XY as a hinge in

* See footnote, p. 442.

the direction of the arrow until it hangs vertically below XY. This plane will then be a continuation of the VP and the two planes become *one*. During this rotation of the plane HP, the point *a* describes a quarter-circle, centre *o*, radius $oa = a'A$; and comes into position vertically under *o* on $a'o$ produced. Similarly for the plans of the corners B, C. The point *b* describes a circle, centre o', radius $o'b = b'B$, and comes into position on $b'o'$ produced. The point *c* describes a circle centre o^2, radius $o^2c = c'C$, and comes into position on $c'o^2$ produced. Thus, the plans of the three corners of the set-square are the same distance

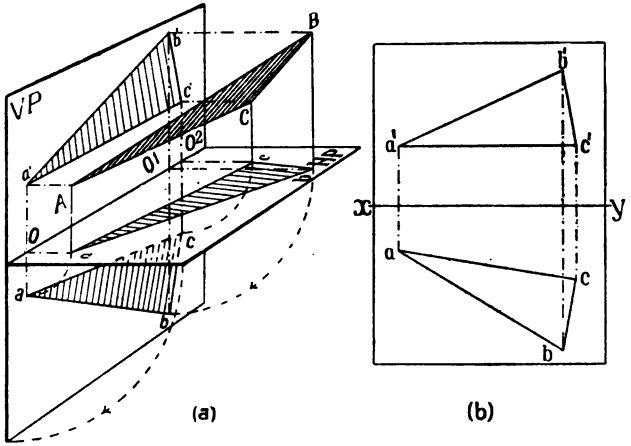


FIG. 412

from the line XY after rotation of the plane HP as they are before rotation ; and in each case *the plan and elevation of the point lie on the same straight line perpendicular to* XY. This fact enables us to draw both *plan* and *elevation* of an object on *one flat sheet of paper*. In Fig. 412 (*b*) are shown the two views (plan and elevation) of the set-square as they would appear, after rotation of the HP into a vertical position, to a person standing directly in front of the plane,

or, as they would appear, if laid flat on a table, to any one standing over the drawing and looking down on it. The portion of the sheet above XY represents the vertical plane and that below XY a horizontal plane. If we suppose the drawing to be made on a sheet of paper stretched flat on a level table, then we can imagine the paper folded along XY so as to bring the part containing the elevation into a vertical position.

A sheet of paper lying on a drawing-board can be folded along any straight line in the paper so as to bring a part of the paper into a position at right-angles with the surface of the board. Consequently, the line XY which marks the position of the VP may be drawn on the drawing-board in any direction we please. If, however, we draw XY across the board, using the T-square placed in its normal position, set-squares can be used to draw the projections of different points; this simplifies the work, and wherever possible we place XY parallel to the edge of the T-square.

428. The angle between two planes is called a **dihedral angle**.

In Geometry, the planes of projection are supposed to have no thickness, and to extend as far as may be necessary on all sides; thus, the vertical plane extends below the HP, and the horizontal plane extends behind the VP, the two planes forming four *equal dihedral angles* of 90°. These angles are called the *first*, *second*, *third*, and *fourth* angles, as indicated in Fig. 413.

429. Imagine the plane VP to turn on xy as a hinge, in the direction of the arrow through 90°, so as to bring the portion of VP above HP down into HP behind xy, and the portion of VP below HP up into HP in front of xy. The two planes will then wholly coincide.

The whole of a sheet of drawing-paper covering the drawing-board may be considered as being two distinct planes hinged along xy. The portion below xy is that part of HP in front of VP and that part of VP below HP, whilst the part above xy is that part of VP above HP and that part of HP behind VP. Similarly, if there be another vertical plane as V′ this plane can be folded about the line $(x'y')$ in which it intersects the HP until the two planes

V′ and HP coincide, and all that has been said about the two planes HP and VP applies equally to HP and V′.

430. In Fig. 413, four points P, C, B, A, one in each dihedral angle, and the projections of the four points on

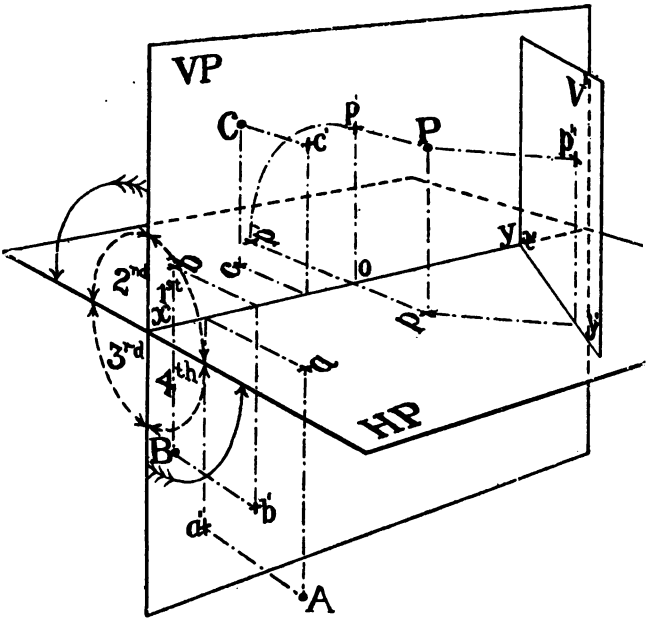


FIG. 413.

HP and VP are shown ; the projection of P on plane V′ is also shown.

During rotation of VP, p' moves down into HP behind xy, its locus being the circle centre o, radius $op' = pP$. Similarly, the elevations of points A, B, C move into HP, and after rotation the elevation c' of C is **above** xy, a distance $= cC$, whilst a', b' the elevations of points A and B are each **below** xy, a distance equal to that of the points below HP.

Fig. 414 shows the projections of the four points in correct positions after rotation of the VP.

From Figs. 413, 414, we learn—

(a) *The plan of a point is the same distance below, or above, the ground line xy as the point itself is in front of or behind the* VP.

Cor. If a point or any figure, rectilinear or curved, is in VP, *its plan is on xy.*

(b) *The elevation of a point is the same distance above, or below, xy as the point itself is above or below* HP.

Cor. If a point or any figure, rectilinear or curved, is in HP, *its elevation is on xy.*

(c) *The* **plan** *and* **elevation** *of a point are in the same straight line, perpendicular to the ground line.*

431. The student will note that the projection (p'') of P on the vertical plane V′ is in the line from p $\perp^r$ to $x'y'$ (Fig. 414). This is clearly as it should be, since the plane V′ is quite independent of VP. Observe also that the distance of p'' from $x'y'$ is equal to the height of the point P above the horizontal plane. A glance at Fig. 413 will show that this must be so, for any number of vertical planes could be grouped around the point P, and P would still remain the same distance from HP, and therefore its elevation on all such planes would be the same height above the respective ground lines. Now, so long as the relative position of the plane HP and the point P remain fixed, the point p will continue to be the projection of P on this plane. Hence, *so long as a figure is fixed relatively to* HP, *it can have only one plan, but elevations can be obtained on any number of vertical planes, and all the elevations of any particular point in the figure will be the same height above or below the respective ground lines.*

432. In Fig. 415, points A, B, C, D, E, F, G, H and K are represented by plan and elevation on HP and VP which intersect in xy.

The positions of the several points relative to the planes of projection may be deduced from their projections. Thus —point A—the plan a is 1 in. below xy, whilst the elevation a' is 2 ins. above xy, therefore the point A is 1 in. in front of VP and 2 ins. above HP. Reasoning

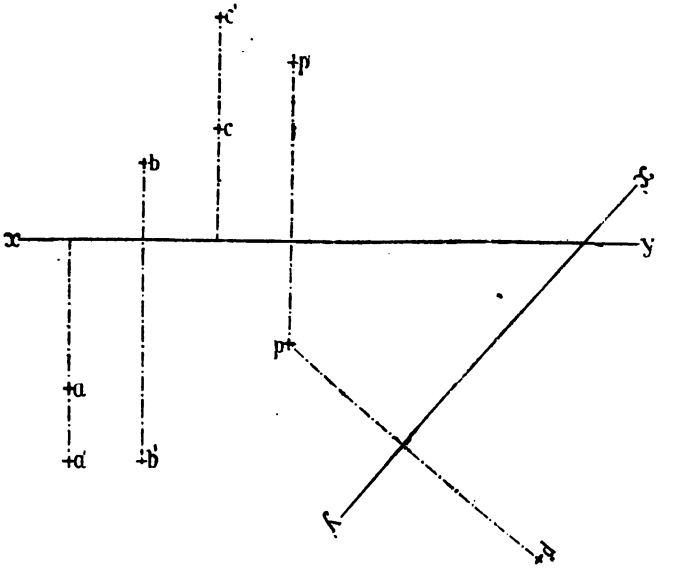


FIG. 414.

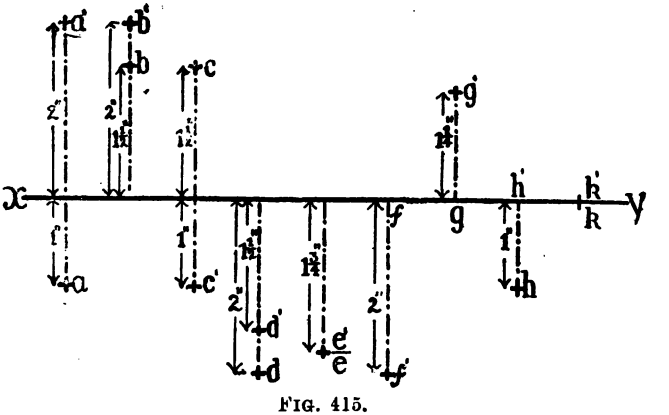


FIG. 415.

in a similar manner in regard to other points, we see that—

B is 1·5 ins. behind VP and 2 ins. above HP.
C is 1·5 ins. behind VP and 1 in. below HP.
D is 2 ins. in front of VP and 1·5 ins. below HP.
E is 1·75 ins. in front of VP and 1·75 ins. below HP.
F is in VP and 2 ins. below HP.
G is in VP and 1·25 ins. above HP.
H is 1 in. in front of VP and in HP.
K is in xy, *i.e.*, in both planes of projection.

433. **The Straight Line and its Projections on Two Rectangular Planes.*** The projection of a straight line on any plane is the straight line which joins the projections of the ends of the line.

A line which is inclined to one plane only (HP or VP) and parallel to the other is called an **inclined line.**

The angle between a line and plane is the angle between the line and its projection on that plane.

434. *A line may be inclined to a plane at any angle not exceeding* 90°. *When the line is parallel to the plane, its projection on that plane has the same length as the line. When the line is inclined to the plane, the projection of the line is shorter than the line; the greater the angle between the line and plane the shorter the projection. When the angle is* 90°, *the line projects into a point.*

For, consider a line OE (Fig. 416) in the plane α which is $\perp^r$ to plane β. If OE is the position of the line, and oe its plan, OE is the hypotenuse of a rt. $\angle$ triangle having the plan Oe and the projector eE for legs, hence OE must be greater than Oe. Again, let OE′ represent the line; then its plan is Oe', and the projector $E'e'$ is greater than Ee, consequently Oe' is less than Oe, and much more less than OE′. Similarly, if the line be revolved into position perpendicular to plane β, its projection is the point O.

If oe *be the projection of a line* OE *and* θ *be the inclination of the line, then* $\mathrm{OE} = \dfrac{oe}{cos.\ \theta}$.

435. A line which is inclined to both HP and VP is called an **oblique line.** *The sum of the inclinations* $(\theta+\phi)$ *of a line to fixed* HP *and* VP *cannot exceed* 90°.

For, consider a line AB (Fig. 417) lying in a plane perpendicular to both HP and VP, its plan is aa', its elevation is Ba'. The angles

* See Note. Art. 457.

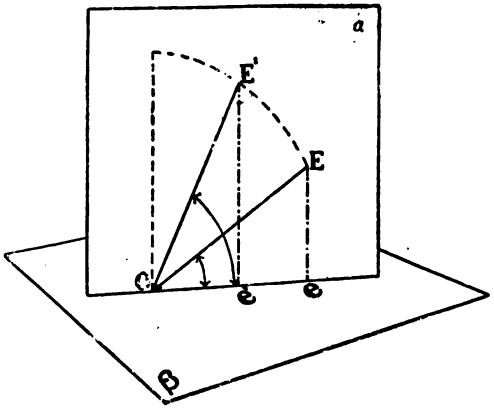


FIG. 416.

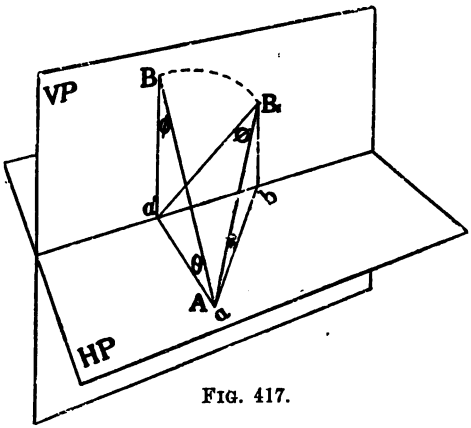


FIG. 417.

which the line makes with HP and VP are θ and ϕ, and since $Ba'a$ has a right-angle at a'. $\theta + \phi = 90°$. Let the end A remain fixed, and the end B move in the plane VP to a new position B_1. The plan of the line is now ab; its elevation is $a'B_1$.

Comparing the triangles ABa', AB_1a', we have $AB_1 = AB$, $a'B = a'B_1$ and aa' common; hence $\angle AB_1a' = \angle aBa'$. In the triangles aB_1b, aBa', $aB_1 = aB$, $\angle abB_1 = \angle aa'B$,

but $bB_1 < Ba'$,

$\therefore \angle B_1ab < Baa'$.

Similarly, if the end B be moved still nearer to HP the sum $\theta + \phi$ will be still less, and ultimately when AB is $\parallel$ HP, θ vanishes and the sum is ϕ.

436. Figs. 418, 419 (a, b . . . f) are plans and elevations of lines occupying different positions relative to the planes of projection xy. These projections reveal certain information regarding the position of the lines in space.

Thus in (a). Both the plan ab, and the elevation $a'b'$ are $\parallel$ xy, therefore AB is parallel to both planes, and to xy, being h distance in front of VP and v distance above HP.

(b) The plan ab, and the elevation $a'b'$ coincide in xy; therefore AB is in xy.

(c) The plan ab is parallel to xy; the elevation $a'b'$ is inclined to xy at α; a' is below, and b' is above xy. Therefore AB is parallel to VP at a distance h in front of same, and inclined to HP at α; the end A is v' distance below, and end B is v distance above, HP.

By reasoning in this manner, show that the position of the line in the other diagrams is as under:

(d) Parallel to HP at v distance above; inclined to VP at β. A is in VP; B is h distance in front.

(e) Vertical at h distance in front of VP.

(f) Inclined to both planes. A is h distance in front of VP and v above HP; B is h' in front of VP and v' above HP.

(f^1) Inclined to both planes. A is x distance in front of VP and the same distance below HP; B is y distance in front of VP and the same distance below HP.

(f^2) Inclined to both planes. A is in xy; B is in front of HP and above xy.

(f^3) Inclined to both planes. A is in HP and in front of VP; B is in VP and above HP.

(f^4) Inclined to both planes. A is in xy.

In f, f^1, f^2, f^3, f^4 *the line is oblique, and neither the length of the line nor the angles between the line and the planes of projection are revealed in the projections.* A construction is necessary to determine these elements.

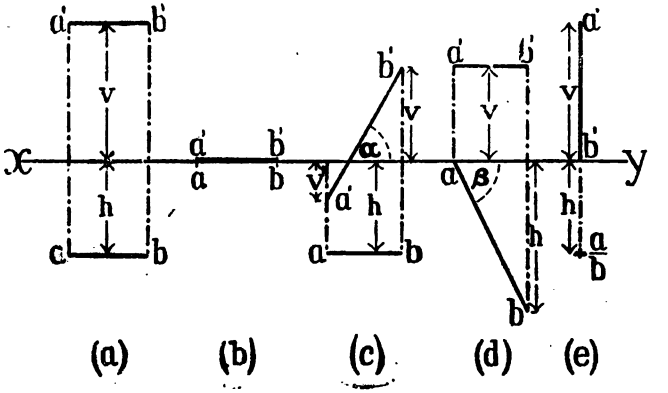


FIG. 418.

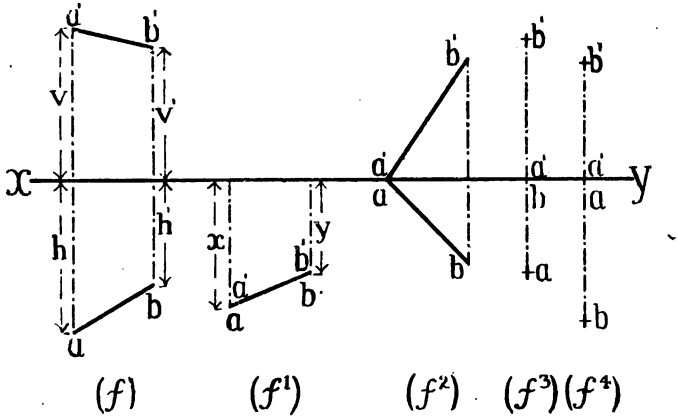


FIG. 419.

437. **Experiment.** Fig. 420 (a). On stout drawing-paper, or Bristol-board, draw a straight line xy. Take any points A and B on opposite sides of xy, and draw perpendiculars b'B, aA. Join b'A, Ba. Cut half through along xy and fold the paper making a model of projection planes HP and VP. Fig. (b). Pass a lady's hat-pin through the paper at A and B. The hat-pin may represent a line in space. The points AB are called **traces** of the line; the trace B in HP, the **horizontal trace** (HT); the trace A in VP, the **vertical trace** (VT).

438. A little consideration will show that a line which occupies a position parallel to a plane has *no* trace on such plane, but if a line is *not* parallel to a plane the line—prolonged if necessary—will have a trace on the plane.

Fig. 421 shows lines occupying different positions relative to planes HP and VP. The positions of the lines are as follows:—(a) ∥ to both planes, and has no trace. (b) ∥ VP and has HT only. (c) ∥ HP and has VT only. (d), (e), (f), inclined to both planes, and have HT and VT. In (f) the line passes through xy, consequently its HT and VT coincide.

439. Examination of the model, Fig. 420 (b), shows that the pin has but *one projection on each* of the planes HP, VP. Consequently, **a straight line is completely fixed in position by its projections on two rectangular planes.**

Also, the VT of the line is in the elevation of the line, and the HT is in the plan; and, since the plan of the VT and the elevation of the HT are both in xy, therefore the *traces of a straight line completely fix the position of the line except in the case where both traces coincide in xy.*

440. **Problem.** Fig. 422. *Determine the traces of a line* AB *shown in plan and elevation at ab, a'b'.*

Extend the plan to meet xy, and draw a projector meeting $a'b'$ prolonged in VT. From the point where $a'b'$ produced crosses xy, draw a projector meeting the plan produced in HT.

441. **Problem.** Fig. 423. *The horizontal and vertical traces of a line are at* H *and* V. *Draw plan and elevation of the line.*

Draw Vv, Hh', ⊥ to xy. Join Hv, h'V.

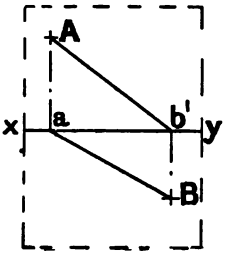


Fig. 420. (a)

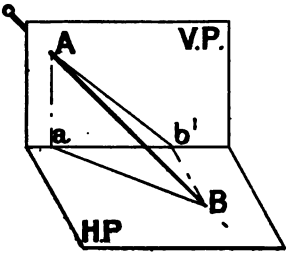


Fig. 420. (b)

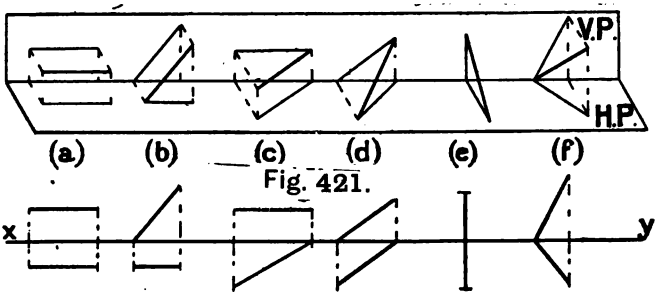


Fig. 421.

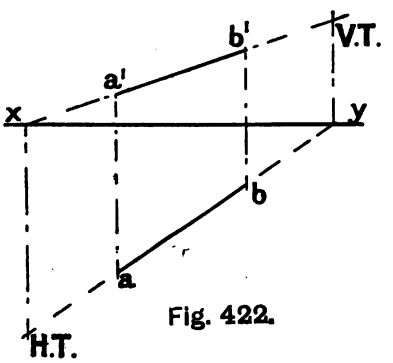


Fig. 422.

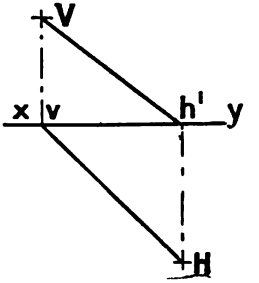


Fig. 423.

442. No matter what position a straight line occupies relative to fixed HP and VP of reference, there must always exist four independent relations, viz.: (1) and (2) a definite projection on each plane; (3) and (4) a definite angle between the line and each plane.

If any two of these are known, the others are determinable.

443. **Problem.** Fig. 424 (a) and (b). *Given the projections ab, a'b' of a line*, AB, *to determine the angles* θ *and* ϕ *which the line makes with* HP *and* VP.

Analysis. Fig. (b). Let a line from A parallel to HP meet the projector Bb in R. Then since AR $\parallel$ ab, we have

$$AR = ab\ ;\ BR = Bb - Rb = Bb - Aa$$

$$\angle BAR = \theta\ ;\ \angle BRA = 90°.$$

Similarly, if a line from B parallel to VP meet the projector Aa' in P; BP $= b'a'$; AP $=$ Aa' $-$ Pa' $=$ Aa' $-$ Bb'. $\angle ABP = \phi$; $\angle APB = 90°$.

Construction. Fig. 424 (a). At b (the plan of the highest end) draw bB $\perp$ ab and equal to the difference in the heights of the ends of the line, that is $b'r'$. Join Ra. Then $\angle Bab = \theta$, the angle between the line and HP.

At a' (the elevation of the end farthest from VP) draw $a'B' \perp a'b'$ and equal to the difference in the distance of the ends of the line from VP, that is ap. Join $B'b'$. Then $\angle\ a'b'B' = \phi$, the angle between the line and VP.

444. **Problem.** Fig. 425. *Given the plan ab of a line* AB, *and the angle* $\theta°$ *which the line makes with* HP. *To determine its elevation and inclination to* VP.

Note.—In order to make this problem definite we must know in which direction the line slopes, that is, whether A is above or below B. Further, the height of some point in the line must be given or assumed. Suppose the line slopes upwards from A to B, and let A be ·75 in. above HP. See Notes on p. 442.

Analysis. Fig. 424 (b). Let AB be the line in position, ab its plan, and $a'b'$ its elevation.

Applying the preceding analysis, we see that the height of B above A will be revealed in the right-angled triangle ARB which has one side $= ab$, and angles adjacent to that side $= \theta$ and 90° respectively.

Construction. Fig. 425. At a draw aB, making the

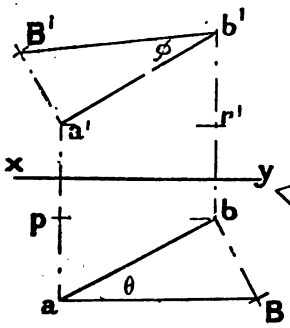


Fig. 424. (a)

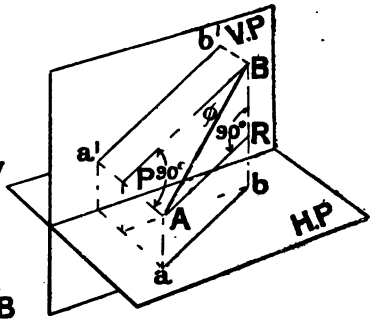


Fig. 424. (b)

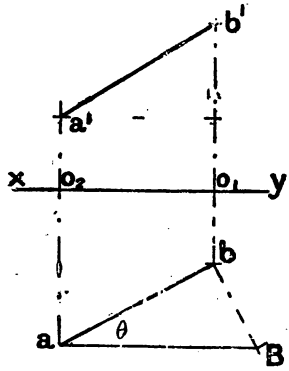


Fig. 425.

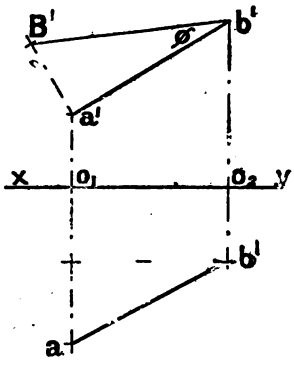


Fig. 426.

given angle θ with ab, to meet the perpendicular from b in B. Then Bb shows how much the end B is above A. Draw projectors from a and b; make $o_2a' = \cdot 75$ in. the given height. Make $o_1b = Bb + \cdot 75$ in. Join $a'b'$, which is the required elevation. The angle ϕ between the line and VP may be found as in the preceding problem.

If the elevation of the line, and the angle ϕ between the line and VP *are given,* the solution is the same as that just explained: it is illustrated in Fig. 426.

445. **Problem.** *Given the projection of a line* AB *on one plane and the angle which the line makes with the other plane. To determine the projection on the other plane and the other angle.*

Let the elevation of the line, and the angle θ, between the line and HP, be given.

As in the preceding problem we must know or assume the position of same point in the line. Suppose B is 1 in. in front of VP.

Analysis. Fig. 427. If a point B in the line be made the apex of a right circular cone, the axis of which is $\perp r$ to the plane HP, and which has a base angle equal to $\theta°$, the surface of this cone must contain AB and every other straight line that can be drawn through B to make the given angle with HP.

If we assume the base of this cone at level A, its radius will be AR and its plan a circle, centre b, radius AR. The elevation of the circular base is a straight line drawn through a' parallel to xy.

Now a projector from a' crosses the plan of the circular base in two points a, a_1,* and ba, ba_1 are plans of two lines on the cone which have the same elevation $b'a'$; hence there are two lines which satisfy the conditions of the problem.

Construction. Fig. 428. Draw projectors from a', b'. Make $o'b$ = 1 in. (the given distance of B in front of VP.) Draw $b'A'$ making $90° - \theta°$ with $b'o'$, intersecting a line from $a' \parallel xy$ in A'. With centre b, and radius RA', describe a circle intersecting the projector from a' in points a, a_1. Join ba, or ba_1.

The angle ϕ may be found as in Problem 443.

Similarly, *if the plan ab and the angle* ϕ *between the line and* VP *is given.* If any point in the line be made the apex of a cone with its base on, or parallel to, VP, and having base angle equal to ϕ, generators of this cone, which have ab for plan, are lines which satisfy the given conditions. These lines are determined as already explained.

The solution of the problem is given in Fig. 429.

* If the projector from a' is tangent to the circle there will be only one solution. If the projector does not intersect the circle at all, the problem is impossible with the given data.

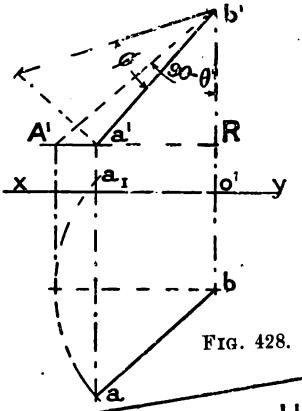


FIG. 428.

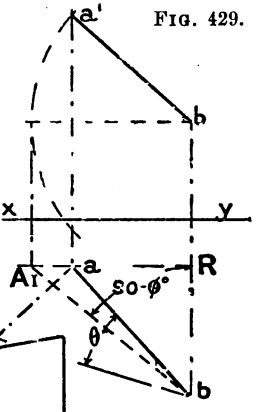


FIG. 429.

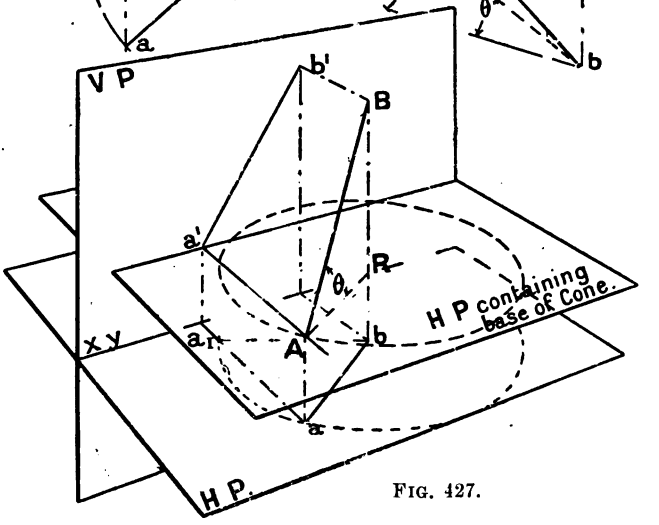


FIG. 427.

446. **Problem.** Fig. 430. *A line* AB 3 *ins. long is inclined to* HP *at* $\theta = 42°$, *and to* VP *at* $\phi = 35°$; *determine its projections.*

Let the end B be fixed in position, a definite distance above HP and in front of VP, b', b, say, and draw a projector from b'. At b' set out the angle $90° - \theta = 48°$, and make $b'A' = 3$ ins. the given length. With centre b and radius AR, describe a circle. At b' set out the angle $A'b'n = \phi = 35°$. Draw $A'n \perp b'n$. With centre b' and radius $b'n$ describe an arc intersecting A'R in a', a''. Join $b'a'$, $b'a''$. From a', a'' draw projectors intersecting the circle centre b in points a, a_1, a_2, a_3. ba, ba_1, ba_2, ba_3 are four lines which satisfy the given conditions.

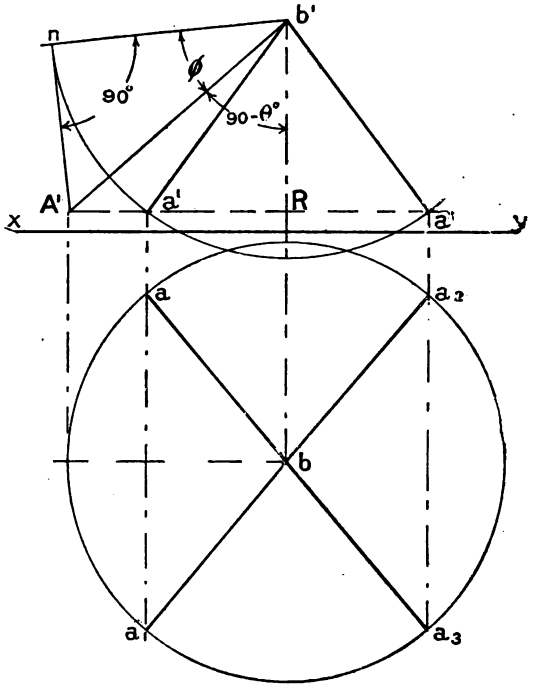


FIG. 430.

For all of these lines are generators of the cone whose apex is B ; slant height = 3 ins., base angle θ = 42° and axis BR ⊥r to HP. Therefore each line has the prescribed length and inclination to HP.

Also $b'n$ is the length of the projection of $b'A'$ on a plane to which it is inclined at 35° and since two of the four lines on the cone have elevations $b'a'$ and the other two have elevations $b'a''$ each equal to $b'n$, the four lines are inclined to VP at the given angle ϕ.

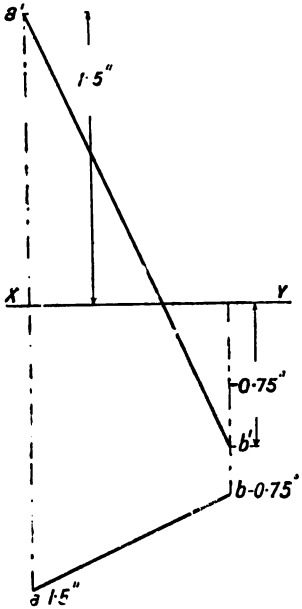


FIG. 431.

447. **Horizontal Projection or Indexed Plans.** In Fig. 431, ab is the plan of a piece of straight wire AB. If we know that the end A is 1·5 ins. above HP, and the end B is 0·75 in. below HP, we can indicate those heights by drawing an elevation of the wire, as $a'b'$, on any XY. As an alternative method, we may write against the respective plans a, b the numbers 1·5, − 0·75, which correspond to the levels of the ends of the wire. (The minus sign is prefixed to the number 0·75 to indicate that the point B is below HP; all distances below HP being distinguished by this sign.) These numbers attached to the ends of the plan indicate without ambiguity the positions of the ends of the wire relative to HP.

The number which represents the level of a point is called the **index** of the point.

A plan on which the heights of points are indexed is called an **indexed plan.**

The method of representing figures by indexed plans is called **horizontal projection.**

448. **Problem.** Fig. 432. *A line* AB *is given by an indexed plan* $a_0\ b_{16}$ *unit* 0·1 *in. Determine the elevation of* AB *on the vertical plane denoted by* XY. *Also draw an elevation on the vertical plane denoted by* X′Y′.

Firstly. Draw projectors from *a* and *b* perpendicular to XY. The elevation of the points will lie in these lines. Because the index of *a* is zero, the elevation *a′* is in the ground line.

The index of *b* is 16, therefore mark a point *b′* on the projector from *b*, 16 units above XY. Join *a′b′*, which is the elevation of AB on the vertical plane XY.

Secondly. Draw projectors from *a* and *b* perpendicular to X′Y′, mark *b″* 16 units above X′Y′, and mark *a″* on X′Y′. Join *a″b″*, which is the projection of AB on the vertical plane X′Y′.

The length of AB and its inclination to HP is shown in the elevation *a′b′*, on XY, but not in the elevation on X′Y′. Why is this? *Answer.* Because AB is parallel to the VP on XY, but inclined to the VP on X′Y′.

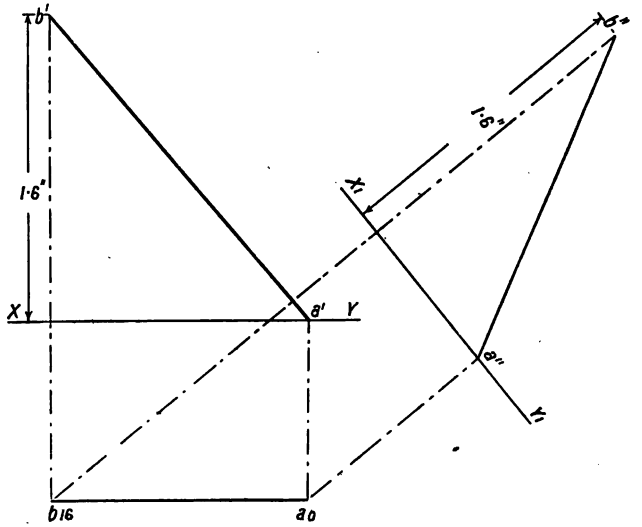


FIG. 432.

449. **Problem.** Fig. 433. *It is required to find the length of a steam pipe to connect with two others the ends of which are shown in plan and elevation at a, a′; b, b′.*

Join *ab*, and from *a* the plan of the higher point draw a

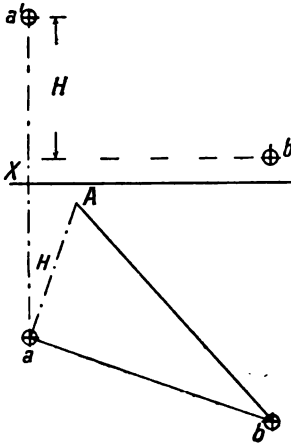


FIG. 433.

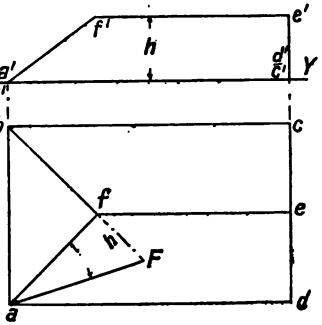


FIG. 434.

perpendicular to *ab*. Make *a*A = the difference in the heights of *a′* and *b′* above XY, and join A*b*. Then A*b* is the length of pipe required.

450. **Problem.** Fig. 434. *Given the plan and elevation of a roof. Find the length of the hip rafter af, a′f′.*

From *f* draw a perpendicular to *af* and make *f*F = *h*. Draw F*a*, which is the length required.

Exercises.

(1) Draw in plan and elevation two lines AB and CD. Line AB is parallel to VP and inclined to 50° to HP, the end A being ½″ above HP. Line CD is parallel to both planes and is 2″ from XY and 1″ above HP. Hint: Get a view perpendicular to XY.

(2) Determine graphically the length of the diagonal of a cube 1½ in. edge.

451. **Problem.** No Fig. *A line ab* 2 *ins. long is plan of a line* 3 *ins. long*; *the end a is indexed* 5. *Draw an elevation on a given* XY. *Unit* 0·1 *in.*

Firstly, take any XY ‖ the plan *ab*, and draw projectors ⊥r this ground line; mark *a'* 5 units above XY, with centre *a'* and radius 3 ins.; describe a circle which will intersect the projector from *b* in two points *b'*, *b''*. Then *b'* or *b''* will reveal the level of B according as AB is required to slope upwards or downwards. Knowing the level of B, the elevation of the line on the given ground line is found as in Problem 448.

452. **Problem.** Fig. 435. a_0 b_{22}, c_{10} d_{15} *are plans of two lines which intersect in* P. *Index the end b and the point* P.

Take XY in any convenient position, and obtain an elevation *c'd'* of line CD. Obtain, also, the elevation *a'* of point A. Draw projectors from *p* and *b*; the former meets *c'd'* in *p'*, which must be the elevation of P (since P is on CD, and its elevation must be on the projector from its plan *p*). Draw *a'p'*, and produce to meet the projector from *b* in *b'*. Then *b'* is elevation of B, and the levels of P and B are revealed in the distances of *p'* and *b'* from XY.

The student will observe that if two lines intersect, both their plans and elevations intersect, and the line joining the projections of the *common point* must always be ⊥r XY.

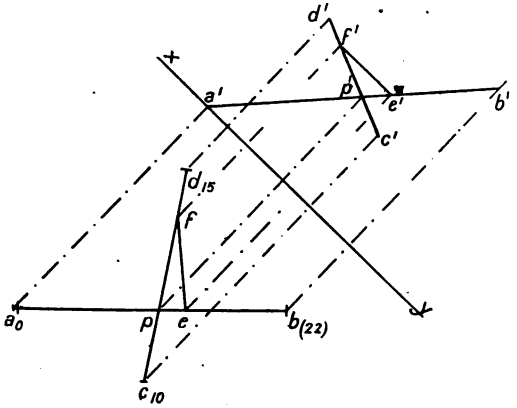


FIG. 435.

453. **Problem.** Fig. 435. *a_0, b_{22}, c_{10}, d_{15} are indexed plans of two lines. Draw a line* EF *at level* 14, *having one end in each line. Unit* 0·1 *in.*

Obtain an elevation of AB, CD on any ground line (XY). Draw a line at the given height 14 which intersects $a'b'$, $c'd'$ respectively in $e'f'$. A projector from e' meets ab in e; a projector from f' meets cd in f. Join ef, which is plan of the required line.

454. **Problem.** Fig. 436. *Being given the projections of a line* AB, *find the shortest distance of the line (produced if necessary) from* XY.

Take X_1, Y_1, $\perp^r$, XY, and obtain a new elevation $a''b''$ of the line; then the elevation of XY on this new VP is the point p, and pc'' drawn $\perp^r$ $a''b''$ is the length of the shortest line that can be drawn to connect AB and XY. To determine the actual position of such line, we know that the plan of C must lie on the plan of ab, it is therefore at c on the projector from c'. Also the line is $\parallel$ VP on X_1Y_1. Therefore its plan is cp $\perp^r$ XY. The elevation of the line on the original VP is $p'c'$.

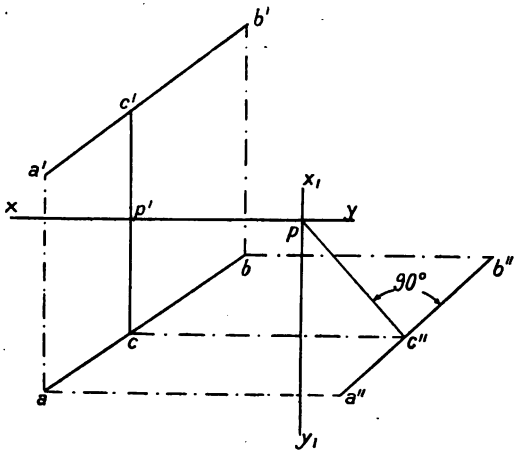


FIG. 436.

455. **Indirect Solution of Problems.** We often encounter problems in which to give a *direct solution* some part, at least, of the construction falls in an inconvenient position. In such cases, the work can often be simplified by the use of an indirect solution.

Ex. *A line* AB 3 *ins. long is inclined to* HP *at* 15° *and to* VP *at* 30°, *the end* A *is in* XY. *Draw its projection.*

If a line is parallel to $\frac{\text{VP}}{\text{HP}}$ *its* $\frac{plan}{elevation}$ *is parallel to* XY. (Art. 426).*

Hence two or more parallel lines in space have parallel projections on any plane, and since (Art. 434) the projection of a line AB inclined at θ is equal to AB cos ϕ, it follows that *the projections of parallel lines are in the same ratio as the lengths of the lines themselves.*

We may therefore draw, *in any position*, the projections of a line 3 ins. long having the given inclinations to the planes of projection, and then draw through the fixed point a, a' lines parallel and equal to these projections.

Notes.

456. Fig. 437. To define completely the position of a point A in space three geometrical conditions are necessary, such as its distance from three planes (Art. 340).

457. To define completely the position of a straight line AB five geometrical conditions are necessary, and in general a straight line can be determined to satisfy five geometrical conditions. For, the end A is fixed by its distance from each of the three planes, H, V, V', *i.e.*, three geometrical conditions. Suppose now that the height of the opposite end B above the plane H be fixed. Then clearly the inclination of AB to plane H is fixed, and if AB be turned about the fixed point A into every possible position consistent with the conditions given, the line will generate a right circular cone whose axis is perpendicular to plane H, and the locus of B will be the circular base of such cone in a plane parallel to H at the given height above. Suppose we have the further condition, that B is to be a definite distance (h) in front of the plane V. There are clearly but two positions that B can occupy to satisfy the given conditions, and in each case the distance of the point from the third plane V' is

* In this paragraph certain letters and words are written one above the other. The paragraph is intended to be read twice: On the first reading omit all words and letters in the bottom line; on the second reading include the bottom line and omit the top line. This method of double writing has been adopted in several instances in order to "force home" the fact that the data which defines the position of an object relative to HP can be applied to define the position of the object relative to VP by interchanging the terms *elevation* for *plan*, and VP for HP.

fixed and unalterable ; as also are the angles between the line AB and the two planes V and V'. Hence *five geometrical conditions are necessary and sufficient to define the position of a finite straight line.*

458. In practical geometry we have to deal almost solely with the *shape* of figures, *i.e.*, we have to make projections of figures of known form, or we have to ascertain the *true form* of an object from its projections, which are definite forms. These interdependent forms are wholly independent of the particular position in *space* occupied by a particular point, *e.g.*, the projection of a line AB on a given

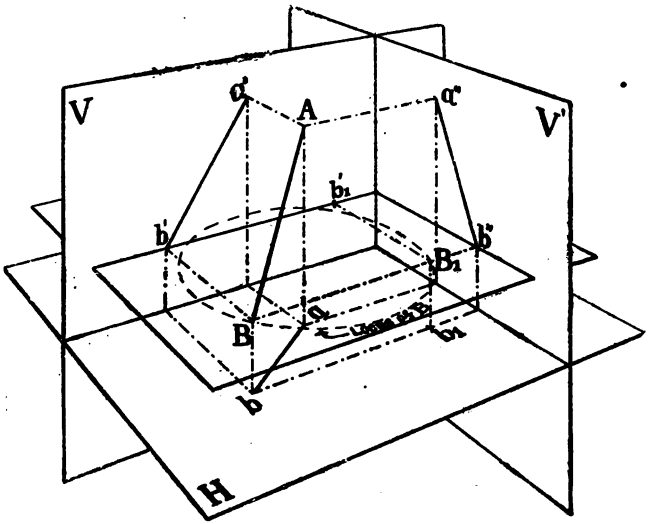


FIG. 437.

plane depends solely on the *angle between the line and the plane*, and so long as this angle between the line and plane remains the same the projection of the line will be the *same length*, no matter what position in space the line occupies. We are thus usually able to free our problems of some of the restrictions which are involved in locating an object in space, and to commence drafting operations in any position we please.

459. When a point or any object is situated in the first dihedral angle, its plan is below XY, and its elevation above. When in the third angle, the plan is above XY and the elevation below ; when in the second angle, both plan and elevation are above XY ; when in the fourth angle, both plan and elevation are below XY. In the two

former positions, XY separates the plan from the elevation and both views are distinct and easily read, but in the second and fourth angles where the plan and elevation overlap, the result is often an almost inextricable maze of lines, which even the experienced geometer has difficulty in mentally separating. For this reason, draftsmen always work in the first or the third angle, according as they may wish to place plan below or above elevation on the drawing sheet. In England, draftsmen usually follow the first course, but in many American offices the latter system is employed. In this book, objects are mostly placed in the first angle. The student of geometry must be able to work in all four angles, because even when operations are conducted chiefly in one angle, some part of a construction may come into either angle. In many books, considerable work is done in the second and fourth angles, but this course is both unnecessary and undesirable. Unnecessary because planes of projection can be assumed anywhere we please, and these planes can always be placed below and behind the object; undesirable because the overlapping already referred to tends to mystify the student and make the subject, which at the best is a difficult one to grasp, still more difficult.

Exercises.

Draw plan and elevation of a line AB 3 in. long when placed

(1) Parallel to HP and to VP and 1 in. from xy.
(2) In HP and inclined at θ (40°) to VP.
(3) „ VP „ „ „ HP.
(4) Parallel to and 1 in. above HP and ⊥r VP.
(5) „ „ in front of VP and ⊥r HP.
(6) Inclined to HP at 50° and to VP at 20°.
(7) „ „ 35° „ „ 45°, and having one end in xy.

How many solutions are there for (6) and (7)?

(8) Show by plan and elevation two equal parallel lines inclined to both planes of projection.

(9) The plan of a line AB is 2 in. long, its elevation is 3 in., the projectors of its extremities are 1½ in. apart, and A is 1 in. above HP and 1 in. in front of VP. Determine the length of the line, and its inclinations to the planes of projections. Show also the traces of the line.

(10) A line 3 in. long is inclined to the HP at 40°, and to the VP at 30°. Determine the traces of the line if one end of the line is 2·5 in. from VP.

(11) A line AB, 10 cm. long, has its end A in the horizontal plane, and AB is inclined at 35° to the latter. Determine the length of its plan ab, and the height of B. You may calculate these if you wish.

Represent AB by a figured plan. Draw an elevation on a vertical plane which contains B and is 3 cm. from A. (B.E., 1904.)

CHAPTER XXVI.

PROJECTIONS OF SIMPLE FIGURES.

460. **Representation of Plane Figures.** Some facts previously discussed are, for convenience, recapitulated:—

The projection of a rectilinear figure on a given plane is obtained by projecting the angular points of the figure on to the plane, and joining up these points in proper order.

The projection of a curved figure on a given plane is obtained by projecting numerous points in the curve on to the plane, and drawing a fair line through the points.

461. Since a line in space is represented by its projections on two rectangular planes (Art. 439) or by an indexed plan (Art. 447), it follows that any plane figure may be represented by plan and elevation on any planes of reference, or by an indexed plan.

Theorem. (a) *If any plane figure is parallel to a plane, the figure and its projection on that plane are alike in all respects.*

Theorem. (b) *If a plane figure is inclined to a plane, the projection of the figure on that plane is* **not** *the same shape as the figure.*

Theorem. (c) *The projection of any plane figure, rectilinear or curved, on a plane perpendicular to the plane of the figure is a straight line.*

462. From Art. 461 we see (*a*) When a plane figure is parallel to the $\frac{\text{VP}}{\text{HP}}$ the $\frac{\text{plan}}{\text{elevation}}$ of the figure is a straight line parallel to XY, and the $\frac{\text{elevation}}{\text{plan}}$ gives precise information as to the size and shape of the figure and the angle which any line in the figure makes with the $\frac{\text{HP}}{\text{VP}}$.

(*b*) When a plane figure is perpendicular to the $\frac{\text{HP}}{\text{VP}}$ the inclination of the figure to the $\frac{\text{VP}}{\text{HP}}$ is shown by the angle between the line $\frac{\text{plan}}{\text{elevation}}$ and XY.

463. In Fig. 438 (*a*), (*b*), (*c*) a triangle ABC is shown by plan and elevation. These diagrams show the position of the triangle relative to the planes of projection.

Thus in Fig. (*a*), (1) the plan *abc* is a straight line; therefore the figure is vertical. (2) The plan is parallel to XY; therefore the figure is parallel to VP, and the elevation *a'b'c'* shows the size and shape of the figure, the height of each corner, and the inclination of each side to HP.

Fig. (*b*). The plan *abc* is a straight line inclined to XY, at the angle ϕ. Hence the figure is vertical, and its plane is inclined to VP at the angle ϕ.

The distance of the corners from VP are shown by the distance of their respective plans from XY. The elevation shows that AC is level, and B is the highest point in the figure. In this case, the inclination of AB, or BC, to HP is **not** shown in the elevation.

Fig. (*c*). The plan is a straight line perpendicular to XY. Hence the plane of the figure is vertical and perpendicular to VP, and the elevation is a straight line, being in the prolongation of the plan. B is seen to be the highest point; whilst the coincidence of *c'* and *a'* shows that CA is perpendicular to VP. The reader will now examine Figs. (*d*), (*e*), (*f*), and satisfy himself that the position of the triangle in these three diagrams is correctly given below:—

Fig. (*d*). Perpendicular to VP and parallel to HP. Side AC is parallel to VP. BC is inclined to VP at α.

Fig. (*e*). Perpendicular to VP, and inclined to HP at θ AC is parallel to VP.

Fig. (*f*). Perpendicular to both HP and VP. The side AC is vertical.

464. A geometrical figure of three dimensions may be

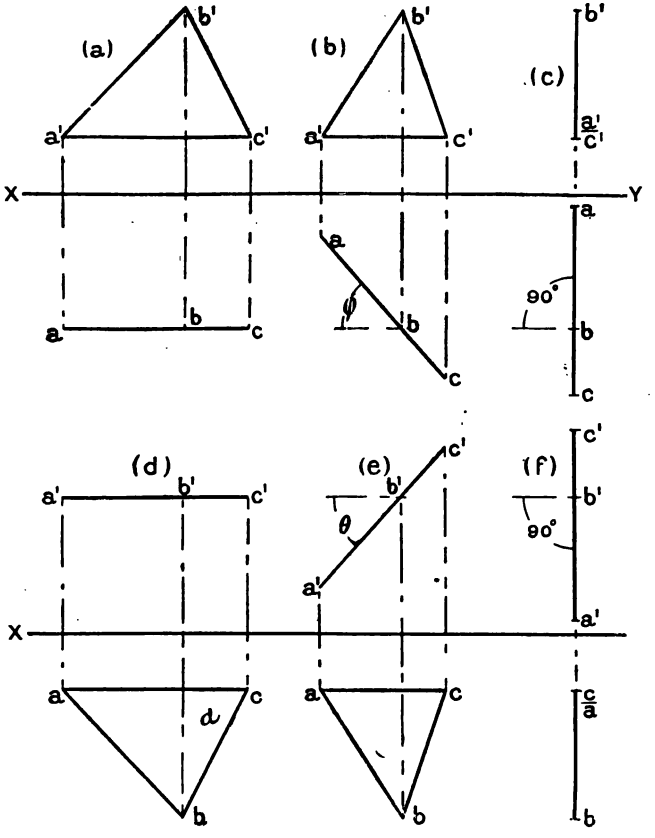


FIG. 438.

represented by **plan** and **elevation** on fixed planes of reference, or by a plan only provided numerals are attached to the plans of some definite points giving the position of the points relative to HP as explained in Art. 447.

In the case of a **polyhedron**, *i.e.*, any geometric figure bounded on all sides by planes, the projections of the figure are obtained by projecting the angular points of the figure and joining up these points by straight lines representing the *edges*, or intersections of the plane faces, of the solid.

465. If we examine a spherical object—say a tennis ball—we notice that it appears the same shape from every point of view. The projection of a sphere on any plane is a circle of diameter equal to that of the sphere; hence *all projections of a given sphere are equal circles.*

466. **Representation of the cone.** Any cone is represented in projection by the projection of its base and apex and lines from the apex tangent to the base.

A *right circular cone* may also be represented by the projection of its apex and any inscribed sphere, and lines from the apex tangent to the sphere; or, by two inscribed spheres and their tangents. See Fig. 452.

467. **Representation of a cylinder.** Any cylinder may be represented in projection by the projections of its ends and tangents parallel to the axis of the cylinder. A right circular cylinder may also be represented by any two inscribed spheres, or—what amounts to the same thing—the axis of the cylinder and an inscribed sphere.

468. If we look at a solid from any point of view there will always be some lines or points hidden from our view by some part of the solid. The position of such lines are indicated in projections by dotted lines. On working drawings there should be shown only such lines and points as help to make the subject clear.

469. The sphere is the only geometric figure that has equal projections on all planes irrespective of its position relative to those planes.

The *shape* of the projection of every other figure is affected by the angular position of certain lines in the figure relative to the plane of projection. The projections of the triangle in Art. 463 illustrate this.

470. In representing an object by projections, the drafts-

man gives those views which supply, in the clearest manner, information as to magnitude and position of all important features. In some cases, one plan and one elevation give all the information required, whereas in others several views are necessary.

471. It will readily be understood that the difficulty experienced in making projections of a given object depends largely on the position of the object relative to the planes of projection. When the prescribed position of the object is such as to render the making of projections somewhat difficult, it is often convenient to make a plan, and an elevation of the object when placed in a position which makes the drawing of these projections as easy a problem as possible, and then to obtain other views, by one of two methods.

First Method:—*Let the original* VP *and* HP *remain fixed and rotate the object about a convenient axis into the desired position and then make a projection on the fixed planes.*

This is called **revolving the figure.**

Second Method:—*The object is supposed to remain fixed and a new plane is placed in proper position to receive the desired projection.*

This is called **changing the ground line.**

Facility in performing these operations is essential to progress in descriptive geometry.

We give examples explaining both methods.

472. *First Method.* Fig. 439. Suppose the vertical triangle shown in (*a*), connected to a vertical wire which passes through the corner B and penetrates HP in *b*. Let this wire serve as an axis about which the triangle may rotate. Then, during rotation, the wire does not change its position, consequently B does not move, nor does the elevation (*b'*) and plan (*b*) move: But every other point in the plane of the triangle moves round the axis, and the locus of any particular point is a horizontal circle.

Thus, the locus of A is a circle of radius *o'a'*; its plan is an equal circle having *b* for centre; its elevation lies in the straight line *a'o'*, parallel to *xy*. Similarly, the locus of C is a circle of radius *n'c'*, its plan being a circle of radius *n'c'* centred in *b*; and its elevation a horizontal line through *c'*.

After rotation through an angle θ, the plan and elevation of the triangle are as shown at Fig. (*b*). After rotation through a right-angle, its plan and elevation are both straight lines as shown in Fig. (*c*).

Let us now refer to Figs. (*d*), (*e*), (*f*), which represent the triangle in positions perpendicular to VP.

If the triangle rotate about a straight line (B*b'*) in its plane perpendicular to VP, any point in the figure (and not in the axis of rotation) will describe a circle, the radius of which will be equal to the distance of the point from the axis; and the plane of the circle will be parallel to VP. After rotation through an angle ϕ (Fig. *e*) the plan (*b*) and elevation (*b'*) of B will remain stationary, whilst A and C move into new positions, *a*, *a'*; *c*, *c'*; the plan and elevation of the triangle now being *abc*; *a'b'c'* (Fig. *e*). When the triangle has been rotated through a *right* angle, its plan and elevation are both straight lines, as shown in Fig. (*f*).

Hence, *when a plane figure is perpendicular to* $\frac{\text{HP}}{\text{VP}}$ *and inclined to* $\frac{\text{VP}}{\text{HP}}$ *at a given angle* ϕ, *the projections of the figure may be determined by drawing, in the first place, the* $\frac{elevation}{plan}$ *and* $\frac{plan}{elevation}$ *of the figure when it occupies correct position relative to the* $\frac{\text{HP}}{\text{VP}}$, *but is parallel to the* $\frac{\text{VP}}{\text{HP}}$ $\left(\text{*where its* } \frac{elevation}{plan} \text{ *exhibits the position of every part of the figure relative to the* } \frac{\text{HP}}{\text{VP}}\right)$, *and afterwards revolve the figure about an axis perpendicular to the* $\frac{\text{HP}}{\text{VP}}$ *until its line* $\frac{plan}{elevation}$ *makes with xy the given angle* (ϕ) *between the figure and* $\frac{\text{VP}}{\text{HP}}$; *copy the* $\frac{plan}{elevation}$ *in this new position, and project a second* $\frac{elevation}{plan}$ *therefrom.*

The two cases are illustrated in the next Art.

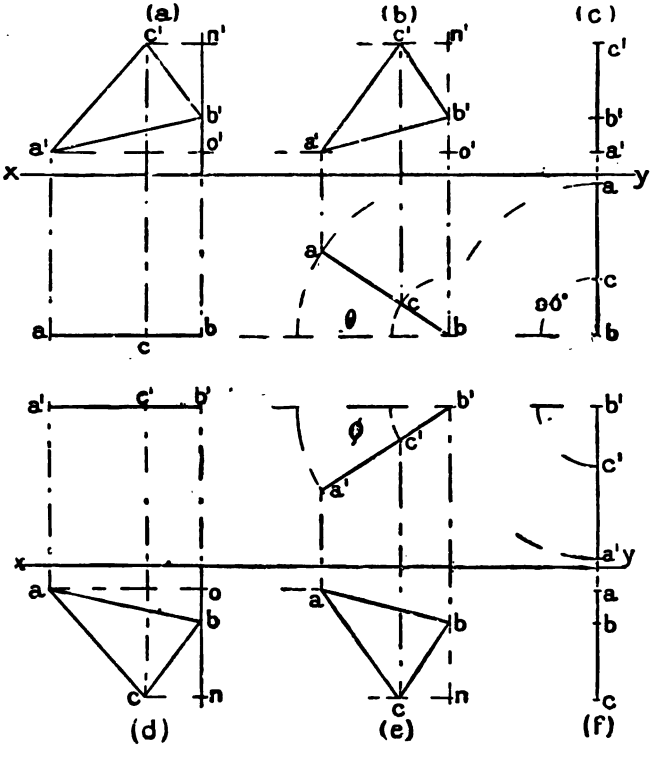


FIG. 439.

473. **Problem.** *Fig.* 440. *Draw the plan and elevation of a triangle* ABC, *having given* AB = 3 *in.*; BC = 2 *in.*; AC = 2·5 *in. The triangle is perpendicular to* HP *and inclined to* VP *at* 40°; *the corner* A *is in* HP *and* 1 *in. in front of* VP; *the side* AB *is inclined at* 35° *to the* HP.

Construction. Draw xy, and take a' anywhere in same: Draw $a'a$ perpendicular to xy and equal 1 in. Draw

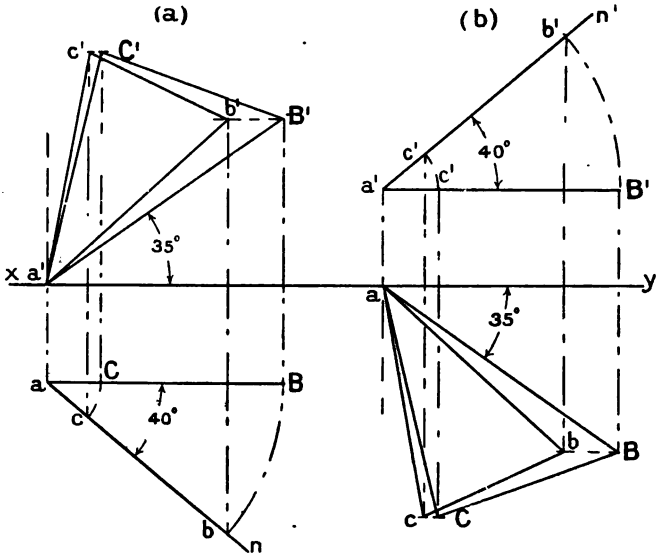


FIG. 440.

$a'B'$ 3 in. long and inclined at 35° to xy. Draw a perpendicular from B' on to a line through a parallel to xy, meeting same in B. Make $a'C' = 2{\cdot}5$ in.; and $B'C' =$ 2 in. (the given lengths of AC; BC). Draw a projector from C' on to aB at C; then aBC, $a'B'C'$ are plan and elevation of the given triangle when in position parallel to VP. Rotate the figure about a line perpendicular to HP and passing through the corner A: Then the plan (a) and

elevation a' of A will remain stationary. Draw from a a line an making 40° (the given angle) with aB. With a as centre and radius aB describe an arc meeting an in b. The elevation of the locus of B is a straight line through B' parallel to xy : The elevation b' is located by a projector from the plan b meeting the parallel to xy drawn from B'. With a as centre, and radius aC, describe an arc meeting an in c. From c draw a projector to meet a parallel to xy drawn from C' in c'. Join a', c' ; c', b' ; b', a'. Then abc, $a'b'c'$ are plan and elevation of the triangle satisfying all the conditions given in the problem.

If we transpose the terms HP, VP, in Problem 473 we obtain the following : *Draw the plan and elevation of a triangle* ABC *having given* AB = 3 *in.;* BC = 2 *in.;* AC = 2·5 *in. The triangle is perpendicular to* VP *and inclined to* HP *at* 40° ; *the corner* A *is in* VP *and* 1 *in. above* HP ; *the side* AB *is inclined at* 35° *to* VP.

The solution of this problem is identical with that just given : it is illustrated in Fig. 440 (b). Observe that this Figure (b) is obtained by turning (a) through 180° about xy.

Exercises on Arts. 472—476.

(1) An equilateral triangle abc of 2 in. side is the plan of a triangle ABC. The corner A is $\frac{3}{4}$ in. above HP and $\frac{1}{2}$ in. in front of VP, C is $1\frac{1}{2}$ in. above HP and in front of VP, B is in HP. Revolve the triangle into HP. Also revolve the triangle into VP.

(2) Make a projection of the triangle given in the preceding question on a plane parallel to the side AB. Also make a projection on a plane perpendicular to AB.

(3) Determine the circle that can be inscribed in the triangle ABC of Question 1, and show it by its plan and elevation.

(4) A rectangle $abcd$ is the plan of a square ; if $ab = cd =$ 2 in., and $ad = bc = 3$ in., what is the size of the square, and what is the inclination of the square to HP ? Show the plans of the inscribed and circumscribed circles.

(5) The plan of a circle of 2 in. diameter is an ellipse having one axis $1\frac{1}{4}$ in. long. What is the length of the other axis of the ellipse, and what is the angle between HP and the plane of the circle ?

474. Ex. 2. **Curved figures.** We give examples of a circle perpendicular to one plane of projection and inclined at a given angle to the other. As before, the two cases are reciprocal and one description will suffice for both.

Problem. Fig. 441 (*a*). *A circle of radius* 1·5 *in. touches the* VP, *and its plane is perpendicular to the* VP *and inclined to* HP *at a fixed angle* (45°).

Construction. Draw xy, and any perpendicular $o'd$ meeting xy in d: Make dc below xy equal 1·5 in. (the given radius of circle). With c as centre and radius cd describe a circle. Through c draw AB—a diameter parallel to xy. At any convenient distance above xy, draw A′B′ parallel to xy; and draw perpendiculars from AcB on to this line, meeting same in A′c'B′.

Let the circle rotate about an axis through its centre and perpendicular to VP; then $\frac{e'}{c'}$, e, c, d the projections of $\frac{}{d'}$ points E, C, D do not move; whilst every other point in the circle has a circular movement parallel to VP. Through c' the elevation of the *axis*, draw a line making 45° with A′B′ (45° being the given angle between figure and HP). With c' as centre, and radius c'B′ = c'A′, revolve B′ into position b'; and A′ into position a'. Perpendiculars from b' and a' on to AB determine the plans of the points in b, a. The elevation of the circle is the straight line $a'b'$, because the plane of the figure is perpendicular to VP. The plan of the circle is an ellipse having ed, ab for major and minor axes, and may be drawn as explained in Art. 241. The plan of any point P in the curve is determined thus :—

Draw a perpendicular on to the initial elevation A′B′ at P′; fold P′ into $a'b'$ at p', and draw a projector from p' to meet the parallel to xy from P in p. Then p, p' are plan and elevation of the point P.

In a similar manner, a number of points may be obtained and a fair curve through the plans of all the points is the desired plan of the circle.

Fig. 441 (*b*) illustrates the solution of the problem when the plane of the circle is perpendicular to HP.

475. When drawing curves freehand, a few tangents to the curve are of great assistance. The tangent at any point in the projection of the circle is readily found by aid of the following principle :—

If a curve and its tangent at any point P *be projected on*

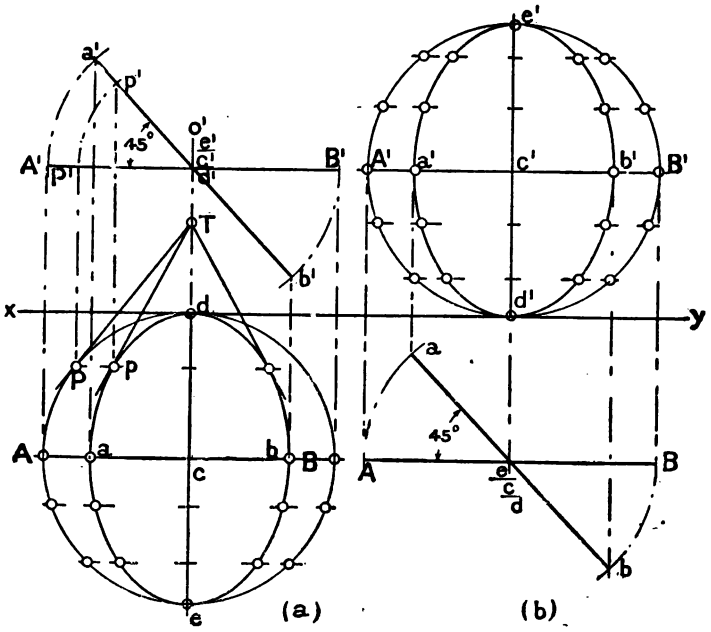


FIG. 441.

to any plane, the projection of the tangent is tangent to the projection of the curve at the projection of the point P.

Thus, in Fig. 441 (*a*) take any point P on the circle and draw the tangent of the point, and let it intersect the axis of revolution in T. Then, as the figure revolves about the axis, the point T in the tangent does not move, but the point P moves to *p* and a line T*p* is tangent to the projection of the circle at the given point.

476. *Second Method.* In Art. 431, it is shown that, so long as the position of a point (P) relative to a fixed HP remains fixed, the point has only one projection on that plane, but any number of vertical planes may be grouped around the point, and the projections (elevations) of the point on such planes **are all the same height above the respective ground lines.**

The application of these principles are further illustrated

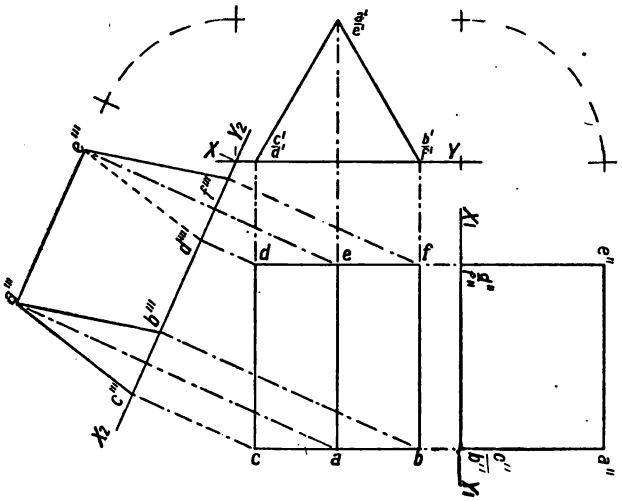


FIG. 442.

in Fig. 442, where a triangular prism is shown by a *plan*, and elevation on three vertical planes, represented by the ground lines XY, X_1Y_1, X_2Y_2.

Note that every elevation of a particular point is the same height above the respective ground-lines. Also, that all the elevations are projected from the same plan, and that the plan may be projected from any elevation, but one *elevation* **cannot** be projected from *another* elevation.

In further illustration, let three truly plane surfaces—say, three thin sheets of metal—be rigidly connected and

arranged to form the model pictured in Fig. 443,* the plane α being perpendicular to both β and ϕ. Let a triangle ABC be rigidly connected to the plane α by wires or other suitable means: Let the plane of the triangle be perpendicular to plane α. Let the model rest with the plane α on a table. Then, α is a horizontal plane whilst β and ϕ are both vertical. The triangle projects on to plane α in the straight line $\frac{a}{c}\ b$, on to plane β in $a'b'c'$; and on to plane ϕ in $a''b''c''$.

Now, since the plane α is horizontal, $\frac{a}{c}\ b$ is a plan of the triangle: And, since β is a vertical plane, $a'b'c'$ is an elevation. Similarly, $a''b''c''$ is an elevation on a vertical plane ϕ.

In Fig. 446, the projections of the triangle on the three planes are given.

Let a hole be made in the plane α of our model, and let the model be hung on a nail against a truly vertical wall-face as illustrated in Fig. 444. Then the plane α (to which the triangle remains rigidly connected) becomes a vertical plane, and if the model be rotated on the nail, and the plane α be kept in contact with the wall surface, the planes β and ϕ will always be perpendicular to the wall plane, and always have the same position relative to one another and to the triangle attached to α.

In Fig. 444, the plane β is horizontal, and in accordance with convention, the projection on α is an **elevation**, whilst the projection on β is a **plan**.

Now the projections on the three planes remain precisely as before; because the relative positions of the planes and triangle have not been changed. Therefore, every relationship which we have seen exists between the *plan* of the triangle in Fig. 443 and the projections (elevations) on the planes β and ϕ, also holds now between the *elevation* of the triangle on the wall plane (α) and the projections on planes β and ϕ.

Let the model supported on the nail be revolved so as

* This Fig. will be found facing p. 458.

to bring the plane ϕ into a horizontal position, as shown in Fig. 445. Then, in accordance with convention, the projection of the triangle on plane ϕ is a **plan**, whilst the projection on plane β is an **elevation**.

From the foregoing we conclude that the operation of projecting a given geometrical figure on to a plane perpendicular to a fixed VP is *practically identical* with that of getting an elevation on a plane perpendicular to a fixed HP.

Rule. *To determine a new* $\frac{plan}{elevation}$ *of a figure on any plane perpendicular to the first* $\frac{VP}{HP}$ *and represented by* $x'y'$, *draw from points in the first* $\frac{elevation}{plan}$ *projectors perpendicular to* $x'y'$ *and mark the new* $\frac{plans}{elevations}$ *of the points the same distance* $\frac{in\ front\ of}{above}$ $x'y'$ *as their first* $\frac{plans}{elevations}$ *are* $\frac{in\ front\ of}{above}$ xy. *Then join up the new projections in proper order.*

The student will see that no matter what is the position of plane ϕ during the time the plane β occupies a horizontal position, it will always be possible to give such rotation to the model as will suffice to bring ϕ into a horizontal position. And, as each plane perpendicular to a fixed VP is represented in orthographic projection by the line (xy or $x'y'$) in which the plane intersects VP, it is only necessary to turn the drawing-board through some angle so as to give to the particular ground line that direction most easily associated with horizontality in order to effect the rotation of the model illustrated above. Beginners usually find it easiest to imagine lines drawn parallel with the base of the drawing-board as being horizontal lines; but, if the drawing-board lies level on a table, every line on the drawing-board is a level line, and this fact may help the student to realise that we may take any direction on the drawing-board for horizontal lines.

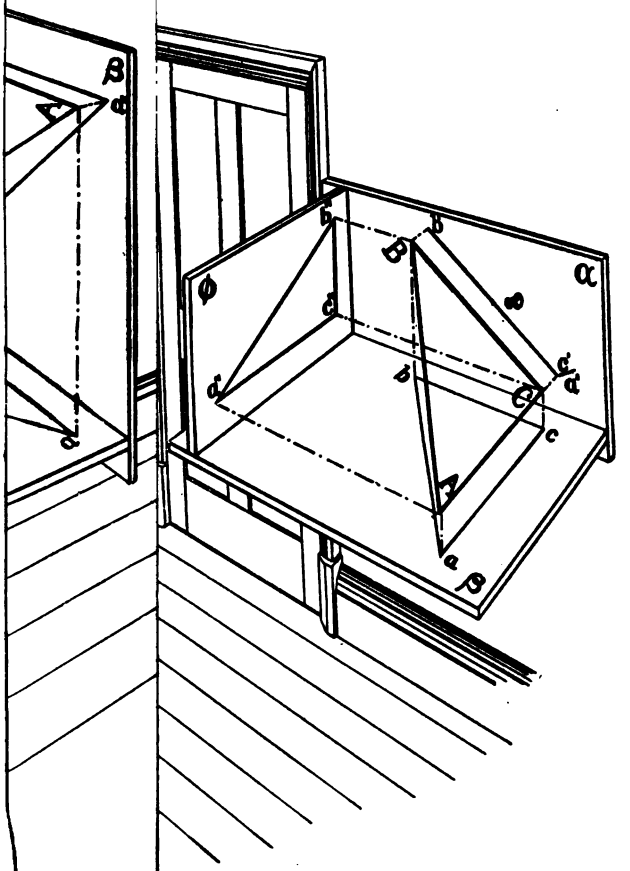


FIG 444.

[*To face p.* 458.

477. **Problem.** Fig. 447. *Given a curve by plan and elevation* $\frac{\text{elevation}}{\text{plan}}$ *a, b . . . g; a' b' . . . g', draw a new* $\frac{\text{elevation}}{\text{plan}}$ *of the curve on x'y'.*

Remember. A new $\frac{\text{elevation}}{\text{plan}}$ is projected from the $\frac{\text{plan}}{\text{elevation}}$.

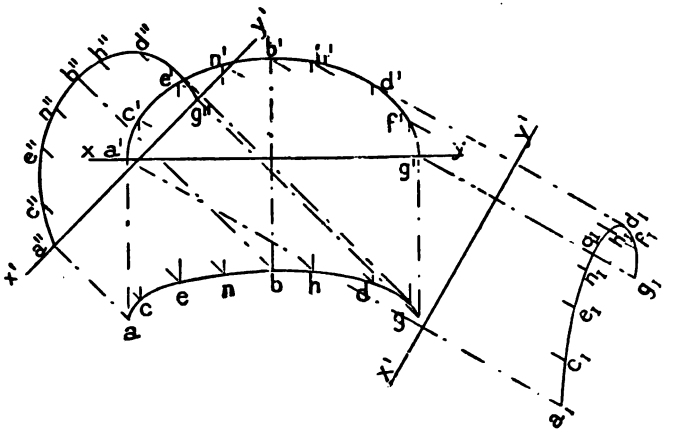


FIG. 447.

Construction. From the points $\frac{a,\ b,\ \ldots\ g}{a',\ b',\ \ldots\ g'}$ draw projectors perpendicular to $x'y'$; make the distance of $\frac{c''}{c_1}$ $\frac{\text{above}}{\text{in front of}}$ $x'y'$ equal the distance of $\frac{c'}{c}$ $\frac{\text{above}}{\text{in front of}}$ xy. Make the distance of $\frac{b''}{b_1}$ $\frac{\text{above}}{\text{in front of}}$ $x'y'$ equal the distance of $\frac{b'}{b}$ $\frac{\text{above}}{\text{in front of}}$ xy. In a similar manner, determine $\frac{a'',\ d'',\ e',\ f'',\ g''}{a_1,\ d_1,\ e_1,\ f_1,\ g_1,}$ and draw a fair line through all the points. This is the required new $\frac{\text{elevation}}{\text{plan}}$.

478. **Problem.** Fig. 448. *A cube of* 2*-in. edge rests on a face in* HP. *Draw its plan and make* (1) *a projection of the cube on a* VP *parallel to a diagonal of the cube;* (2) *a projection on a plane perpendicular to that diagonal.*

1. The plan of the cube will be a square having sides equal to the edge of the cube, because the top face is vertically over the bottom face. Hence, draw a square of 2-in. side as $\frac{a}{c}\,\frac{e}{f}\,\frac{b}{h}\,\frac{d}{n}$.

To obtain a projection parallel to a diagonal. We observe that a diagonal of the cube joins opposite corners in opposite faces of the cube. See Fig. (*a*), where CB is a diagonal of the cube of edge AC, AB being a diagonal of a face.* And, since AC is vertical, the diagonal CB will project its true length on a VP parallel to the diagonal *ab* of the top face.

Hence, draw XY ‖ *ab*, and project the corners *a, e, b, d, c, f, h, n* on to this plane and join up.

2. *To obtain a projection on a plane* $\perp^r$ *to* AH we have only to draw X_1Y_1 in any position $\perp^r$ to the elevation *a'h'* and project each corner of the cube, making the distance of each corner below X_1Y_1 equal to the distance of the respective corners below XY, and join up in proper order.

The complete projection is a regular hexagon with its diagonals.

Octahedron.—The surface of the *octahedron* is composed of eight equilateral triangles. A development of the surface is shown in Fig. 449 (*b*). The sketch (Fig. (*a*)) shows that the octahedron consists of two equal square pyramids base to base.

Problem. *Fig.* 449. *Draw projections of an octahedron* 2*-in. edge, which rests on a face in* HP.

Let the face ABE be placed in HP, then an equilateral triangle *abe* of 2-in. side is the plan of that face. As ABDC is a square its plan will be a rectangle on *ab*.

To determine the length of the projections ad, bc, observe that the opposite faces EAB, NDC are parallel, as also are the faces EDC, NBA, and if lines be drawn from N and

* The edge of a cube, diagonal of its face and diagonal of the cube, are in the ratio $1 : \sqrt{2} : \sqrt{3}$.

E to O and R the mid-points of CD, AB, the figure NOER is a rhombus, sides of which are equal to the altitude of the

FIG. 448.

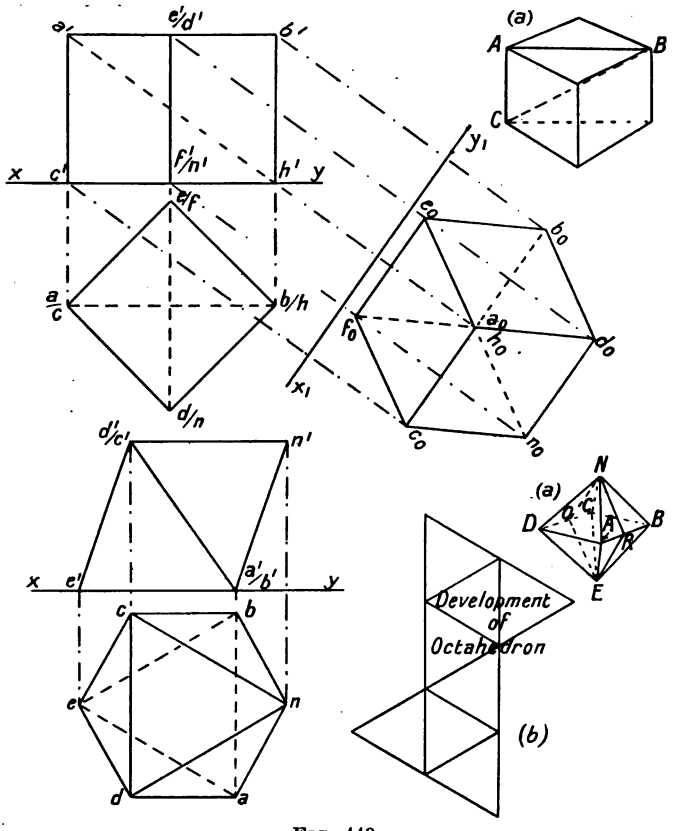


FIG. 449.

face triangle. The plane of the rhombus is $\perp^r$ AB, and will project its true shape on a VP $\perp^r$ to the plan *ab*.

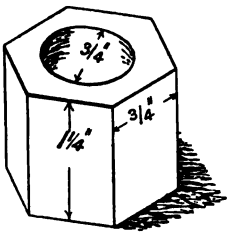


FIG. 450A.

Hence, draw XY $\perp^r$ *ab*, project *abe* on to XY at *b'a'e'*. Construct the rhombus on side *b'e'* making *a'd* = *ab* and join *c'a'*. Then $\frac{e'd'}{c'}\ n'\ \frac{a'}{b'}$ is complete elevation of the octahedron. From this elevation the plan is readily projected at *a d e c b n*.

The complete plan is a regular hexagon with lines joining alternate corners.

479. **Problem.** *Fig.* 450A *is a sketch of a regular hexagonal prism having a cylindrical hole bored through its centre. The diameter of the hole and the dimensions of the prism are figured on the sketch.*

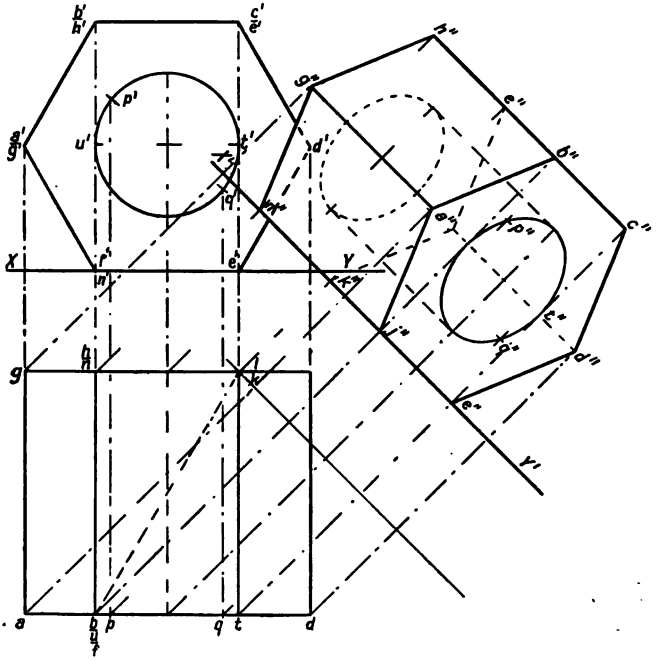


FIG. 450B.

Draw the projections of the solid, when a rectangular face is in HP, *and a diagonal of that face is inclined to* VP *at a given angle, say* 75°.

It will be convenient to draw first the elevation of the object on a VP parallel to the ends, from which a plan can be projected, and then project a second elevation from this plan.

First draw a hexagon $a'b'c'd'e'f'$, Fig. 450B, having sides equal to 0·75 in. (the given length). Centre a circle of the given diameter in the centre of the hexagon. Draw XY through a side $f'e'$. (Then the hexagon and circle is the elevation of the figure when it rests with a rectangular face in HP and its polygonal ends are parallel to VP, in which position the plans of its long edges will be perpendicular to XY and equal (1·25 in.) the given length.) Make ga = 1·25 in., and draw ad parallel to XY. Project the centre and the diameter TU of the circle which is parallel to HP on to ad in plan. We now have the plan and elevation of the figure when one end is in VP. Draw fk a diagonal of the face in HP and draw X'Y' making 75° (the given angle) with fk. Then X'Y' is the ground-line of a VP which is inclined at the given angle to the diagonal FK. The elevation of the figure on X'Y' is projected from the plan. Thus, from a draw a projector perpendicular to X'Y' and mark a'' above X'Y' equal to the height of a' above XY.

In precisely the same way, obtain the projections of all the angular points, and join them up in proper order by straight lines. To obtain the elevation of the circular hole, determine the projection of the vertical and horizontal diameter of each circular end and some intermediate points, as P, Q, and draw a fair curve through the projections of such points. The projections of circles are ellipses.

The hole and a portion of the hexagon at one end is hidden by the solid, and should be shown by dotted lines.

Exercise.

(1) Draw the plan and elevation of a square prism which rests on one long face on the HP, and has its long edges inclined at 30° to the VP. Nearest corner of prism is 1″ from XY.

480. **Problem.** Fig. 450. *A regular tetrahedron of* 2 *in. edge rests on a face in* HP. *Draw its plan and make* (1) *a projection of the tetrahedron on a* VP *parallel to a slant edge;* (2) *a projection on a plane perpendicular to that slant edge.*

NOTE.—A *tetrahedron* is a triangular pyramid, and is thus bounded by four triangles. A *regular tetrahedron* is bounded by four *equilateral triangles.* A development is given in the Fig.

The plan of the tetrahedron when resting on a face is an equilateral triangle of 2 in. side, and lines drawn from the centre v of the triangle to each corner. Hence, draw an equilateral triangle ABC of 2 in. side, find its centre v and join vA, vB, vC.

To obtain a projection on a plane parallel to a slant edge, AV *say* :—Draw xy in any position parallel to the plan Av, and draw projectors from the apex v and corners A, B, C. (We note that BC is perpendicular to Av and xy and that the projector from C is also the projector from B.) The elevation of A is on xy at a'; the elevations of B and C are also on xy and coincide in $\frac{c'}{b'}$. To determine the elevation of the apex v: Because Av is parallel to xy it will be seen in true length in elevation. Hence, describe an arc from centre a' with radius AB (= edge of tetrahedron) crossing the projector from v in v'. Join $v'a'$, $v'\frac{c'}{b'}$. This triangle is the complete elevation, because the elevation of vB and vC coincide.

To determine the projection on a plane perpendicular to Av: draw $x'y'$ anywhere perpendicular to $a'v'$: draw projectors from v', a', $\frac{c'}{b'}$ and make $o\frac{v}{a} = a'$A = the distance of A in front of xy. Make $o'b$, $o'c$ respectively equal to the distance of the points B and C in front of xy. Join $\frac{v}{a}b$, $\frac{v}{a}c$. Then $\frac{v}{a}bc$ is the required projection.

Observe, that the line $b\frac{v}{a}$ in the projection on $x'y'$ is a measure of the distance of the corner B from the edge Av., *i.e.*, the shortest line that can be drawn from B on to the edge Av is equal to $b\frac{v}{a}$. Similarly $c\frac{v}{a}$ is a measure of the distance of B from the edge Av. In this case $c\frac{v}{a} = b\frac{v}{a}$, but this would *not* be the case if the edges vB, vC were unequal in length.

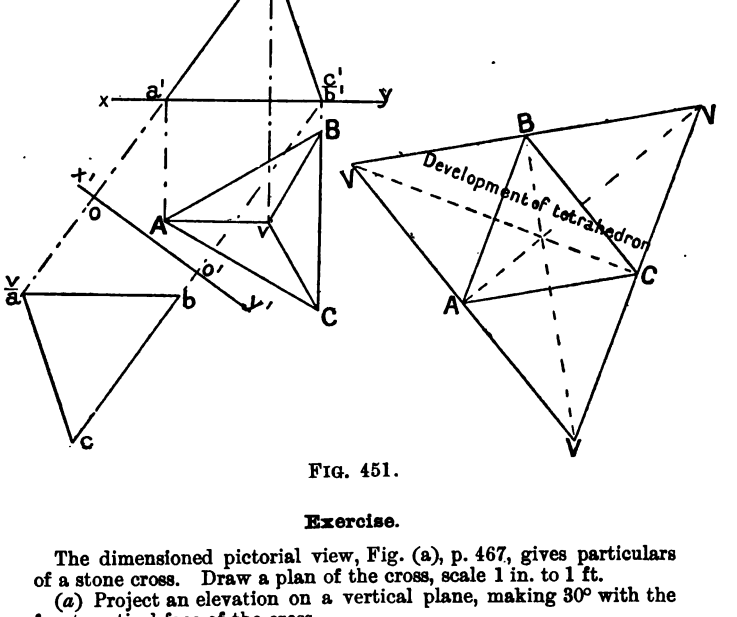


FIG. 451.

Exercise.

The dimensioned pictorial view, Fig. (a), p. 467, gives particulars of a stone cross. Draw a plan of the cross, scale 1 in. to 1 ft.

(*a*) Project an elevation on a vertical plane, making 30° with the front vertical face of the cross.

(*b*) Draw the projections of the cross when it has been tilted over so that AB, CD are in HP.

(*c*) Make a projection on a plane perpendicular to AC.

(*d*) Make a projection on a plane perpendicular to AD.

481. **Problem.** Fig. 452. *The figure abcde is the plan of the centre line of a stair rail* ABCDE. *Two unfinished elevations are given; one of the portion* ABC, *the other of the remainder* CDE. *Complete both elevations.*

B. E. 1911.

The diagram referred to in the question is given in (*b*). It is reproduced in (*a*).

Select points as *f*, *g*, *h*, on the plan *bc* of the curve, and obtain the elevations of these points on *b'c'* at *f'*, *g'*, *h'*. Obtain new elevations of points FGHBA on *x'y'* at *f''*, *g''*, *h''*, *b''*, *a''*, making the height of each point above *x'y'* equal to its height above *xy*. A fair line through the points *f''*, *g''*, *h''*, *b''*, *a''*, completes the elevation on *x'y'*. Similarly take points *n*, *m* on the plan *cd*. Find their elevations at *n'*, *m'* and project new elevations on *xy* at *n''*, *m''*, *d''*, *e''*, making the height of each point above *xy* equal to its height above *x'y'*. A fair line through *n''*, *m''*, *d''*, *e''* completes the elevation on *xy*. Observe that *e*,'' *d''* is a straight line tangent to the curve at *d''*. Similarly, *a''* *b''* is a straight line tangent to the curve at *b''*, because *ed* and *ab* are tangents to the plan of the curve (refer to Art. 475). Observe also that projectors *n''n*, *f''f* are both tangents to the projected curves. The points N, F are *critical* points; they are easily determined by lines through the centre of the plan circle parallel to the given ground lines.

Exercise. Fig. (b), p. 467. A geometrical model is made by soldering together at O three pieces of wire OA, OB, OC. You are given the plan with OA, OB resting on the horizontal plane, and OC inclined upwards, the height of C being indexed in millimetres.

(*a*) Draw an elevation of the model on an *xy*, so chosen as to exhibit an edge or a profile view of the plane containing AOC. What is the inclination of this plane to the horizontal?

(*b*) Draw the plan of the model after it has been turned about OA as axis until OC rests on the horizontal plane. What is now the height of B? B.E., 1907.

(The diagram has been reduced. Make OA = 75 mm.; O*c* = 45 mm.; OB = 67 mm.; angle AO*c* = 38°; angle AOB = 62°).

Hint. Take *xy* ⊥r OA.

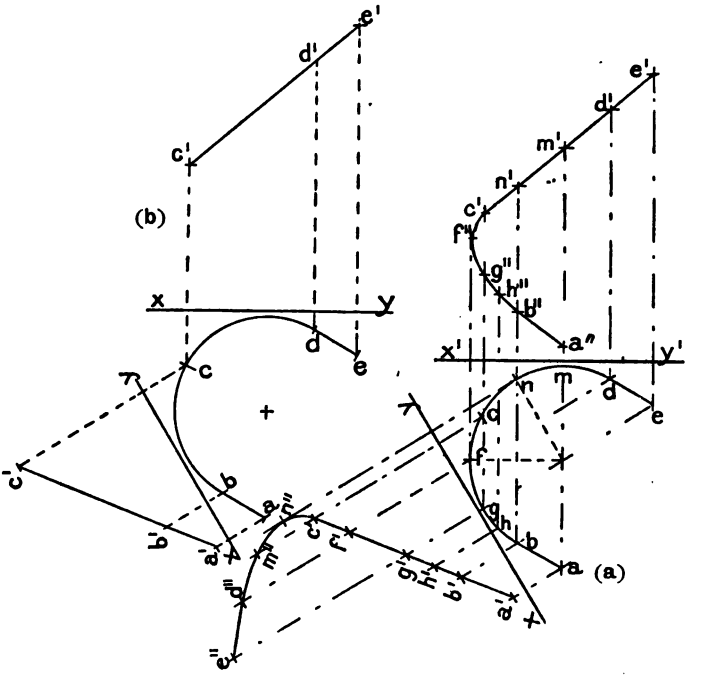


FIG. 452.

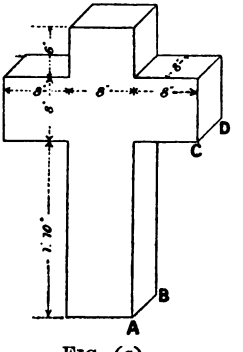


FIG. (*a*).

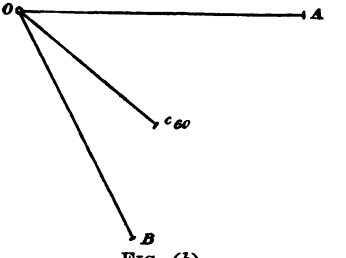


FIG. (*b*).

482. **Problem.** Fig. 453. *Draw plan and elevation of a cone having given apex angle* $=55°$, *axis* 3·5 *in. long, when the* **plan** *ac of the axis is* 2·25 *in. long and inclined at* 40° *to xy.*

Let *ac* be the given plan of axis 2·25 *in.* long and making the given angle with *xy*.

Analysis. The projection of the cone on a plane parallel to the axis is an isosceles triangle in which the axis is shown in real length perpendicular to the base, the apex and base angles are shown in true form, and the base of the triangle is a line elevation of the circular base of the cone. From this view, the plan of the cone is readily projected, and in turn an elevation on any VP can be projected from the plan, heights being transferred from the triangular elevation.

Construction. Make the plan of axis an auxiliary *xy*. With centre *a* and radius *a*C 3·5 in. (the given length of axis) describe a circle meeting the projector from *c* in *c'* : Draw *ac'* and *b'd'* perpendicular thereto. Set out on each side of *a c'* at *a* an angle of $\frac{55}{2} = 27\frac{1}{2}°$ (55° being the given apex angle) the arms of which meet *b'd'* in *b'*, *d'* : Project *d'* and *b'* on to *ac* at *d*, *b*. On the projector from *c'* make *ce*, *cf* each equal *c'd'* or *c'b'*. Then *bd*, *ef* are minor and major axes of the elliptical plan of the circular base. Draw the ellipse and tangents to same from *a* to complete the plan.

Obtain an elevation on *xy* of the point A at *a''*, and the lines BD, EF, at *b''d''*, *e''f''*, taking heights from the auxiliary elevation. These lines are conjugate diameters of the elliptical elevation of the base of the cone. Draw the ellipse and tangents to same from *a''* to complete the elevation.

Exercise. Draw the projections of a cone (dimensions as in Problem 482) *when the axis is inclined to* HP *at* 40°, *and to* VP at 25°.

Hint.—First determine the projections of a line AC 3·5 in. long, having the given inclinations (Problem 446), then apply Problem 482

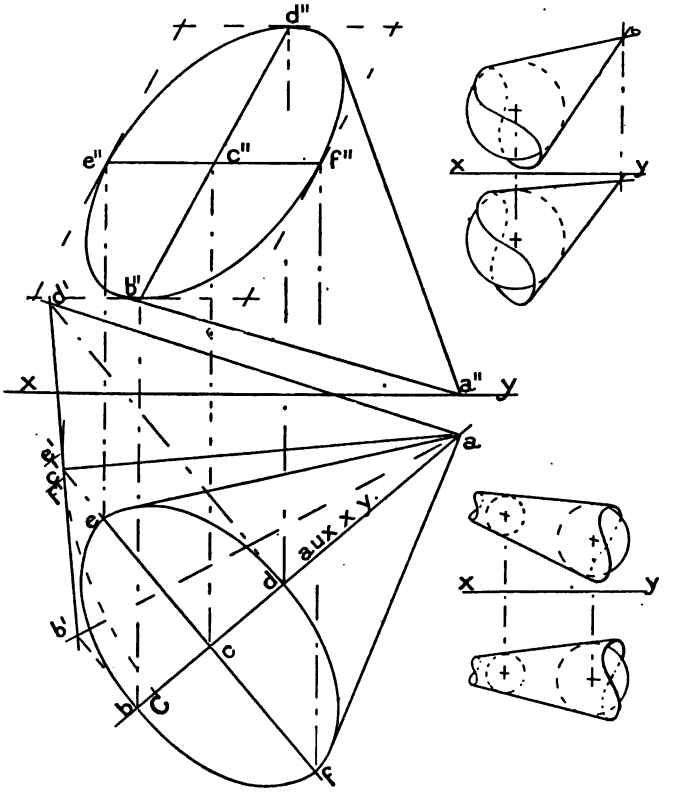


FIG. 453.

Exercises.

(1) Draw the projections of a 2-in. cube, when one face is horizontal and an adjacent face makes 35° with VP.

(2) Make a projection of the cube in Question 1 on (a) a plane ⊥r to a diagonal of one face ; (b) on a plane ⊥r to a diagonal of the cube.

(3) The cube in Q. 1 is pierced by a cylindrical hole 1″ diameter which passes through the centre of the cube, the axis of the boring being ⊥r to a face A of the cube. Draw projections when (*a*) a diagonal of the face A is horizontal and the face is inclined to HP at 25° ; (*b*) a diagonal of face A makes 50° with HP and 25° with VP ; (*c*) a diagonal of the cube is vertical ; (*d*) a diagonal of the cube is situated as in (*b*).

(4) Draw projections of a tetrahedron 1¾ in. edge, and show a development of the surface.

(5) Draw the projections of a 9 in. × 4½ in. × 3 in. prism when (*a*) one diagonal of the prism is inclined at 50° to HP ; (*b*) a diagonal is vertical ; (*c*) a long edge rests on the HP and the large face at that edge is inclined to the HP at 30°. Also draw an elevation on XY making 50° with the lowest edge of the prism. Scale ¼.

(6) An equilateral triangle ABC of 1½ in. side is the base of a pyramid the axis of which is 2½ in. long. Make a projection of the pyramid when a large face is in HP. Also, when a large face is in VP. What are the angles between the slant edges and the base ? Show also a development of the pyramid.

(7) A square pyramid, size of base 1¼ in., axis 2½ in., has its base in VP, one corner of the base in *xy*, and a side of the base inclined at 35° to HP. Draw plan and elevation. Draw also an elevation on a VP parallel to a slant edge of the pyramid.

(8) Draw a development and make a paper model of each of the following solids—(*a*) a cube, (*b*) regular tetrahedron, (*c*) regular octahedron, edge of each being 2 in., (*d*) square prism, (*e*) hexagonal prism, (*f*) octagonal prism, edge of base, and height in each case being, 1·5 and 2·5 respectively, (*g*) square pyramid, (*h*) hexagonal pyramid, (*k*) octagonal pyramid, the base and height in each case being as in (*d*) above. Determine the surface and volume of each solid. Compare carefully the results obtained in (*a*), (*b*), (*c*) ; in (*d*), (*e*), (*f*), and in (*g*), (*h*), (*k*).

(9) A line $a_0 b_{15}$, 3 in. long, is the plan of the axis of a right circular cone whose apex angle is 50°. Draw the plan of the cone and an elevation on a vertical plane making 30° with the plan of the axis.

(10) Make an elevation of the line $ab \ldots e, a'b' \ldots e'$, Fig. 452, on any ground line not parallel to *xy* or *x'y'*.

(11) Make a projection of the cone shown in Fig. 453 on a plane perpendicular to any generator of the surface.

(12) A circular cylinder, radius of base 1½ in., axis 3 in., rests on HP, with axis horizontal and inclined to VP at 35°. Draw its plan and elevation. Also draw projections of the cylinder when the axis is inclined to HP at 40° and to VP at 35°.

CHAPTER XXVII.

THE PLANE IN SPACE.

483. Any two non-parallel planes intersect in a straight line.

If we have fixed in position HP and VP of reference, a plane in space may be parallel to either one of these planes and inclined to the other; or, it may be inclined to both planes. If the plane be inclined to both HP and VP, it will have a line intersection in both.

The line in which a given plane intersects a plane of projection is called the **trace of the plane**: the trace in HP

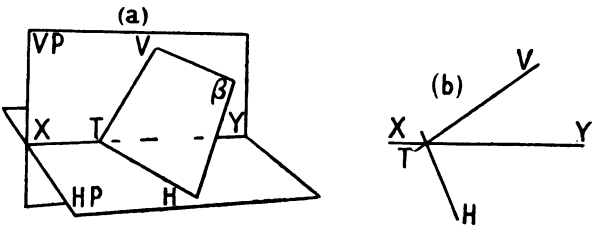


FIG. 454.

is called the **horizontal trace** (HT); the trace in VP is called the **vertical trace** (VT).

Fig. 454 is a sketch of a plane β intersecting fixed HP and VP. The lines VT, HT in which the plane β intersects the given VP and HP completely represent the plane, and in projection the traces only of a plane are used to indicate the plane's position. See Fig. (*b*).

In the pictorial view (*a*), the plane β is limited in extent so as to render its position more easily seen. But planes extend as far as necessary always, and therefore the traces of a plane are drawn any length we please, and extend above *or* below a given XY.

Figs. 455 (*a*), (*b*) show a plane in space occupying different positions relative to a fixed HP and VP, the former being pictorial sketches, and the latter orthographic projections of the plane traces.

The different positions of the plane marked (1) to (9) may be divided into four groups.

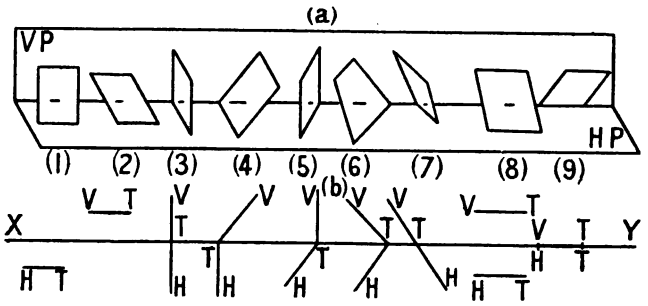


FIG. 455.

(i.) Parallel to one plane of projection; (1), (2) are in this group.

(ii.) Perpendicular to both planes, (3).

(iii.) Perpendicular to one plane, and inclined at some angle less than 90° to the other; (4), (5) are in this group.

(iv.) Inclined at some angle other than 90° to both of the planes; (6), (7), (8), (9) are in this group.

In (9) the plane contains XY, the intersection of VP and HP, and thus the plane intersects both planes of projection in XY. In this case, some additional data is required to fix the position of the plane; its trace on a third plane will suffice.

Observe that the traces of a plane always meet in XY, unless the traces are parallel to XY.

484. A plane which is $\perp^r$ to one plane of projection and inclined to the other at some angle other than 90° is called an **inclined plane.** A plane which is inclined at angles other than 90° to both HP and VP is called an **oblique plane.**

485. Some tangential properties of the right cone, which are extensively used in problems on planes.

Experiment. Fig. 456. Place a cone with its base on the drawing-board, and rest a card on the cone. Observe that the card has line contact with the cone. It contains the apex of the cone, and the bottom edge HT is tangent to the circular base of the cone. Also, the angle between the card and the drawing-board must be equal to the base angle of the cone.

Imagine the card to be a plane in space; then the bottom edge HT of the card is the trace of the plane on the plane of the drawing-board. Hence *if the circular base of a right cone rests upon (or is parallel to) a fixed plane, any plane which rests on the cone will have line contact with the cone, and contain the apex. The trace of the plane will be tangent to the circular base of the cone, and the angle between the planes will be equal to the base angle of the cone.*

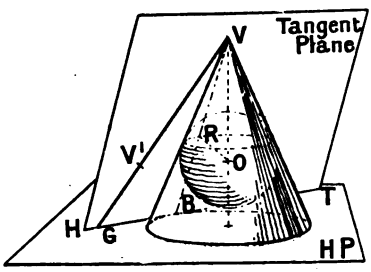


FIG. 456.

The plane represented by the card in Fig. 456 is called a **tangent plane** of the cone.

If from any point O in the axis of the cone (Fig. 456) OR is drawn perpendicular to a generator VB, a sphere centre O, radius OR will lie in the cone and touch the cone in a circle. The card also touches the sphere at a point on the circle of contact. If the card be moved into a new position but remain tangent to the cone, it will continue to touch the sphere at some point in the circle of contact.

From a point G in the bottom edge of the card, draw GV. Then the plane represented by the card contains the line GV in addition to being on the cone. If from any point V′ in the line GV a line be drawn to touch the sphere inscribed in the cone, and this line be made to move around the sphere and pass always through V′, it will generate a second cone enveloping the sphere, and the plane of the card will be tangent to this cone also. Thus we see that *a plane can be found which will be tangent to two cones if both cones envelop the same sphere.* There are two such planes, and they *always contain the line which joins the two apices of the cones.*

Fig. 457 shows two cones enveloping a sphere, the centre of which is in the intersection of the planes HP, VP. In this case, only a quadrant of the sphere is in the first dihedral angle, and the axes of the cones lie one in each plane. As the student proceeds, he will often find it convenient to place the sphere in this position.

Fig. 458. Imagine a fixed point V in space to be the common apex of two cones. If a card be placed on one cone and rolled on this cone, in either direction, its movement will be arrested when it comes in contact with the other cone, and, since it contains always the apex of the second cone, it will, when brought into contact with some other part of that cone, have line contact with the same. Hence, *a plane can be found tangent to two cones which have a common apex.*

(*Note.*—Neither cone must lie wholly inside the other.)

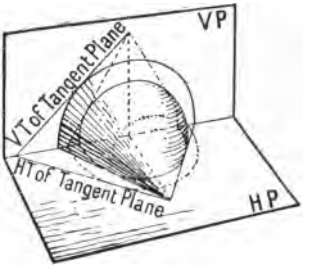


FIG. 457.

Again (Fig. 459), let any two cones have parallel and distinct axes and equal base angles; then *corresponding rectilinear elements in the two cones are parallel, and a plane can be found tangent to both cones.* In this case, the lines of contact are parallel.

486. **A plane satisfying given conditions relative to two fixed planes of reference.**

If we except those cases in which a plane is parallel to either HP or VP, we find that the following relations exist

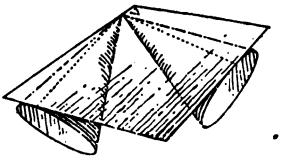


FIG. 458.

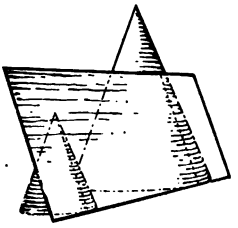


FIG. 459.

between a plane in space and fixed HP and VP of projection.

(1) A definite HT; (2) a definite VT; (3) a definite angle θ with HP, which cannot exceed 90°; (4) a definite angle ϕ with VP, which cannot exceed 90°; (5) a definite angle between the traces of the plane.

If any two of these relations are known the others are determinable.

487. **Problem.** Fig. 460. *Given the traces of a plane determine* (a) *the angle between the traces of the plane,* (b) *the inclinations of the plane.*

First part. **Analysis.** If the plane be rotated on HT as an axis into HP, or on VT into VP, the angle between the traces may be measured.

Construction. Take any point P in VT (elevation p', plan p) and draw pP perpendicular to HT. With centre T and radius Tp' describe an arc meeting the perpendicular from p in P. Join TP. Then PTH is the true angle

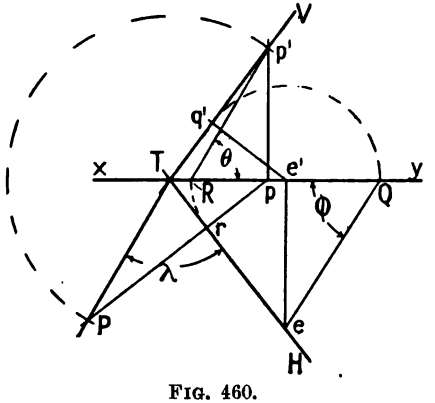


FIG. 460.

between the traces. For Tp', being a segment of the VT, is seen in true length.

Second part. With centre p and radius pr describe an arc meeting xy in R. Join p'R. From any point e in HT, draw a projector on to xy in e'. Draw $e'q' \perp r$, VT, and with centre e', radius $e'q'$, describe an arc meeting xy in Q. Join eQ. Then $p'Rp = \theta$ is the angle which the plane makes with HP, and $eQe' = \phi$ is the angle which the plane makes with VP.

It will be observed that pp' is the axis of a cone whose base is on HP and base angle $= \theta$. Similarly ee' is the axis of a cone whose base is on VP and base angle $= \phi$. The former cone touches the plane in PR, and the latter in EQ. The plan of EQ is not shown.

488. **Problem.** Fig. 461. *Given* HT *of a plane and the angle* λ *between the traces. Draw the* VT.

Analysis. Set out the given angle λ between the traces in HP at HTN, and rabat the angle about HT, bringing TN into the vertical plane on XY.

Construction. Draw TN, making the given angle λ with TH. From any point N in TN draw Nn perpendicular to HT meeting XY in n. With centre T and radius TN describe a circle intersecting the projector from n in n'. Join n'T, which is the required trace.

In Fig. 462, a similar construction is applied to the case where the VT of the plane is given.

489. **Problem.** Fig. 463. *Given* HT, *the horizontal trace of a plane, and the angle between the plane and* HP. *To find the* VT *of the plane :*

From any point r in HT, draw a perpendicular intersecting XY in p. With centre p and radius pr describe an arc meeting XY in R. From R draw Rp', making the given angle θ with XY to meet the projector from p in p'. Join Tp', which is the required VT.

490. **Problem.** Fig. 463. *Given* VT *the vertical trace of a plane and the angle* θ *which the plane makes with* HP. *Find* HT

From any point p' in VT draw a projector $p'p$. From p' draw p'R making $90° - \theta$ with $p'p$. Draw TH tangent to the circle struck from p of radius pR. This is the required HT of the plane. A similar construction will solve the problem when the HT and the angle between plane and VP are given.

491. **Problem.** Fig. 461. *Given the angle* λ *between the traces and* θ *the inclination of the plane to* HP. *Represent the plane.*

Set out the given angle λ in HP at HTN. From any point N in TN draw Nn $\perp^r$ to, and intersecting HT in r. Draw rN$_1$, making the given angle θ with rn. With centre r and radius r N describe an arc cutting r N$_1$ in N$_1$. From N$_1$ drop a $\perp^r$ on to Nn, meeting same in n'. Draw XY through Tn. Make n n' $\perp^r$ XY and equal to nN$_1$. Draw Tn'. Then HTn' is the required plane.

For imagine the right-angled triangle $rn\mathrm{N}_1$ to rotate on rn into a vertical position, and the plane HTN to turn on HT until it rests on the triangle $rn\mathrm{N}_1$. Then since $\mathrm{N}n$ is $\perp$ HT, and $r\mathrm{N}_1 = r\mathrm{N}$, and $nn' = n\mathrm{N}_1$ the line $r\mathrm{N}$ will coincide with $r\mathrm{N}_1$, $n\mathrm{N}_1$ will be vertical on the point n, and the line TN will be vertical over Tn.

In Fig. 462 a similar construction is applied to the case where λ and θ the angle between the plane and VP is given.

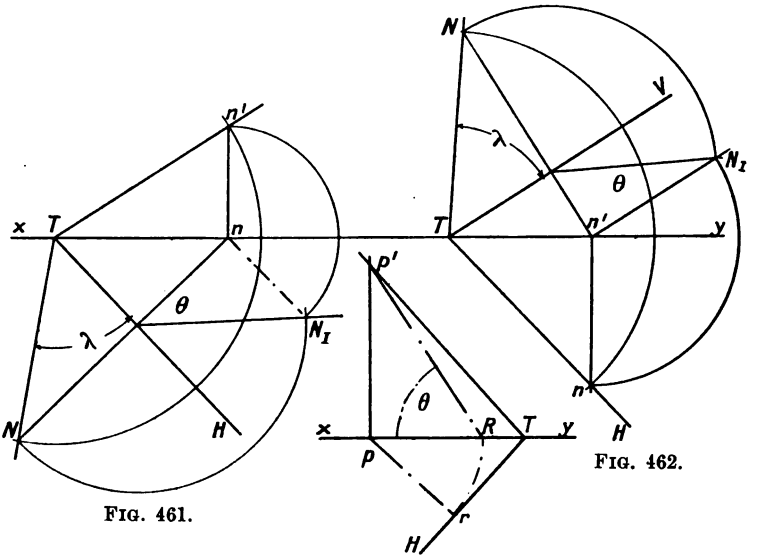


FIG. 461.

FIG. 462.

FIG. 463.

492. **Problem.** Fig. 464. *Given the angles θ and ϕ which a plane makes with* HP *and* VP, *show the plane by its traces.*

Note $\theta + \phi$ must be between 90° and 180° both inclusive.

Let $\theta = 62°$ and $\phi = 55°$.

Analysis. If we take any sphere in any position, and envelop this sphere with two cones, the first having axis

perpendicular to HP and base angle equal to the given angle $\theta = 62°$; the second having axis perpendicular to VP and base angle equal the given $\phi = 55°$, a plane tangent to both cones will satisfy the given conditions. The drawing is simplified by centering the sphere in XY. The two projections of the sphere then overlap, and one circle represents both plan and elevation: the axis of one

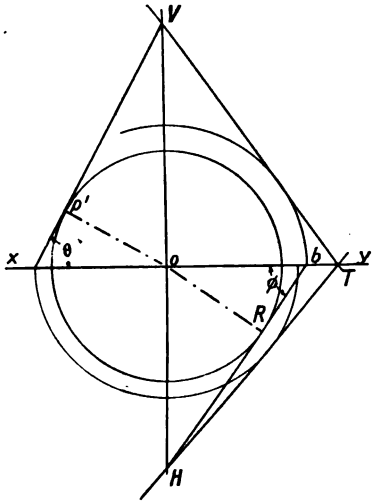


FIG. 464.

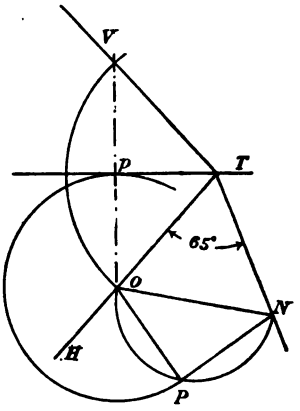


FIG. 465.

cone is in HP, and the axis of the other is in VP. See Fig. 457.

Construction. Draw a line $\perp^r$ XY, crossing XY in *o*. From any point H in this line, draw H*b* making 90° − 55°, with HV and meeting XY in *b*. With centre *o* and radius *ob* describe a semicircle. Draw *o*R perpendicular to H*b*. With centre *o* and radius *o*R describe a circle. Draw *op'* making 90° − 62° with *o*V meeting the circle radius *o*R in *p'*. Draw *p'*V perpendicular to *op'* meeting *o*V in V. Draw VT tangent to the circle radius *ob*. Join TH. Then VTH is a plane

satisfying the given conditions. Observe that TH touches the circular base of the cone whose axis is oV and base angle θ.

Experiment. Apply the construction of the preceding problem when $\theta + \phi = 90°$. Take $\theta = 50°$, $\phi = 40°$. We find that the circular trace of one cone passes through the apex of the other. Consequently the traces of the plane are both parallel to XY.

493. **Problem.** Fig. 465. *A plane is equally inclined to both planes of projection; the angle between its traces is* 65°. *Construct the plane.*

Analysis. Consider the plane tangent to two cones enveloping a sphere centred in XY as in Fig. 457. Since the base angles of the cone are equal, if their bases are made equal, their axes are equal, and the apex of each cone is the same distance from XY; hence the traces of the plane will be equally inclined to XY.

Construction. Set out the given angle 65° at HTN. Take points N and O on TN and TH equidistant from T. Construct an isosceles right-angled triangle OPN on ON as hypotenuse. With centre O and radius OP describe a circle: from T draw Tp tangent to this circle. Make pV = pO. Join TV. Then VTH are the traces of the plane on planes of reference which have Tp for XY.

Exercises.

(1) Represent the following planes by their traces :—
(1) Parallel to HP and 1 in. above it.
(2) „ VP „ ¾ in. in front of it.
(3) Perpendicular to both HP and VP.
(4) „ VP and inclined at 45° to HP.
(5) „ HP „ „ 35° „ VP.
(6) Inclined to HP at 45°, and to VP at 45°.
(7) „ „ „ „ 70°.
(8) „ „ 50°, „ „ 50°.

Show the planes of (5), (6), (7), and (8) by their scales of slope.

(2) A plane is inclined to HP at 30°, and contains xy. What angle does it make with VP, and how would you represent this plane? (Answer: 60°.) Show trace of the plane in profile view, or, show scale of slope.

(3) The VT of a plane is inclined to xy at 55°, and the plane is inclined to HP at 60°. Draw its HT and show the angle between the plane traces.

(4) The traces of a plane are in one straight line and make 35° with xy. Determine the angle which the plane makes with the planes of projection, and show the angle between the traces of the plane.

494. **Theorem.** *Parallel planes have parallel traces on any plane to which they are inclined, and the projections of these traces on any plane are parallel.*

Ex. Fig. 466. Let two parallel vertical planes V′ and V and two horizontal planes H_1 and H be intersected by a plane β. The two vertical traces are parallel, and the two horizontal traces are also parallel.

Hence (a) *If any line in a given plane be parallel to the* VP *it will be parallel to the* VT *of the plane; its elevation will be parallel to* VT *and its plan be parallel to* XY.

(b) *Any horizontal line in a given plane will be parallel to the* HT *of the plane; its plan and elevation will be respectively parallel to* HT *and* XY.

(c) More generally. *If two planes intersect, any line in one plane and parallel to the other will be parallel to the intersection of the two planes.*

495. **Problem.** Fig. 467. *To convert a given oblique plane* VTH *into an inclined plane.*

Analysis. If we take a plane X′Y′ $\perp^r$ to the HT of the given plane, such plane will be perpendicular to the given plane and to HP. And if this new plane be substituted for the original VP of projection, the plane VTH becomes a plane inclined to HP and $\perp^r$ to (new) VP; and its trace on this latter plane is an edge, or profile view, and the angle between the new VT and X′Y′ is the angle between VTH and HP. Similarly, if we take a plane X_2Y_2 $\perp^r$ to the VT of the given plane, such plane will be perpendicular to the given plane, and to VP, and if this new plane be substituted for the original HP of projection, the plane VTH becomes a plane inclined to VP and $\perp^r$ to (new) HP; and its trace on this latter plane is an edge, or profile view, and the angle between the new HT and X_2Y_2 is the angle between VTH and VP.

Construction. Draw X′Y′ anywhere $\perp^r$ HT meeting same in *o* and XY in *v*. Project *v* on to VT at *v′*. Draw vV_2 $\perp^r$ X′Y′ and make $vV_2 = vv'$. Join oV_2, which is an edge elevation, or profile view, of the plane VTH.

Or Fig. 468. Draw X_2Y_2 $\perp^r$ VT meeting same in *o*, and XY in *h′*; project *h′* on to HT at *h*. Draw $h'H_2$ $\perp^r$ X_2Y_2, and make $h'H_2 = h'h$. Join oH_2, which is an edge view of the plane.

496. **Problem.** Fig. 469. *To determine the distance between two parallel planes* VTH, V′T′H′.

Analysis. If the planes be cut by a plane ⊥ᵣ to both

FIG. 466.

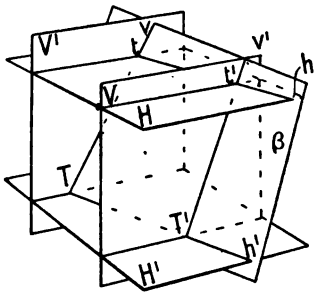
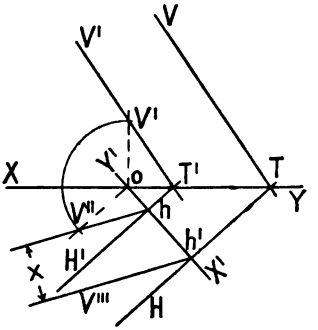


FIG. 469.



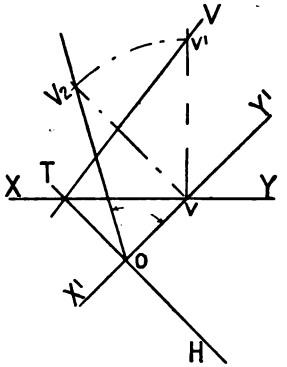


FIG. 467.

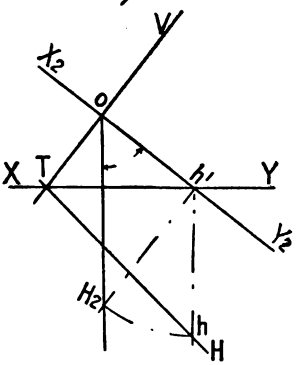


FIG. 468.

the distance between the planes will be revealed in the distance between their traces on that plane.

Construction. Draw X′Y′ ⊥ᵣ HT. Make $oV'' = oV'$, and draw $h'V''' \parallel hV''$. Then X is the distance between the planes.

497. **Scale of Slope.** Instead of representing a plane by its traces on a fixed HP and VP, we may show in plan the direction of level lines in the plane and the position of two or more points of known height in the plane, and index these points as in Art. 447.

Imagine a line of fixed length PQ, Fig. 470, free to rotate about a fixed point Q in FH but remaining always in the plane VFH. In one complete revolution, the point P will describe a circle in the plane VFH about Q as centre, one-half of the circle being above HP, the other half below. When QP lies along QH, it must be horizontal, but after the least turning movement in an anti-clockwise direction, the point P rises whilst Q remains fixed, and thus QP becomes inclined to HP at some angle θ. As the line continues to revolve, its inclination changes. During one-quarter of a complete turn, P continues to ascend to P_4. During the next quarter turn, the inclination decreases as the point P descends to P_5 in QF, where, again, it becomes horizontal.

In the next quarter turn, P continues its descent to a point P_6 which is in P_4Q produced, and the same distance below HP as P_4 is above. Any further rotation brings QP nearer to the horizontal. Then, clearly of all lines that can be drawn in the plane VFH, none can have a greater inclination to HP than QP_4, QP_6, which is perpendicular to the horizontal trace (FH) of the plane, and has for its projection the line p_6Qp_4, also perpendicular to FH. Therefore, any line in the plane and perpendicular to the horizontal trace (FH) makes with HP an angle equal to the inclination of the plane, *i.e.*, such line and plane have the same slope. If through the plan *a* of a point A of known height (15 units) in the plane VFH, *ae* be drawn perpendicular to FH and index figures 15 and *o* be placed against the plan *a*, and the horizontal trace (*o*) of AE, the line *ao* may be divided into a number of equal parts equivalent to the difference in the number of unit levels of A and *o* (15 in this case). Then the points of division will be the plans of points in OA and the plane at unit levels 1, 2, ... 15 above HP. Similarly, if the same divisions be set off along *oe*, the points of division will be plans of points in AE at unit levels below HP. (The points

below HP are distinguished by prefixing the sign — (minus) to the indices.)

The graduated line *eoa* constitutes a Scale of Slope of the plane VFH ; the slope being given by the ratio $\frac{\text{unit}}{o1}$. If the value of this unit be known, a Scale of Slope of a plane completely represents the position of the plane.

The scale of slope is usually represented by two parallel lines, one thicker than the other, the thick line being

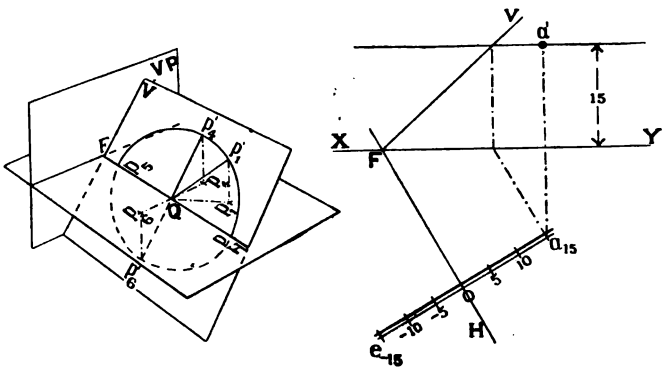


FIG. 470.

placed on the left side of the scale, reading upwards, as in Fig. 470. This mode of representation serves to distinguish scales of slope from other indexed lines.

Fundamental facts may be restated briefly :—

(a) *Level lines in the plane are at right-angles to the scale of slope.*

(b) *The* HT *of the plane passes through the point indexed O (zero) on the scale.*

(c) *The steeper the slope of the plane, the nearer together will be the plans of the unit levels in the plane, and the shorter the scale of slope.*

498. **The traces of a plane on three mutually perpendicular planes of reference.**

A plane β, Fig. 471, which is inclined to each of three mutually perpendicular planes YOX, ZOX, YOZ has a line trace on each plane, and these traces form a triangle in one solid angle illustrated by VFH in the Fig. If the plane β

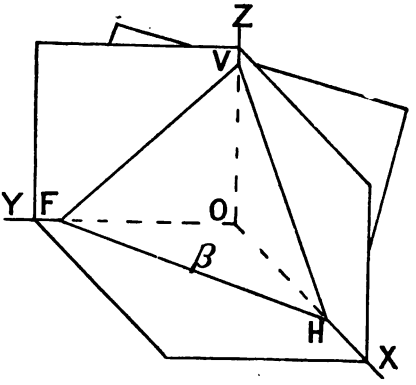


FIG. 471.

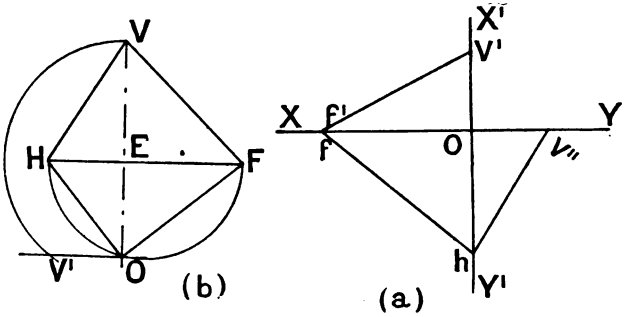


FIG. 472.

is fixed relatively to the plane YOX and the plane ZOX, the angle between β and YOZ will also be fixed, and the shape, *but not the size*, of the triangle VFH will be fixed independently of the position of YOZ.

The plane β will be fixed if OV, OF, and OH be known ;

or if VF, FH, HV, or *any three independent elements* of the triangles forming the pyramid OVFH, be known.

Ex. Fig. 472 (*a*). XY, X'Y' *are ground lines of two vertical planes at right-angles to each other. Show the traces of a plane which intersects* HP *in* HF 2 *in. long, the* VP *on* XY *in* VF $1\frac{3}{4}$ *in. long, and the* VP *on* X'Y' *in* HV $1\frac{1}{2}$ *in. long.*

Set out the given triangle VFH to true shape in a separate drawing at (*b*). On HF, the side corresponding to the horizontal trace of the plane, describe a semicircle. Draw VEO $\perp^r$ HF meeting HF in E and the semicircle in O. Draw OV' $\perp^r$ OV. With centre E and radius EV describe an arc meeting OV' in V'. Join OF, OH. Then OF, OH, OV' are the measures of the intercepts of the ground lines made by the required plane. Hence make O*f*, O*h* (Fig. *a*) respectively equal to OF, OH (Fig. *b*). Make OV'', OV' each equal OV' (Fig. *b*). Join *hf*, *f*'V', *h*V'', which are the traces of the plane.

Exercises.

(1) Cut off a corner of a cube such that the triangle obtained has sides of length $1\frac{1}{2}$ ins., $1\frac{3}{8}$ ins., and 2 ins. Determine the angles which the plane of the triangle makes with each cut edge of the cube. (See Art. 498.)

(2) Determine the traces of a plane inclined at 50° to HP, and 60° to VP, and containing a point P which is $1\frac{1}{2}$ in. above HP and $2\frac{1}{4}$ ins. in front of VP.

(*Hint.* Make P the common apex of two cones, one having base on HP; the other having base on VP. For another solution see Problem 534.)

(3) The real angle between the traces of a plane inclined at 50° is 70°; represent the plane by its traces.

(4) A line making 50° with *xy* is the vertical trace of a plane inclined at 60° to the ground. Determine the horizontal trace.

(5) Draw projections to the line TN which lies in the plane VTH (of Ex. 3) and bisects the angle between the traces. (*Hint.* Rabat the plane into HP. Draw TN, then reconstruct.)

(6) A hill slope is inclined at 28°, the lines of slope being due east, what is the inclination of a path on the hill which goes in a direction N.E.?

(7) Rabat the plane of Ex. 1 about its traces into HP, and VP, carrying thereto the point P.

(8) A point P is situate as in Q. (2). Draw through P a line 6 in. long having one end in each plane of projection. (*Hint.* Take any plane containing P and $\parallel$ *xy*, *i.e.*, whose traces are $\parallel$ *xy*. Rabat this plane and the point P into HP. Draw a line 6 in. long having an extremity in each trace, then draw a parallel to this line through the rabatted point P and reconstruct.)

CHAPTER XXVIII.

FIGURES IN GIVEN POSITIONS.

499. Fig. 473. Consider a fixed point P in a given plane VTH. Clearly innumerable lines can be drawn in the plane and through the point P, *e.g.*:—

(*a*) A line PQ can be drawn which shall be parallel to HP. Such line will be parallel to the horizontal trace of the plane, and its plan and elevation will be parallel to HT and XY respectively.

(*b*) A line PR can be drawn which shall be parallel to VP. Such line will be parallel to the vertical trace of the plane, and its elevation and plan will be parallel to VT and XY respectively.

(*c*) A line PS can be drawn which shall be inclined to HP at the same angle as that between HP and the plane VTH. Such line and its plan will be perpendicular to HT.

(*d*) A line PA can be drawn which shall be inclined to VP at the same angle as that between VP and the plane VTH. Such line and its elevation will be perpendicular to VT.

(*e*) *Two* lines can be drawn which shall make with HP, or with VP, any given angle less than the angle between the plane VTH and HP, or VP.

To make clear the last case, suppose the point P is the apex of two right circular cones, the base of one being on HP and of the other on VP. The HT of the plane intersects the circular base standing on HP in points W, Z, and the two generators PW, PZ of the cone are in the plane VTH and inclined to HP at θ, the base angle of the cone.

Similarly, VT intersects the circular base of the other cone in N, M, and PN, PM are in the plane VTH and inclined to VP at θ the base angle of the cone.

(*f*) Since PQ and PS are respectively parallel and perpendicular to HT, a line can be drawn through P and make any angle between θ and 90° with either trace of the plane.

Observe that the HT and VT of every line in the plane are in the corresponding traces of the plane.

500. **Problem.** Fig. 474. *Given one projection of a point* P *in a plane* VTH. *Draw through the point, and in the plane a line which shall be parallel to* (1) HP, (2) VP.

1. Let the plan p be given. Draw $pv \parallel$ HT meeting XY in v; project v on to VT at v'. Draw $v'p' \parallel$ XY. Draw projector from p intersecting $v'p'$ in p'. Then vp, $v'p'$ are plan and elevation of line required.

Observe (*a*) p' is the elevation of the point P. (*b*) v' is the VT of line PV.

2. Draw $ph \parallel$ XY : project h on to XY at h', and draw $h'p' \parallel$ VT ; ph, $p'h'$ are plan and elevation of line $\parallel$ to VP. Observe, that h is HT of the line PH.

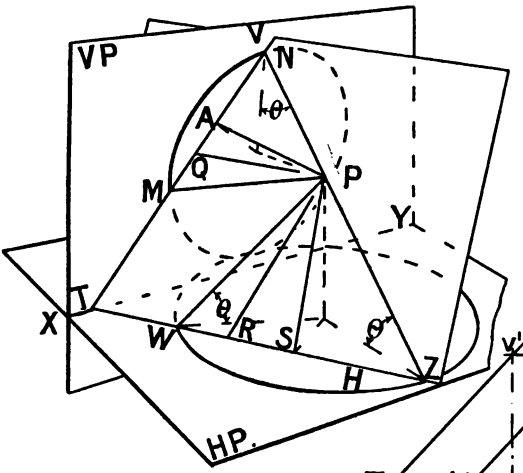


FIG. 473.

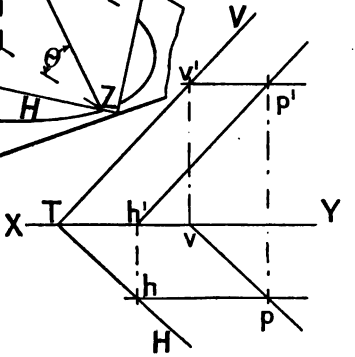


FIG. 474.

501. **Problem.** Fig. 475. *Given a plane* VTH *and a point p, p' in the plane, draw through* P *a line in the plane and making* $\theta = 50°$ *with* HT.

Draw through P a parallel to HP (pv, $p'v'$) and turn the plane about its HT into HP. Thus, draw $vV_0 \perp^r$ HT, intersecting a circle centre T, radius Tv' in V_0. Draw $V_0P \parallel$ HT, and $pP \perp^r$ HT, meeting V_0P in P. Draw Ph making 40°, *i.e.*, the complement of 50°, the given angle. Project h on to XY at h'. Join ph, $h'p'$, the projections of the required line.

502. We can now solve the general problem. *Given a plan, or elevation, of a figure, rectilinear, or curved, lying in a given plane: determine the other projection, and the true shape of the figure.* This Problem we illustrate by four examples.

503. **Ex. 1.** Fig. 476. *Given the plan ab of a line in the plane* VTH, *determine the elevation of the line, its true length, and its inclinations to the plane traces.*

In the diagram, the VT of the given line is found at v', and a line in the plane and parallel to VP is used to locate the elevation b' ; $a'b'$ is the required elevation.

To find the length of AB, and the angle it makes with the traces, fold the plane and line into HP. The length of the line is shown at A_0B_0 ; the angles AB makes with the VT and HT are shown at α, β.

504. **Ex. 2.** Fig. 477. *Given the plan abc of a triangle lying in the plane* VTH. *Determine the elevation of the triangle, and its true shape.*

In the diagram, ab produced meets HT in h and XY in v, whilst ca meets HT in O. By drawing the elevations of OC, HV we readily determine the elevation of the triangle at $a'b'c'$.

Again, the lines OC, HV are readily brought into HP at OC_0, hV_0 and assist in bringing the triangle into HP at $A_0B_0C_0$.

505. **Ex. 3.** Fig. 478. *Given the plan abcdefgh of a curve lying in the plane* VTH, *determine the elevation of the curve, and its true shape.*

When projecting a curve, always select the more important points in the curve and obtain correct projections of such points. Thus, in the present case, the highest and

FIG. 475.

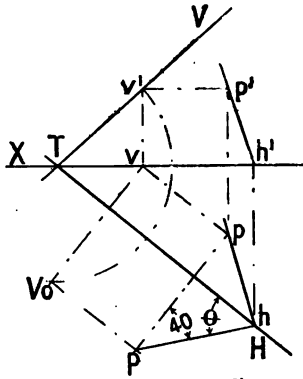
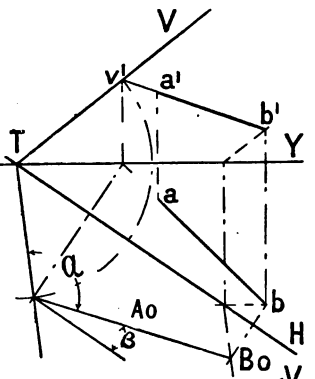
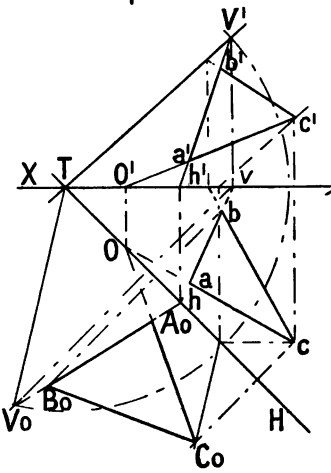


FIG. 476.





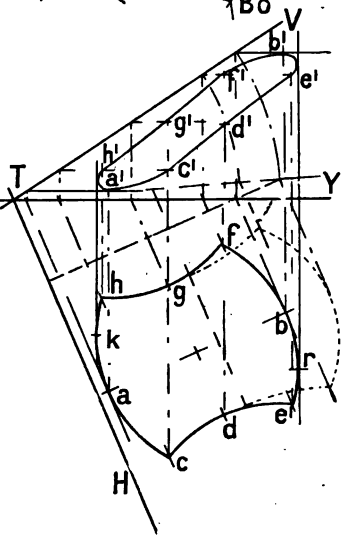


FIG. 477.

FIG. 478.

lowest points B, A, and the points C, D, E, F, G, H are the most important. Determine the elevations of these points and some intermediate points as necessary. Draw a fair line through the elevations of these points. Observe that lines perpendicular to XY which touch the plan of the curve at *r* and *k*, are also tangents to the elevation.

To determine the true shape of the curve, turn the plane and numerous points in the curve into HP. In the diagram, the true shape of the upper half of the figure is shown by dotted lines.

506. **Ex. 4.** Fig. 479. *The circular arcs lk, eg, described from centre c and the tangents tk, t are plans of lines lying in the plane* VTH. *Determine the true shape of* LCKT *and of the curves* LK, EG.

Take x_1y_1 through c $\perp^r$ TH. Draw $kk'' \perp^r x_1\ y_1$ and make $ok'' = kk'$. Then $l''k''$ is profile of plane VTH and projectors $\perp^r x_1y_1$ from points e, c, g, t determine by their intersection with $l''\ k''$ elevations of points E, C, G, T on the profile.

We can obtain correct relative positions of points E, G, on the plane VTH by drawing projectors $\perp^r l''\ k''$ through the elevations e'', c'', and setting off thereon lengths equal to the distances of the plans e, c, from $x_1\ y_1$. Thus, denote $l''\ k''$ by x_2y_2 and draw $k''k_0 \perp^r x_2y_2$ and $= ok$. Similarly, make $l''l_0 = l''l$, and $g''g_0$, $e''e_0$, $t''t_0$ equal the respective distances of the plans g, e, t from x_1y_1. Observe that c'' is a double projection of C, since c is on x_1y_1. Join $c''l_0, l_0t_0, t_0k_0, k_0c''$ which is the true shape of CLTK.

(Observe that k_0t_0 and $l''l_0$ when produced meet in f_0 such that $l''f_0 = l'$T. This is a useful test of accuracy.)

Intermediate points in the curves EG, LK may be found in a similar manner: *e.g.*, the point S (plan s), draw $ss'' \perp^r x'y'$. Draw $s''s_0 \perp^r x_2y_2$ and make $s''s_0 =$ distance of s from x_1y_1. In a similar manner a number of points may be found through which the curves may be drawn freehand.

The true shapes of the curves EG, LK are arcs of *similar* ellipses, since the plans are concentric circular arcs. The axes of these ellipses are readily determined, since in each case the shorter or minor axis is equal to the diameter of the plan circle and occupies position $\|$ HT. The longer or major axis is the line whose plan is the diameter of the plan circle $\perp^r$ HT. Thus $c''b \perp^r x_2y_2$ and $= ce$ is semi-minor axis of the smaller ellipse, whilst $c''b_0 = ce$ is semi-minor

axis of the larger ellipse. To determine the major axes, let the plan circle eg meet x_1y_1 in a. Project this point on to

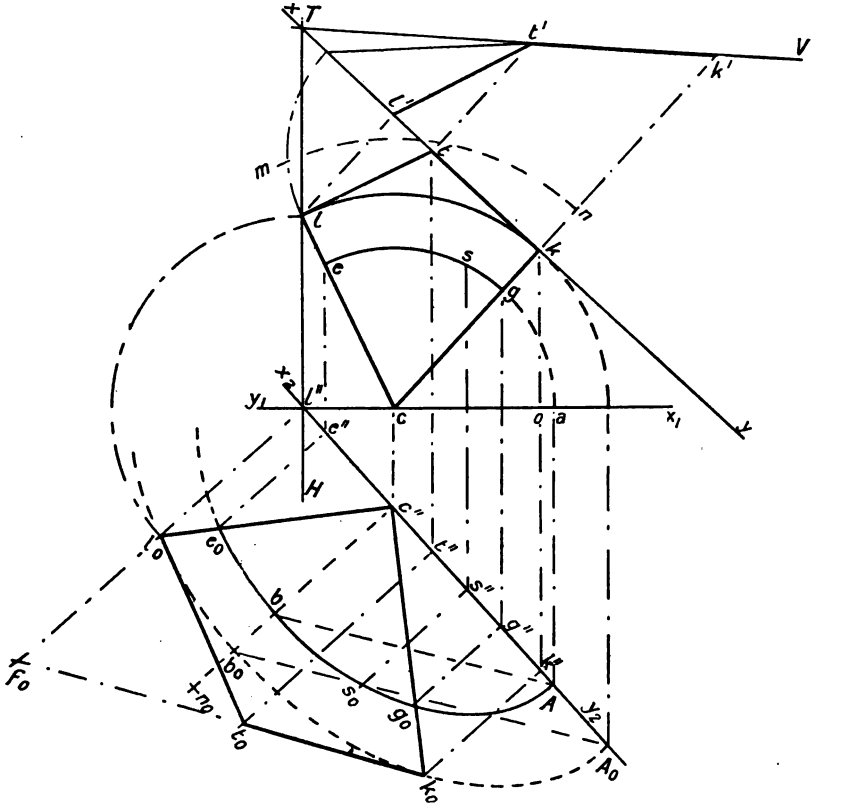


FIG. 479.

x_2y_2 at A. Then c''A is semi-major axis corresponding to $c''b$. The semi-major axis of the larger ellipse is $c''A_0$; it may be found as in the case of $c''A_1$ or by drawing $b_0A_0 \parallel bA$.

The student will observe that any arc concentric with *lk* will be part of an ellipse similar to that on $c''b$, $c''A$, *e.g.*, the dotted arc *mn* is part of the ellipse whose semi-minor axis is $c''n_0 = cn$ and semi-major axis $C''A_2$ (not shown) which may be found from the proportion $c''b : c''A, :: cn''_0 C''A_2$. A practical example of the preceding problem occurs in finding the *face mould* for a handrail wreath joining two straight flights of stairs. In Fig. 479 the two tangents represent the centre lines of the rail over the straight flights, and the middle curve *lk* the centre line of the wreath connecting the straight rails. The width of the rail is the distance between the arcs *eg*, *mn*, which are plans of the inner and outer face of the rail. The rail is cut out of a plank, the flat of which is taken parallel to the plane of the tangents TL, TK, and the face mould is the section in which this plane cuts the hollow cylinder represented in plan by the curves *eg*, *mn* centre *c*. The plane of the tangents is called the *plane of the plank*. The lines $c''k_0$, $c''l_0$ are called *springing lines*. The form of the diagram depends both on the plan and pitch of the stairs.

507. **Problem.** Fig. 480. *Given the projections of a point* (P) *in a given plane* VTH. *Draw the projections of a line* PQ *lying in the plane; passing through* P *and inclined to one plane of projection at a given angle.*

Let the inclination of the line to HP be given, 40°, say. (*The angle between the line and the plane of projection cannot exceed the angle between the given plane and that plane of projection. Art.* 498.)

Analysis. If P is made the apex of a cone having axis ⊥ to HP and base angle equal 40°, the required line must lie on that cone. This cone will, in general, be cut by the plane VTH in two lines, each of which is a solution of the problem.

Construction. From the elevation p' draw $p'Q'$, making 50° (the complement of the given angle 40°) with the projector $p'p$. Draw $pQ \parallel XY$; drop a projector from Q′ on to this line in Q. With centre p and radius pQ describe a circle which intersects HT in q and h; draw projectors from q and h on to XY at q' and h'. Then pq, $p'q'$; or ph, $p'h'$ are plan and elevation of required line.

Fig. 481 shows the construction applied to solve the problem, when the inclination of the line to VP is given.

508. **Problem.** Fig. 482. *Draw plan and elevation of a line of given length* (X) *lying in the plane* VTH, *and having its extremities in the planes of projection and inclined to* HP *at* θ (50°).

FIG. 480.

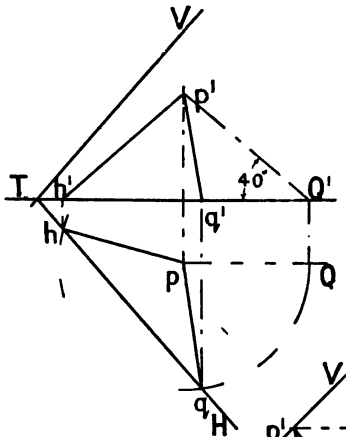
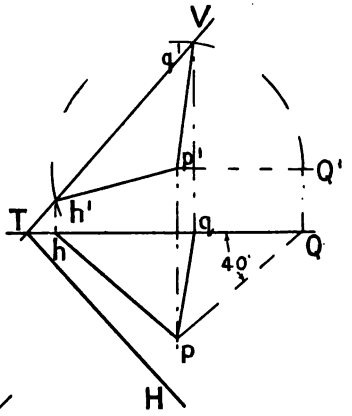


FIG. 481.



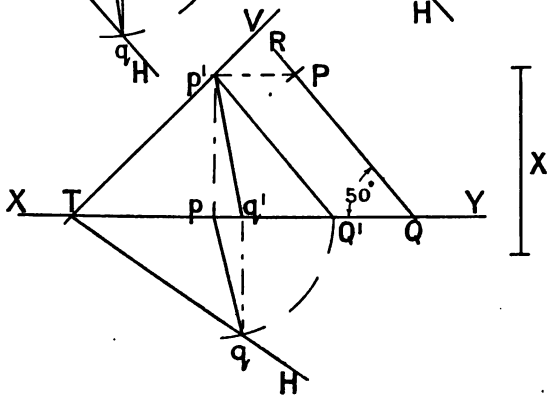


FIG. 482.

From any convenient point Q in XY draw QR, making the given angle 50° with XY. Make QP on QR equal to the given length X ; from P draw a parallel to XY intersecting VT in p' ; draw p'Q' parallel to PQ. Project p' on to XY at p; with centre p and radius pQ' describe a circle crossing HT in q. Then pq is plan of a line satisfying the conditions given ; $p'q'$ is the elevation.

509. We are now in a position to solve the general problem. *To draw the projections of a plane figure when given the angles which the plane of the figure makes with the planes of projection, and the inclination of a straight line in the figure.*

To illustrate this problem, we give two examples.

Ex. 1. Fig. 483. *An equilateral triangle* ABC, 2 *ins. side, is inclined to* HP *at* 50°, *the corner* A *is in the ground, and the side* AB *makes* 40° *with* HP.

Note.—As the data does not specify the position of the triangle relative to VP we can place the plane of the figure $\perp^r$ VP, and draw the VT and HT of the plane without construction. VTH is the plane inclined to HP at 50°.

From any point $q'q$ place a line QA in the plane inclined at 40° (the given inclination of AB by Art. 507). Fold the plane with this line into HP at QA. Set out the triangle ABC to given size. Fold the plane back into its proper position, carrying with it the corners B, C of the triangle. The elevations of these points are at b', c', their plans are b, c.

Why is the elevation of the triangle a straight line?

Answer.—Because the plane of the triangle is $\perp^r$ VP.

Ex. 2. Fig. 484. *A square of* 2 *ins. side is inclined to* HP *at* 50° *and to* VP *at* 55°. *One side of the square makes* 46° *with* VP, *and one corner* (A) *is* 1 *in. above* HP. *Draw its projections.*

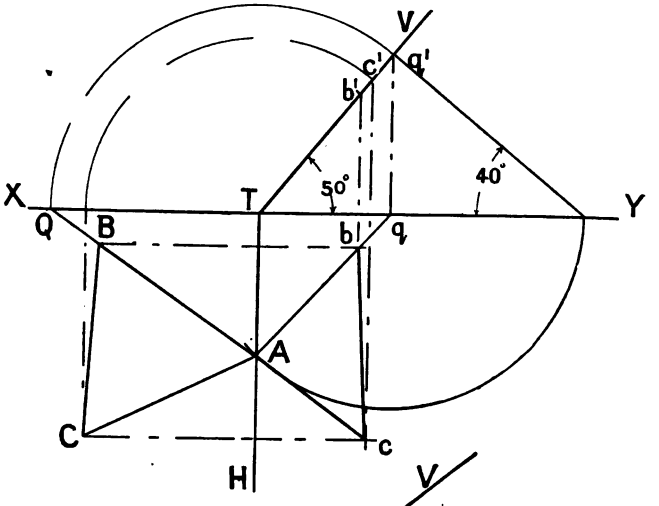
First determine a plane making the given angles 50° and 55° with HP and VP, Art. 492; place any line PQ in this plane making 46° with VP, Art. 507. Take a point aa' in this plane 1 in. above HP and draw ad, $a'd' \parallel qp$ $q'p'$. Fold the plane and line AD into HP and set out the square making the point A one corner. Then bring the plane back into its proper position.

Observe, that both projections of the square are parallelograms, because the square is a parallelogram.

Exercise.

Draw projections of the solid shown in Fig. 450, when its polygonal ends are inclined to HP at 50°, a corner of the lower hexagon is on HP and an edge is inclined at 35°.

FIG. 483.



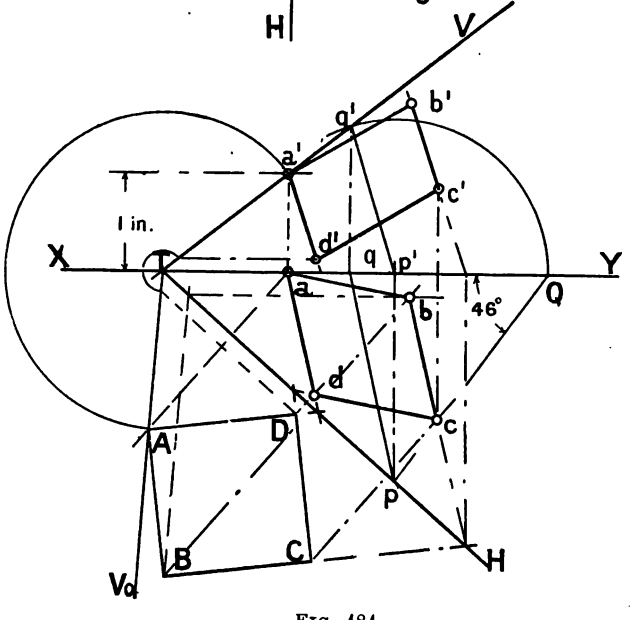


FIG. 484.

510. **Problem.** Fig. 485. *abc is a plan of a triangle.* AB *is inclined at* 40° *and* AC *at* 35°. *Determine the true shape of the triangle.*

Draw $aA' \perp ab$. Draw bA', making the given angle 40° with ab meeting aA' in A'. Draw $aA_1 \perp^r ac$ and equal in length to aA'. From A_1 draw A_1O making the given angle 35° with, and intersecting ac (produced, if necessary) in O. Then Ob is a level line in the plane of the triangle. For, if the two triangles $A'ab$, A_1aO be folded about ab and aO respectively into vertical positions the points A_1 and A' will coincide, because $aA' = aA_1$, and the angles $A'ab$, A_1aO are both right-angles.

Thus $A'b$ is real length of AB, and A_1O is real length of AO, hence, with centre b and radius bA', and with centre O and radius OA_1, describe circles intersecting in A. Join bA and draw AC through O, meeting a line from $c \perp^r$ to bO in C. Then AbC is true shape of triangle.

511. **Problem.** Fig. 486. *Given a triangle* ABC. *Determine the plan of the figure when* AC *is inclined at* 30° *and* AB *at* 40°.

This is the converse of the last problem, and the same expedient of rabatted triangles is employed to obtain a solution.

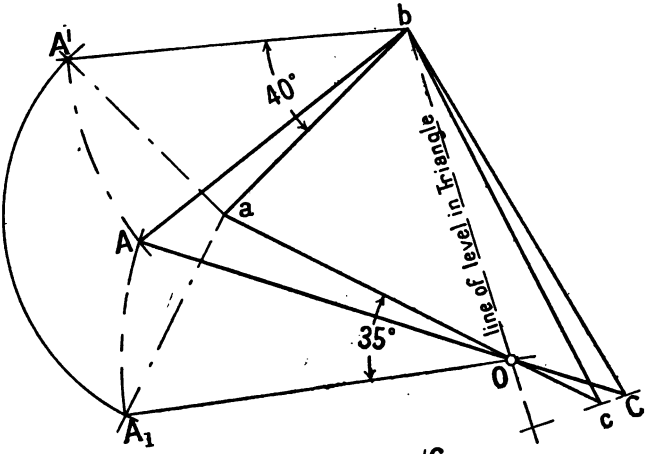
Draw Ah making 60° with AC, *i.e.*, the complement of the angle of inclination of AC. Draw Ch perpendicular to Ah. Draw $Ah_1 = Ah$ and making 50° with AB, *i.e.*, the complement of the given angle of inclination of AB. From h_1 draw h_1o perpendicular to Ah_1, meeting AB in o. Then Co is a line of level in the plane of the triangle. With centre C and radius Ch, with centre o and radius oh_1, describe circles intersecting in a. Join aC. Draw a line from a through o meeting the line from B perpendicular to Co in b. Then acb is the required plan.

Exercises.

(1) Draw the plan of a regular hexagon of one inch side, one side being inclined at 35°, one diagonal of the hexagon being inclined at 40°.

(2) Construct a triangle ABC having given AB = $2\frac{1}{4}$"; perimeter $6\frac{1}{2}$"; angle ACB = 52°. This triangle is the base of a triangular pyramid having P for apex. The slant edge AP = $2\frac{1}{4}$"; BP = $2\frac{1}{2}$"; CP = 2". Draw the projections of the pyramid when AC is inclined at 30°, and AB is inclined at 20°.

FIG. 485.



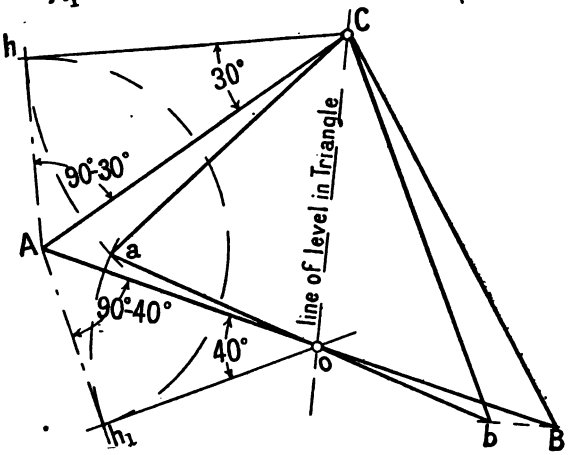


FIG. 486.

512. If three points in any figure, plane, or solid are given in position, the whole figure will be given in position, and therefore arbitrary positions can be given to three points only in a figure; and when three points

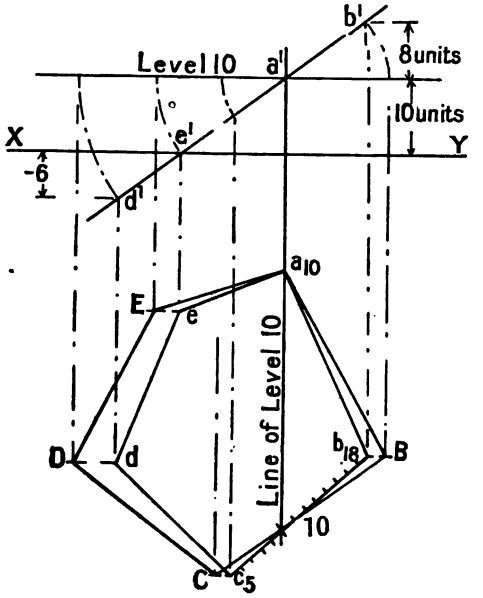


FIG. 487.

are fixed the position of other points in the figure is determinable.

Problem. Fig. 487. *Given the plan of a rectilinear figure abcde, corners a, b, c being indexed. Index the other corners e, d, and determine the true shape of the figure.*

The line *cb* slopes from 5 to 18, so there is some point in this line level with the corner $a = 10$. To locate a point in CB, level 10, divide *cb* into a number of equal parts equal to the difference in the indices of the extremities *c*, *b*, *i.e.*, $18 - 5 = 13$, by Problem 36.

The fifth point on the line from *c* is clearly 5 units above *c*, *i.e.*, $5 + 5 = 10$, and is therefore level with *a*. Draw $a\,10$, which is plan of level 10 in the figure. Take XY $\perp r$ $a\,10$, and obtain an edge view of the plane of the figure, as $a'b'd'e'$. From this elevation the level of any point in the figure may be found. Thus e' is in XY, hence E is in HP and must be indexed 0 (zero), d' is below HP a distance found by measurement to be 6 units ; therefore D is 6 units below HP and must be indexed − 6.

The true shape of the figure is found by rotating the plane about the level line a_{10}. It is shown at *a*BCDE. Notice that lines joining corresponding points in the plan and rabatment of the figure intersect in the rotating axis.

Miscellaneous Exercises.

(1) Draw a plane inclined at 45° to the HP and at 60° to the VP. Draw a line in the plane inclined at 30°, and 2 ins. long between its traces.

Lastly, draw the plan of a regular hexagon lying in the plane of which the above line is a diagonal. (B.E. (2), 1897.)

(2) A right pyramid has for base a regular pentagon of which the diagonals measure 2·5 ins. The vertex is 2 ins. above the base. Draw the plan and elevation of the pyramid, with its base in a plane inclined at 55° to the VP and at 60° to the HP ; one diagonal inclined at 30°, and one end of that diagonal in the vertical trace.

(B.E. (2), 1894.)

Hint. After having obtained projections of the base, apply Problem 521.

(3) The section of a square post by a plane VTH is a parallelogram ABCD. The sides AB, BC are in the ratio 5 : 3 and the angle ABC is 75°. Determine the cross-section of the post. (Refer to Art. 514.)

(4) A man 6 ft. high, going along a northward road, observes that where the road is level his shadow measures $5\frac{1}{4}$ ft. and has the direction north-west. Immediately afterwards the road has a downward gradient of 1 vertical in *n* horizontal, and his shadow now measures $8\frac{1}{4}$ ft. What is the angle which the sun's rays make with the horizontal? What is *n*? (B.E. (2), 1910.)

513. If we know the form of a figure, and have the projections of three non-colinear points, the projection of the whole figure may be found with the help of the property enunciated in Art. 245, viz.: If a square with an inscribed circle be projected on to a plane inclined to the plane of the square, the projection of the square will be a parallelogram, and the projection of the circle will be an ellipse inscribed in the parallelogram. We know (Art. 498) that the shortest diameter in the ellipse or minor axis will be a line of greatest inclination in the plane, and the longest diameter or major axis will be a level line in the plane.

514. **Problem.** Fig. 488. *The parallelogram mnpq is the plan of a square. Determine the length of the side and the inclination of the plane of the square.*

Determine a_1a the major axis, and b_1b the minor axis, of the ellipse which can be inscribed in *mnpq* (Problem 250). With centre *c* and radius *ca* describe an arc meeting *b*B drawn parallel to a_1a in B. Join *c*B; then a_1a is the length of the side, and $\angle$ B*cb* = θ is the inclination of the plane of the square.

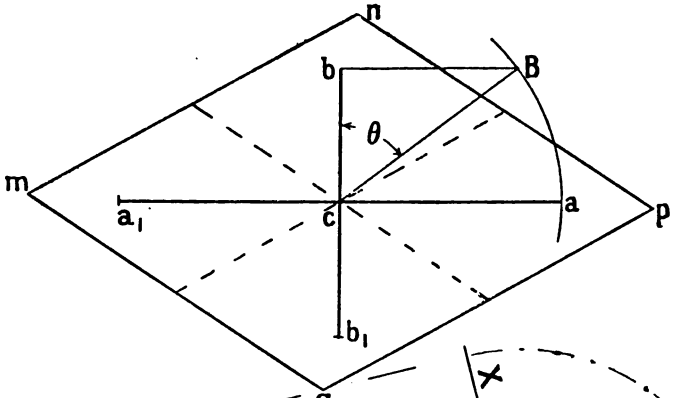
515. **Problem.** Fig. 489. *ab, bc are plans of two adjacent sides of a regular pentagon. Complete the plan of the pentagon.*

On *ac* as base construct an isosceles triangle *a*B*c* making the angle at B = the angle in a regular pentagon, *i.e.*, 108°. On the opposite side of *ac* construct a square *a*D*Ec*. Draw BD intersecting *ac* in *n*. Draw *bn* and produce to meet a line from D parallel to *b*B in *d*. Join *ad* and draw *de*, *ce* respectively parallel to *ac* and *ad*. Then if the square *a*D*Ec* and the triangle *a*B*c* lie in the same plane, the parallelogram *adec* is the projection of the square *a*D*Ec* corresponding to the projection *abc* of the triangle *a*B*c*. For, $\frac{nb}{nd} = \frac{n\text{B}}{n\text{D}}$.

Determine the inclination θ of the plane of the figure (previous Problem) and a profile view of the plane on XY taken perpendicular to the major axis *vv* of the ellipse. Rabat the triangle into HP at AB_0C. On this rabatment determine the other corners D_0E_0 of the pentagon. (D_0 is not shown). Reconstruct the plane carrying the points D and E into correct position. The complete plan

is ad_0e_0cb. Parts of the construction are omitted to avoid overcrowding the diagram.

FIG. 488.



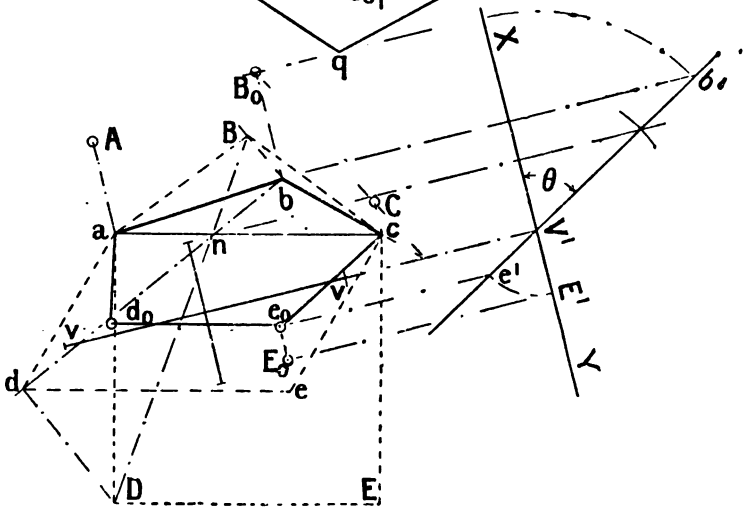


FIG. 489.

Exercises.

The HT and VT' of a plane VTH make 60° and 70° respectively with XY, show in this plane.

(1) An equilateral triangle ABC 2 ins. side, when the corner A is in the HT of the plane and 3·5 ins. from T, the corner B is 2·75 ins. from T.

(2) A circle radius 1·5 ins., its centre being 2 ins. from each of the plane traces.

(3) A pentagon ABCDE of 1·5 ins. side, CD being equally inclined to the plane traces.

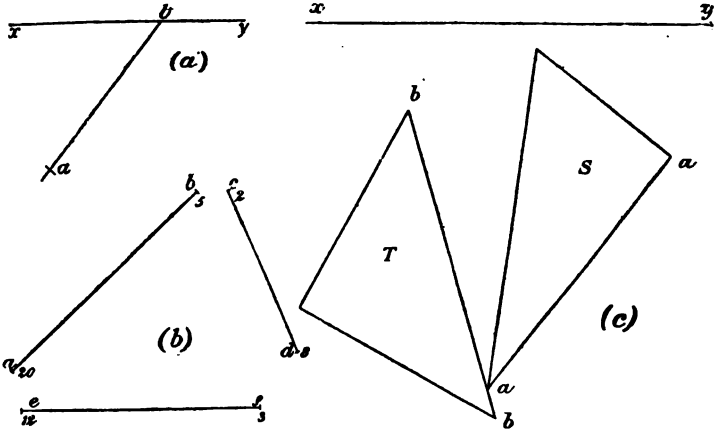
(4) ABC is the base of a pyramid, and V is its vertex. AB = 2½ in., AC = 2 in., BC = 2¾ in., AV = CV = 3 in., BV = 2½ in. Draw the plan of the pyramid when A = 1½ in., B = 2½ in., and C = 1 in. above the horizontal plane. (B. E. (2), 1896.)

(5) Fig. (a).—*ab* is the horizontal projection and *a* the trace of a line inclined at 50° to the horizontal plane. It is the axis of a right cylinder of 1·5 in. diameter and 1·5 in. in length, its lower base resting on the horizontal plane. Draw the plan of the cylinder. (B. E. (2), 1895.)

(6) A line *ab*, 2½ in. long, inclined at 28° to XY, is the plan of a line AB 4 in. long. AB is the base of an isosceles triangle, the apex being in XY. Draw the projections of the triangle.

(7) A hemisphere of 1·25 in. radius is placed with its plain surface vertical, and inclined at 48° to the vertical plane of projection. Draw its elevation. (B. E., 1895.)

(8) A rectangle, sides 1¼ in. and 2½ in., rests with a short side in the horizontal plane, and is rotated about that side until its plan is a square of 1¼ in. side. Determine its inclination to the horizontal



plane, and draw an elevation on a vertical plane parallel to the horizontal sides of the rectangle. (B. E., 1895.)

(9) Fig. (b). Draw the plan of a circle passing through the points of bisection of the three lines *ab*, *cd* and *ef* given by their figured heights. Unit = 0·1 in. (B. E. (2), 1895.)

(10) Fig. (c). Two thin set-squares S and T lie in the horizontal plane as shown. Suppose S to be turned about *aa* so that its plane is vertical, and T to be then turned about *bb* so as to rest against the edge of S.

Draw the plan of the set-squares in their new positions, and draw an elevation on *xy*. At what angle is the plane of T inclined to the horizontal? Also, what is the angle between the sloping edge of S and the horizontal line *bb*? (B. E. (2), 1904.)

(11) Draw the plan of the set-squares in their new positions, and draw the elevation on *xy*. Find and measure the dihedral angle between the planes of the set-squares. (B. E. (3), 1904.)

(12) A small object P is situated in a room at a distance of 17 in. from a side wall, 24 in. from an end wall, and 33 in. above the floor. Find the distance of P from the corner O of the room where these three mutually perpendicular planes meet.

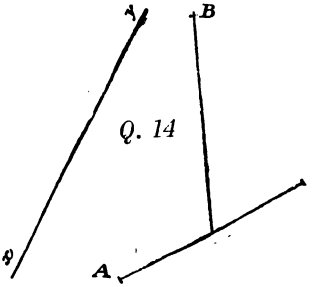
If a string were stretched O to P, find and measure the angles which OP would make with the floor, the end wall, and the side wall respectively. Scale $\frac{1}{10}$th. (B. E. (2), 1901.)

(13) Draw two lines *ab*, *bc*, respectively 1·2 in. and 1·72 in. long, and containing an angle of 81°. These are the plans of two sides of a regular hexagon. Complete the plan of the figure. (B. E. (2), 1901.)

* (14) *Two pieces of wire are soldered together at right angles in the form of the letter* T. *The figure shows the plan when the ends A and B rest on the ground, the plane of the letter being inclined upwards.*

(a) *Determine the true shape of the letter.*

(b) *Draw an elevation of the letter on* xy.



CHAPTER XXIX.

INTERSECTING PLANES.

516. Any two non-parallel planes intersect in a straight line (483). It follows that three planes, no two of which are parallel, intersect in pairs, and the two lines of intersection will, in general, intersect in a point. This principle is the basis of the constructions employed to find the intersection of two planes. Thus, we take an auxiliary plane, in such position that its intersections with the given planes can be drawn without a construction, and find the meeting point of the lines of intersection. This is a point in the required intersection. In a similar manner we determine a second point. The line joining these two points is the required intersection.

If the horizontal traces and the vertical traces of the given planes intersect in points convenient for use, these points determine the required intersection.

517. **Problem.** Fig. 490. *Determine the intersection of two planes given by their traces*, VFH, V'F'H'.

Case I. Fig. (a). The two horizontal traces meet in g; hence G is a point in the intersection, its elevation g' is determined by a projector from g on to xy. The vertical traces meet in k', and therefore K is in the intersection; the plan of K is found by a projector from k' on to xy at k. Join gk, $g'k'$, which are plan and elevation of the required intersection.

Case II. In Fig. (b), one of the planes is vertical and the plan of the intersection falls along the HT of that plane; because that trace is an edge view of the plane. The elevation of the intersection is g' k'.

Case III. In Fig. (c), one of the planes is horizontal and the elevation of the intersection coincides with the VT of

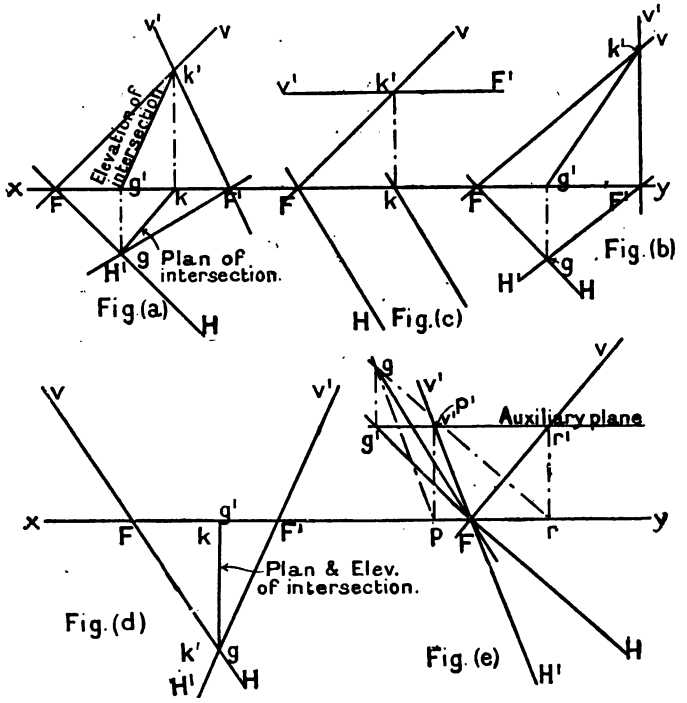


FIG. 490.

that plane. (Reason as above.) The plan of intersection is parallel to FH.

Case IV. In Fig. (d), the VT and HT of each plane project into straight lines. The line of intersection is determined precisely as before. Thus the two horizontal traces meet in g; the elevation of this point is on xy at g'. The two vertical traces meet in a point k', which coincides with the plan g; the plan of K is, of course, on xy and coincides with g', the elevation of G. Consequently the projections of the intersection overlap in gk, $g'k'$. The segment of the intersection line is in the fourth dihedral.

Case V. In Fig. (e), all the traces meet in xy, and the VT and HT of one plane are in the same straight line. Clearly F is both plan and elevation of one point in the intersection. To determine another point; take a horizontal plane at any convenient level, and find the intersections of this plane with the given planes. Thus, the edge view of the auxiliary horizontal plane intersects VF in r' and V'F in p'. The plans of these points are shown at r and p on xy. Lines through r and p parallel respectively to HF, and H'F are the lines in which the auxiliary plane intersects the planes VFH, V'FH'. These lines intersect in g, the elevation of which, g', is in the line elevation of the auxiliary plane; it is therefore on the perpendicular from g. Join Fg, Fg', which are the plan and elevation of the required intersection.

Case VI. In Fig. 491 (f), both planes are parallel to xy. In this case, the intersection of the planes is also parallel to xy; its position is easily found by taking an auxiliary plane perpendicular to xy. Thus, draw $x'y'$ perpendicular to xy. Make $Of'' = OF'$; $Of' = OF$. Join F'f'', Ff' which intersect in $\frac{k''}{g''}$. Through this point draw gk parallel to xy. Then gk is plan of the intersection; its elevation is g', k'.

Case VII. Fig. 492. When two planes are given by their scales of slope 0, 20; $-$ 5, 25, level lines of the same altitude in the two planes determine, by their intersections, a point in the intersection of the planes. Thus lines of level 5 in each plane meet in p, a point in the intersection of the planes. Similarly, lines of level 15 meet in q, a point in the intersection. The line pq is the required intersection.

518. **Problem.** Fig. 493. *Determine the intersection of the three planes given by their scales of slope,* 0, 15; $-$ 5, 20; 0, 17.

Determine, as in last problem, the intersection of 0, 15; 0, 17, and of 0, 17; $-$ 5, 20. The two lines of intersection cross in p, which must be in all three planes, and is therefore the required intersection.

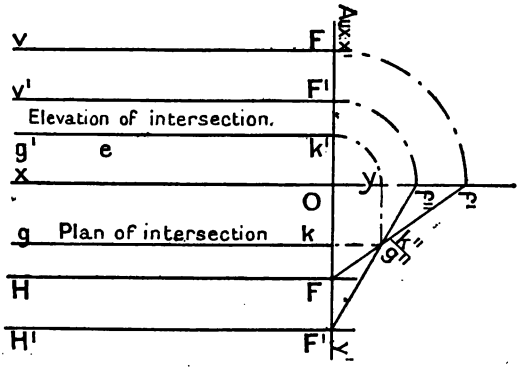


FIG. 491 (f).

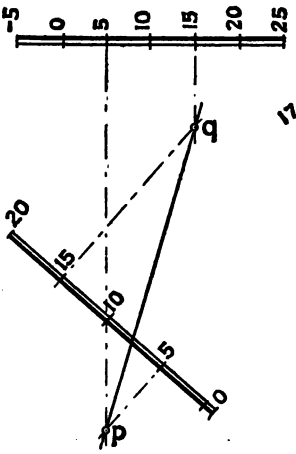


FIG. 492.

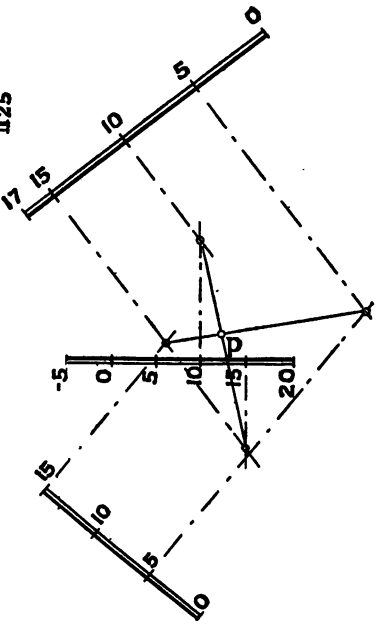


FIG. 493.

519. **Problem.** Fig. 494. *Given a line by its projections ab, a'b', and a plane* VTH. *Determine the point in which the line—prolonged, if necessary—penetrates the plane.*

The plane VTH, the line, and its projections on the planes of projection, are shown in the sketch Fig. (a).

Analysis. Assume a vertical plane α to contain AB. The HT of this plane coincides with (*ab*), the plan of the line, and its VT is the vertical line SS' ; the point S being the point in which the plan *ab* prolonged crosses *xy*. The planes VTH and α intersect in rS' ; and since AB is in α, the point P where BA crosses the intersection rS' must be the point in which BA penetrates VTH.

Construction. Extend the plan, if necessary, to meet HT in r and *xy* in *s*. Run up a projector from *s* to meet VT in *s'* ; project r on to *xy* at r'. Join $r's'$. Then *rs*, $r's'$ are plan and elevation of intersection of VTH and a vertical plane containing the line AB ; $r's'$ crosses $a'b'$ in *p* ; this is the elevation of the required point of interpenetration ; its plan is found by a projector on to *ab* at *p*.

In Fig. (c), the plane VTH is perpendicular to VP, its intersection with the vertical plane containing AB is represented in plan by the plan *ab*, and in elevation by VT, the elevation of the point of interpenetration is at p', where $a'b'$ crosses VT ; its plan (*p*) is found by a projector from *p* on to *ab*.

In Fig. (d), the plane VTH is vertical, its intersection with the vertical plane containing AB is a vertical line standing on the point (*p*) in which the plan (*ab*) of the line crosses the horizontal trace of the plane. The point *p* is also the plan of the point of interpenetration. The elevation of the point is at p'.

520. Comparing Figs. (b) and (c), we learn that less drawing is required to solve the problem above when the plane is shown in profile. We observe, further, that the distance of the plane from any point as A in the line is revealed in Fig. (c) but not in (b). Many problems in which an oblique plane is involved are more quickly solved in an indirect manner, by finding firstly, a profile view of the plane and solving the problem in this view, then projecting essential details on to the given planes of reference.

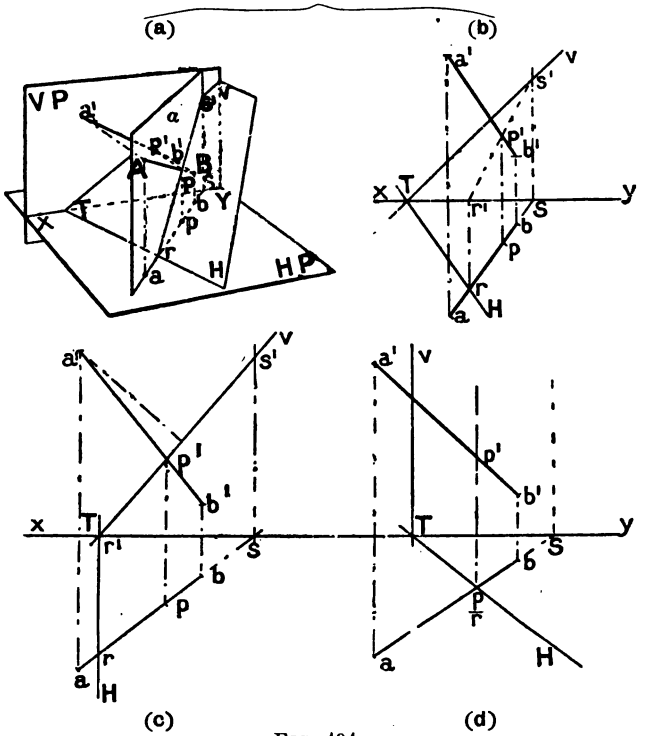


FIG. 494.

521. **Problem.** No Fig. *From a given point a', a in space draw* AP *perpendicular to a given plane.*

Take XY perpendicular to HT of the plane, and obtain a profile view of the plane, and a new elevation *a''* of the point A. Draw *ap* ⊥ HT and *a''p''* ⊥ VT. The point in which *a''p''* intersects the vertical trace in the auxiliary view in an elevation of the point in which AP penetrates the plane, and the distance of this point from *a''* shows the length of the intercept between A and the plane. The plan is projected from this elevation, on to *ap*: from the plan an elevation on any VP may be projected.

522. **Problem.** Fig. 495. *Given a plane* VTH *and the projections of a line* AB. *If the line penetrates the plane in* P, *find the length of* AP, *its projection on the plane, the distance of* A *from the plane, and the angle between the line and the plane.*

Analysis. Let a perpendicular from A on to the plane meet same in Q, and join PQ. Then AQP is a right-angled triangle, and all the information asked in the problem is determined by finding the true shape of this triangle.

Construction. Take $x'y' \perp$ HT and obtain V''T'' a profile view of the plane, and a new elevation of the line at $a''b''$. The projections of the point in which AB pierces the plane are $p''p$. Draw $a''q'' \perp$ V''T'', and $aq \perp$ HT. Project q'' on to the plan aq at q. Join pq. Then pq is the plan, and $p''q''$ the elevation of the *projection* of AP on the plane. The elevation of PQ on the original VP is projected from the plan at $p'q'$.

The distance of A from the plane is shown at $a''q''$. The length of AP and the magnitude of the angle APQ, *i.e.*, the angle between AP and the plane, may be found as follows :—

The projector $p''p$ crosses the plan aq in o. Draw $p''o' \perp$ V''T'' and equal po. With centre q'' and radius $q''o'$, describe an arc intersecting V''T'' in P. Join a''P. Then the length of the intercept AP is shown at a''P. The angle between AB and the plane is $a''Pq'' = \phi$.

523. **The Angle between Two Planes.** We have seen (497) that the angle between two planes is the angle between two lines, one in each plane, drawn from a point in the intersection of the planes and perpendicular thereto.

Problem. Fig. 496. *To determine the angle between two planes* VTH, V''T''H''.

Determine the intersection (pq) of the planes as in Art. 517. Draw an elevation qp'' of PQ on the plan qp as ground line. Through any point c on pq, draw cs' anywhere convenient and perpendicular to qp''. Draw through c a perpendicular to qp, intersecting TH in t and T''H'' in n. Then cs' is the vertical trace, and tcn the horizontal trace of a plane perpendicular to the intersection PQ and intersecting same in S, elevation s'. Fold s' down into HP at S, by describing a circle centre c, radius cs'. Join tS, nS. Then $tSn = (\phi)$ is the true shape of the angle between the planes VTH, V''T''H''.

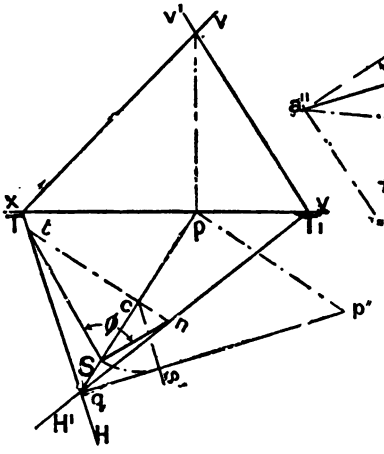


FIG. 496.

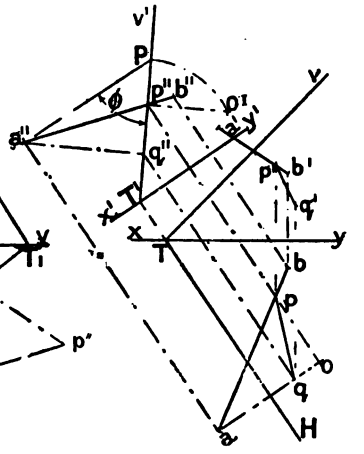


FIG. 495.

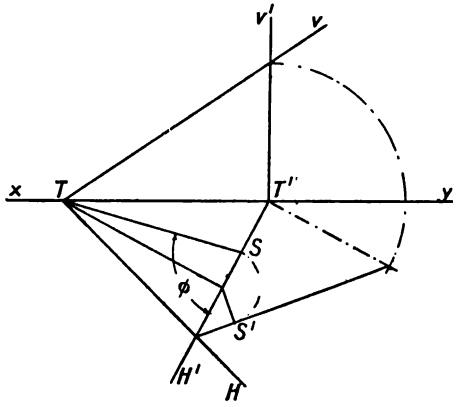


FIG. 497.

In Fig. 497 one of the planes is vertical, in which case the plan of the intersection lies in the HT of that plane.

In Fig. 498 the intersection is such that its plan is $\perp^r$ to the HT of one plane, in which case the construction appears to fail, inasmuch as *nt*—the HT of the plane $\perp^r$ to the intersection—is parallel to the HT of plane VTH. We observe, however, that when the HT's of two intersecting planes are parallel the planes intersect in a level line. Hence the line corresponding to *st* in Figs. 496 and 497, when brought into HP is parallel to HT.

The dihedral angle is ϕ equal to the angle between the two HT's.

In Fig. 499 *the construction is applied to determine the angle between two planes given by* scales of slope 0, 20 : 0, 30.

First determine *pq*, the plan of the intersection (Prob. 517), then complete as before.

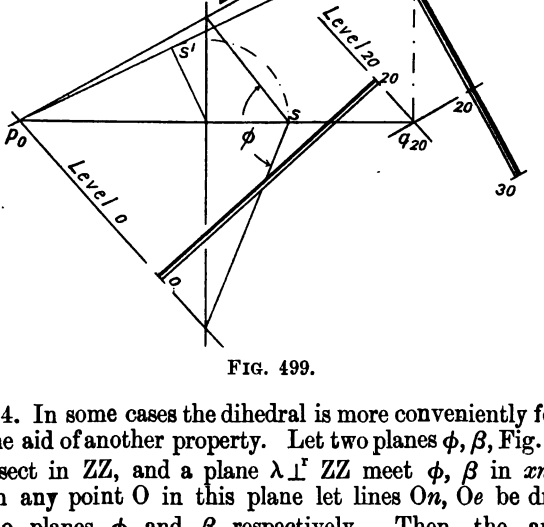


FIG. 499.

524. In some cases the dihedral is more conveniently found by the aid of another property. Let two planes ϕ, β, Fig. 500, intersect in ZZ, and a plane $\lambda \perp^r$ ZZ meet ϕ, β in *xn*, *xe*. From any point O in this plane let lines O*n*, O*e* be drawn $\perp^r$ to planes ϕ and β respectively. Then, the angles *exn*, *e*O*n* are supplementary, hence *the angle between two*

planes is the supplement of the angle formed by two lines drawn from any point perpendicularly one on to each plane.

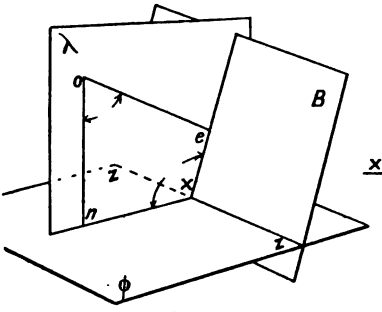


FIG. 500.

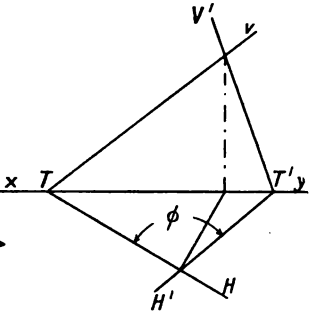


FIG. 498.

525. **Practical Application.** Fig. 501 *is part plan and sectional elevation of a* **hipped roof.** *The elevation shows the* **wall plate** E, *a* **common rafter** R, *the* **ridge** V, *and the* **purlin** P, *the latter much enlarged. In plan these members are shown, and also the* **hip** *and a* **jack** *rafter abutting against the hip.*

To ensure proper fitting of the various parts in such a roof as this, the carpenter must obtain correct solutions of several geometrical problems.

Geometry of the Common Rafter R. We require (a) *The true shape of the cuts to fit on the wall plate, and against the ridge.*

These cuts are shown in true shape in the elevation, the former at E, the latter at V.

(b) *The real length of the rafter.* This also is shown in the elevation.

If we denote the difference in level of the ends of the rafter by h, and the horizontal distance between these points by s, then the length of rafter $L = \sqrt{h^2 + s^2}$,

or $$L = \frac{s}{\cos \theta},$$

where θ = the angle of inclination.

Geometry of the Purlin. The end of the purlin abutting against the hip is the cut made by a vertical plane corresponding with a vertical side of the hip. This cut is inclined to all edges and faces of the purlin, and if the cross-section of the purlin is rectangular (as is usual) the shape of the cut will be a parallelogram.

We require to know the angles which the traces of the cut on two adjacent faces of the purlin make with the edges of the purlin.

These angles are readily found by rotating the surfaces into a horizontal position. Thus, the line of the cut on the top surface is ab, $a'b'$. Draw $x'y'$ through $a' \parallel xy$: Revolve b' into $x'y'$ at B'. The plan of the point is then at B. Draw aB. Then γ is the angle of the cut on the top, and parallel bottom surface, of the purlin. Similarly, the line of cut on the highest adjacent surface is ac, $a'c'$: Revolve c' into $x'y'$ at C'. The plan of the point is then at C. Draw aC. Then δ is the angle of the cut on the other two surfaces of the purlin.

Geometry of the Jack Rafter. The cut at the bottom of this rafter is the same as that of the common rafter R. The cut at the top is oblique. The trace of the cut on a vertical face of the rafter is vertical, and therefore the angle on this face is the same as that of the common rafter at the ridge. The trace of the cut on the top or bottom surface of the rafter makes with the edges of the rafter an angle $\mu = \gamma - 90°$.

Geometry of the Hip. The top of the hip is left square with the vertical sides sometimes. We will assume, however, that it is to be cut to fit the two planes forming the roof surfaces. We require (a) the angle made by the roof surfaces at their intersection in the centre of the hip. This angle is called the **backing bevel**; it is shown at ϕ, and found as in (Art. 523).*

(b) *The length of the hip and the shape of the cuts at both ends.*

The length of the hip may be found from $L = \sqrt{h^2 + s^2 + x^2}$, or graphically by a new view as shown at ko'.

* The angle ϕ between the roof slopes is sometimes given as the angle to which hip and valley tiles should be made. This is not quite correct, for these tiles do not lie parallel to the slope of the woodwork.

The angle α is the bevel to which the hip must be cut to fit on the top of the wall plate, and β is the angle of the cut on the sides of the hip against the ridge. To complete this cut, we must know the angles which the top arris of the hip makes with the traces of the vertical mitres at the ridge.

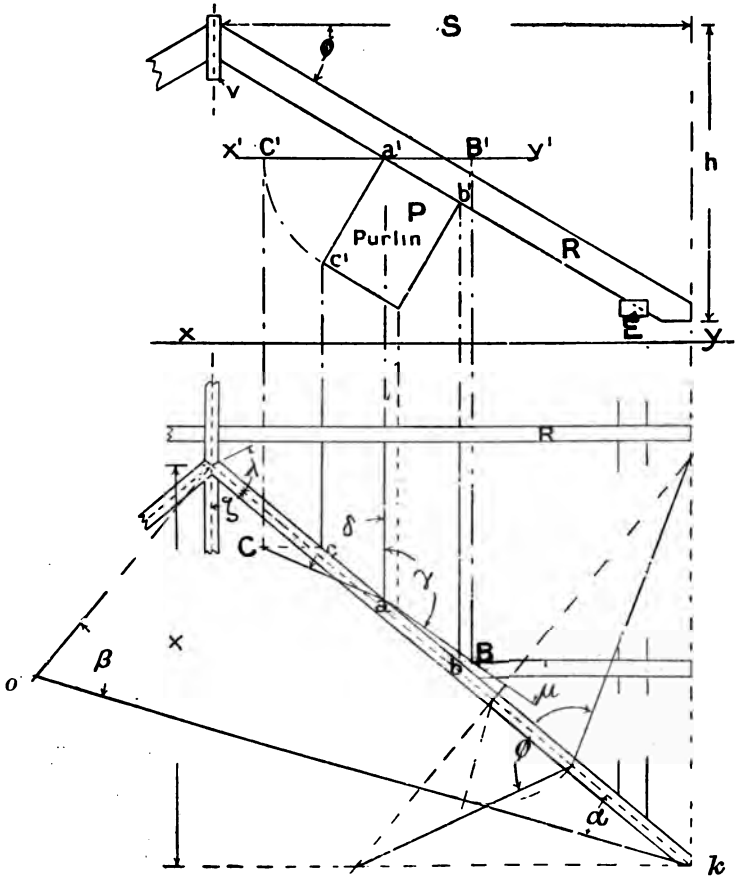


FIG. 501

These angles are shown in **plan** at λ and ζ. The real measure of λ and ζ may be found by rabatting into a horizontal position. The angles are **not** shown in the diagram.

526. **Ex. 2.**—*In Fig.* 502 *the keystone of an arch is shown in plan and elevation at e s r h, a' e' g' h' f' b'. Set out the true shape of the faces* A, B *and* C. *Find the dihedral angle between the faces* A *and* B. (B. E. *L. G.* 1912.)

The true shape of the faces may be found by rabatting each face into HP. Thus :—

Face A.—Draw eG, fH $\perp^r$ ef make eG, eE, fH, fF = the heights of g', e', h', f' respectively above xy, and join by straight lines.

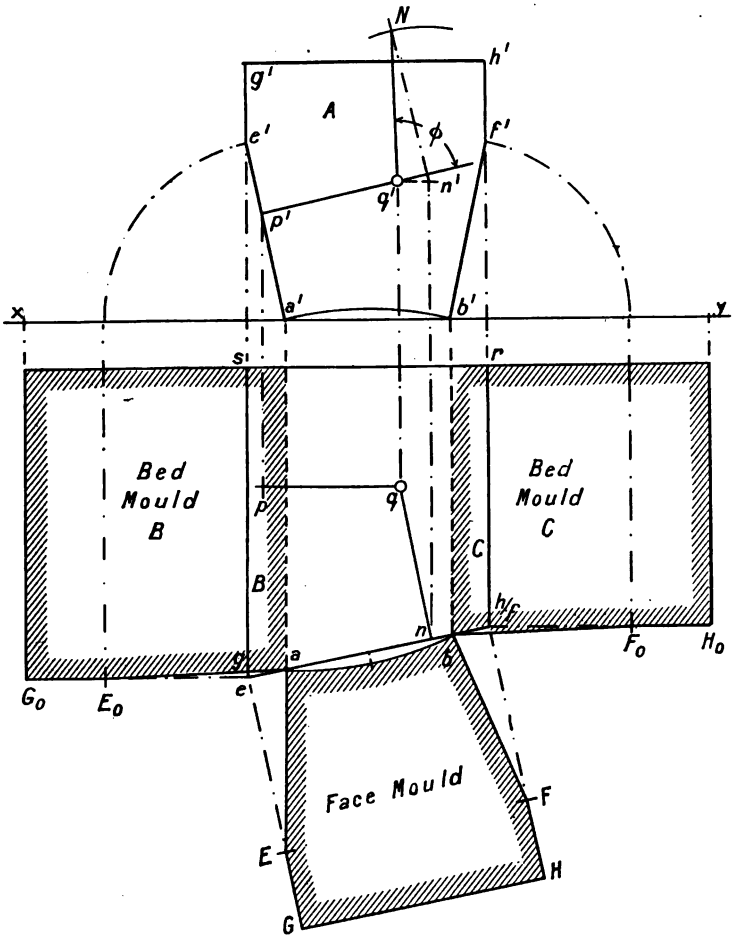
The curve of the *soffit* of the stone in the face A is an arc of a circle in elevation, hence, its true shape is elliptical and may be found by drawing projectors through various points in the plan of the curve and marking off along these the heights of the points measured on the elevation. This construction cannot be clearly shown on a small diagram like Fig. 502.

Face B. — Rabat this face about its lowest edge (shown in plan by dotted line). The point E moves out to E_0, and $E_0G_0 = e'g'$ and is $\parallel xy$.

Face C.—Similarly, rabatting about the lowest edge, $F_0H_0 = f'h'$ and is $\parallel$ xy.

To find the dihedral angle between faces A, B.— From any point q,q' draw qn, $q'n'$ $\perp^r$ face A meeting same in n, n'. Draw qp, $q'p'$ $\perp^r$ face B, meeting same in p, p'. With centre q' and radius qn describe an arc meeting a line drawn from n' $\perp$ $p'q'$ in N. Then $p'q'$N is the true shape of the angle between lines pq, nq, $p'q'$, $n'q'$ and since PQ is $\perp^r$ to face B and NQ is $\perp^r$ face A hence ϕ, the supplement of this angle is the angle between the faces A, B.

Note.—Ex. 2 has been taken from the latest (1912) Exam. of the Board of Education. We may, however, remark that the stonecutter would hardly be likely to use a method involving either a development of the faces B and C, or the dihedral of BA. He would more likely proceed as follows: (1) Work a plane surface corresponding with the top of the stone; (2) a similar surface $\perp$ to the top surface and corresponding to the back; (3) apply to the back surface a mould cut to the form of the elevation of the stone, and to the top surface a mould cut to the plan *efrs*; (4) work a plane surface $\perp^r$ to the top and containing the top front arris *gh*, *g'h'*, and apply to this face the mould *aGHb*. The markings of the front and back *facemoulds* will serve as directors to the stonecutter. Each joint surface is formed of 2 planes. The soffit is cylindrical and may be worked in drafts $\perp$ to the back face of the stone.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g
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F_0
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G_0
E_0
Face Mould
F
E
H
G

FIG. 502.

Exercises.

(1) A plane is inclined at 35° to the horizontal, the lines of steepest upward slope going due East. A second plane has an inclination of 52°, the direction of the lines of upward slope being due North. Represent the planes by scales of slope, unit for heights 0·1 in. Show the figured plan of the intersection of the planes. Find and measure the angle between the planes. (B.E. (2), 1904.)

(2) The roof of a house is rectangular in plan, and two adjacent surfaces are inclined to the horizontal at 30° and 40° respectively. Find the inclination to the horizontal of the "hip," or line in which the two surfaces meet.

What is the magnitude of the dihedral angle between the surfaces? (B.E. (2), 1906.)

(3) Fig. (c). Two planes are given by their scales of slope; unit for heights = 0·1 in. Find the inclination of these planes to the horizontal. Determine the figured plan of the line of intersection of the planes. What is the dihedral angle between the planes? (B.E. (2), 1908.)

(4) Fig. (a). a_0 is the lowest angle of a cube; ab is the plan of one edge, AB (real length = 20 units), of a face which lies in a plane, of which ht is the horizontal trace. Draw the plan of the cube. Also the plan of its section by a plane parallel to the horizontal plane of projection and at a height of 12 units above it. Unit = 0·1 in. (B.E. (2), 1896.)

(5) Fig. (d). Draw (1) a plane (M) inclined at 65° to the horizontal plane, and containing AA'; (2) a plane (N) through p inclined at 50° to the horizontal plane and perpendicular to M. (B.E. (2), 1896.)

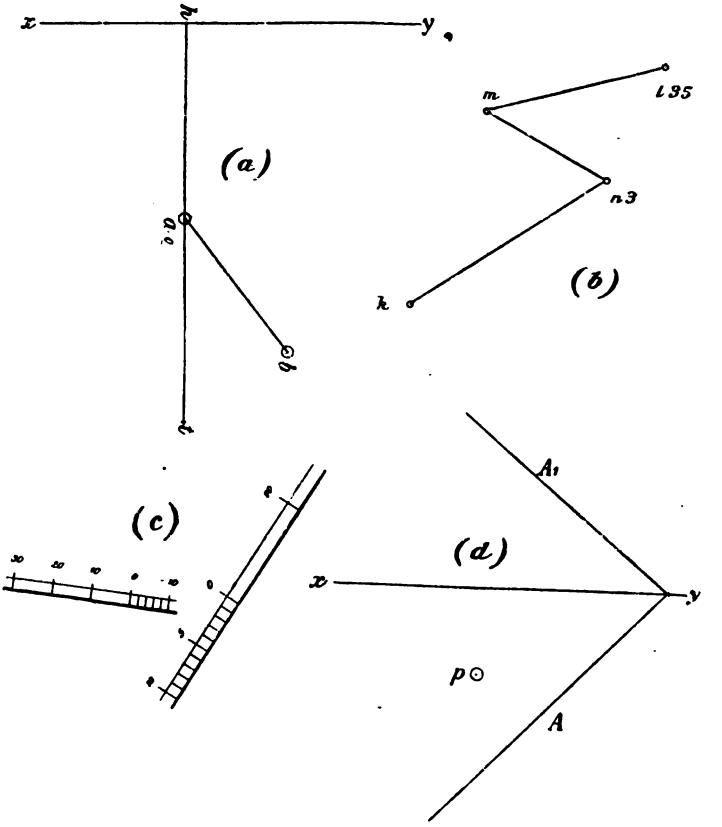
(6) Fig. (d). p is the plan of a point P distant $\frac{1}{2}$ in. (measured perpendicularly to the surface) from the plane AA' and above it. Through P draw (1) a line parallel to the plane AA', and inclined at 45° to the horizontal plane; (2) a line also parallel to the plane AA', but making an angle of 15° with the line inclined at 45°. (B.E. (2), 1896.)

(7) Fig. (b). The plans of three lines LM, MN, NK are given, the heights of L and N being indexed. ML is perpendicular to the plane of MNK. Find the heights of M and K and index their plans. Unit = 0·1 in. (B.E. (2), 1901.)

Solution.—As LM is $\perp^r$ the plane MNK, therefore LM is $\perp^r$ MN and LMK is a right-angled triangle on hypotenuse LN. The projection of angle LMN on any plane parallel to LM will be a right angle (Art. 472). Hence take $xy \parallel lm$, obtain an elevation $n'l'$ of NL on this plane; draw a projector from m which meets a semicircle described on $n'l'$ in two points m', m''. There are thus two possible positions for LM; the elevations being $l'm'$, $l'm''$; $n'm' \perp^r l'm'$ is a profile of the plane MNK and projectors from m and k determine at m', k the heights of M, K. Similarly, if $l'm''$ be taken as the elevation of LM, then $n'm''$ is profile of plane MNK, and the heights of M, K are determined in m'', k''.

(8) Two planes are perpendicular to one another; one of them is inclined to the horizontal at 62°, and their intersection is inclined at 50°. Represent the planes by scales of slope. (B.E. (3), 1910.)

Hint. First find a plane inclined at 62°. In this plane place a line inclined at 50°. Then determine a plane to contain this line and be perpendicular to the first plane. (Problem 535.)



CHAPTER XXX.

PLANES AND LINES.

A PLANE is fixed completely when three non-colinear points in it are fixed, and a plane can be found which shall contain any three non-colinear points, or the equivalent data —two intersecting straight lines or two parallel straight lines.

527. **Problem.** Fig. 503 (a). *Determine a plane to contain the two intersecting lines, ap, a'p'; bp, b'p'.*

The traces of all lines in a plane are in the traces of the plane. Therefore, determine the horizontal and vertical traces of AP, BP. A line through the two horizontal traces is the HT, and a line through the two vertical traces is the VT of the plane containing AP, BP.

528. **Problem.** No Fig. *Determine the plane to contain any three points not in the same straight line.*

Join up the points, forming a triangle, then find the traces of two sides and complete as in preceding problem.

529. **Problem.** Fig. 503 (b). *Determine a plane to contain two parallel lines ab, a'b', cd, c'd'.*

This problem is solved precisely as Problem 527, but the lines, as arranged in the diagram, do not penetrate the VP within the limits of the drawing. Draw any line *pq* across the plans of the given lines; run up projector from *p* on to the elevation *c'd'* at *p'*, also from *q* on to elevation *a'b'* at *q'*, and draw *p'q'*. Then PQ is a line resting on the given lines AB, CD, and consequently in the same plane. The VT of PQ is shown at *v'*. Join *v'* to the point where HT meets XY and thus obtain the required VT of the plane.

530. **Problem.** No Fig. *Determine a plane to contain a given point and a given line.*

Take any point on the given line and join to the given point. The problem is then reduced to Problem 527.

531. **Problem.** No Fig. *Through a given point* P *draw a line which shall meet two given straight lines.*

Solution.—Determine a plane to contain the point P and one line, next, the plane containing P and the other line. The intersection of these two planes passes through P and is the required line.

532. **Problem.** No Fig. *Given a line* AB *and a point* P *not in the line. Draw through* P *lines* PS, PR, PQ, *making*

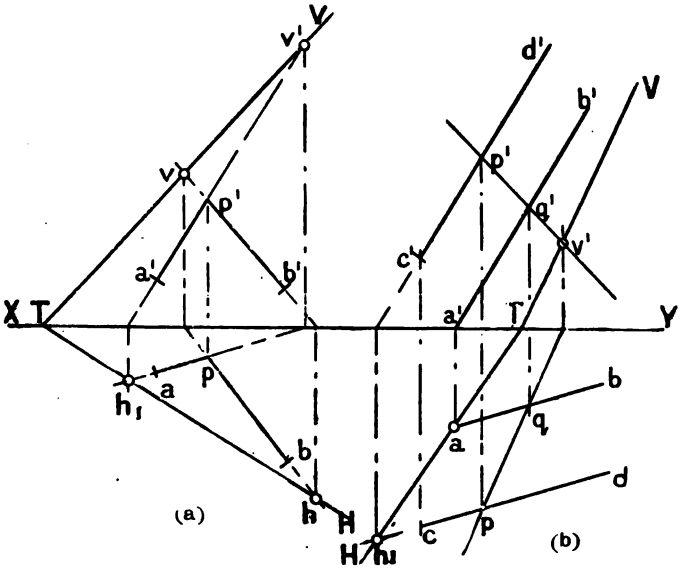


FIG. 503.

(a) *the angle between* PS *and* AB=, 50° say. (b) PR ⊥r AB (c) PQ *a given length.*

Solution.—Determine the plane containing P and AB, rabat this plane into HP or VP. Determine PS, PR, PQ by plane geometry, then reconstruct.

Observe that PR is a measure of the distance of P from AB. Further, PQ cannot be less than PR. If greater there will be two lines. There are always two lines PS.

533. **Problem.** Fig. 504. *Determine a plane to contain a*

given point a, a' and be perpendicular to a given straight line cd, c'd'.

First method. Make the plan *cd* serve as XY of a vertical plane, on which obtain an elevation of A at *a''*, and of CD at *c''d''*. From *a''* draw a perpendicular to *c''d''* intersecting the plan *cd* (produced if necessary) in H. Draw HT perpendicular to *cd*, meeting XY in T. This line is HT of the required plane. A line VT drawn through T and perpendicular to *c'd'*, the elevation of the line, is the vertical trace of the plane.

2nd Solution.—As any line in the required plane has its traces in the traces of the plane (499), the problem may be solved without the auxiliary view, thus—draw $a'\ h' \perp^r c'\ d'$, and $a\ h \parallel xy$. The HT of plane passes through *h* the HT of this line and is $\perp^r$ to the plan *c d*.

534. **Problem.** Fig. 505. *Determine a plane to contain a given point a, a' and be parallel to a given plane* VTH.

This problem may be solved by aid of a profile view as in the preceding, but more easily from the fact that the plane must have traces parallel to those of the given plane and contain the two lines which can be drawn through A parallel to those traces. The construction is shown in the Fig. and should be self-evident.

535. **Problem.** Fig. 506. *Determine a plane to contain a given line ab, a'b' and be perpendicular to a given plane* VTH.

A plane will be perpendicular to a given plane if it contain a line which is perpendicular to that plane. Hence, from any point *a, a'* in the given line, draw *ae, a'e'* perpendicular to the traces of the plane. Then, determine the plane $V_1T_1H_1$ which contains AB, AE (Problem 527). This is the plane required.

In Fig. 507, the plane is given by its scale of slope 0, 15, and the line by indexed plan a_{14}, b_5. To solve the problem take XY $\parallel$ scale of slope, and obtain an edge view of the plane at TV, and an elevation of AB at *a'b'*. Draw *ah* parallel XY and $a'h' \perp^r$ TV. Draw $b'e' \parallel$ XY meeting *a'h'* in *e'*. Project *e'* on to *ah* at *e*. Then b_5e is plan of a line of level 5 in the plane of ABE. The scale of slope is $\perp^r$ to b_5e. The HT of the plane passes through *h* and is $\parallel$ to b_5e. The required plane is 0, 20.

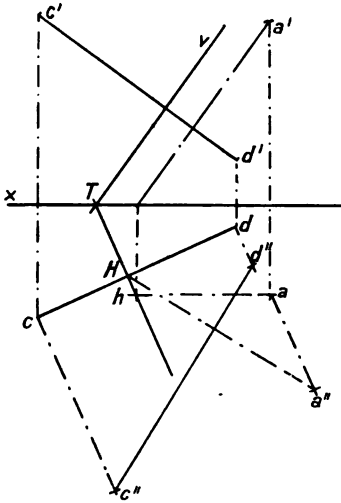


FIG. 504.

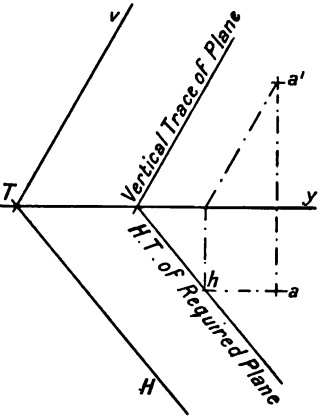


FIG. 505.

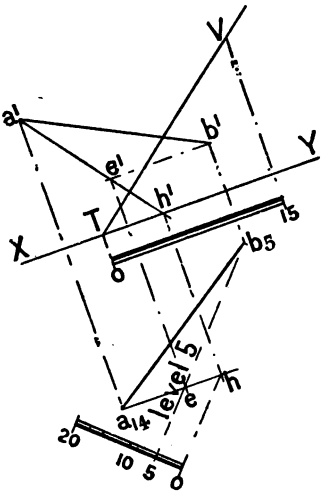


FIG. 507.

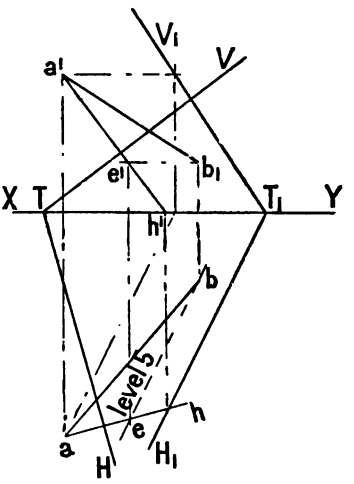


FIG. 506.

536. **Problem.** No Fig. *Determine a plane to make a given angle θ with a given line* AB *and to contain a given point* P *which is not in the given line.*

Find the plane containing AB and P (Problem 530), and draw from P a line making the angle θ with AB (Problem 532). Then find the plane containing this line and perpendicular to the plane of ABP (Problem 535).

537. **Problem.** Fig. 508. *Determine a plane to contain a given line ab, a'b' and make a given angle θ, say* 50°, with HP.

Limitation.—The angle θ cannot be less than the angle between AB and HP.

Analysis. The traces of the plane must contain the traces of AB, and the plane will be tangent to a right cone which has base on HP, base angle $= \theta$, and vertex in the line AB. Two planes can be found.

Construction. From the elevation of any point, as a' in the line, draw $a'o' \perp^r$ XY. Draw $a'f'$ making 90—50° with $a'o'$. With centre a, radius $o'f'$ describe a circle. Determine h, the horizontal trace of AB, and draw hs tangent to the circle centre a meeting XY in T. Determine v' the VT of a horizontal line in the plane through point A, and draw Tv'. Then v'Th is a plane satisfying the given conditions.

The second plane has hs' for HT, and v^2 is a point in the VT. A second point in VT may be found thus; Join v^2 to the elevation of any point, as h', in the HT of the plane. From any point n in hs' draw $ne \parallel hv$. Project n on to XY at n'. Draw $n'e' \parallel h'v^2$ meeting a projector from e in e'. Then e' is a point in the VT of the required plane.

Observe that $vh'v^2$, $en'e'$ are parallel sections of a triangular pyramid, the edges of which are v^2e', hn, and XY.

Special Case. If the line AB lies in either plane of projection, it will be the trace of the required plane in that plane of projection, and the problem then becomes Problem 489 or 490.

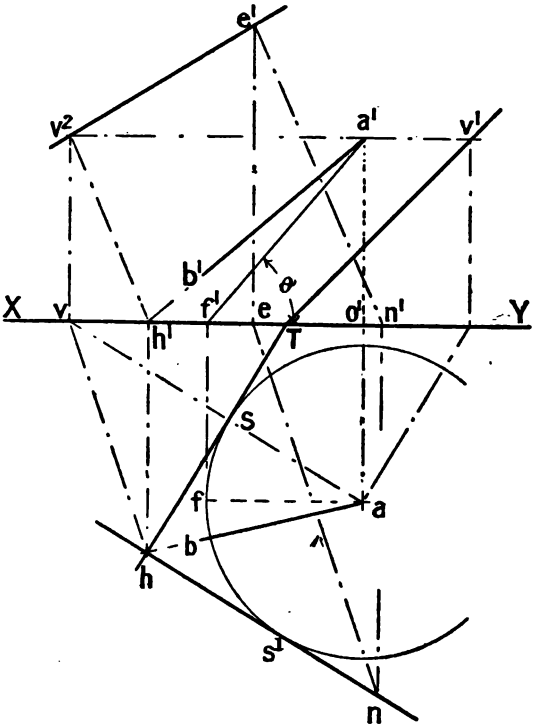


FIG. 508.

538. **Problem.** Fig. 509. *Determine a plane to contain a given line a'b', ab, and make a given angle θ with a fixed plane* VTH.

This is the general problem of which Problem 537 is a special case, and, as in that problem, the angle between the line and plane cannot be less than θ.

Analysis. The plane will be tangent to the right cone, which has a base angle equal to given angle θ, its vertex in the given line AB, and its axis perpendicular to given plane

VTH. A projection on a vertical plane perpendicular to VTH will simplify the drawing.

Construction. In any convenient position, draw X_1Y_1 perpendicular to HT, and obtain an edge view of the plane and a new elevation of some point—say A—in the given line.

Draw $a''o''$ perpendicular to tv''. Draw $a''e''$; $a''r''$, each making with $a''o''$ the complement of given angle θ. Then $a''e''r''$ is the elevation of a cone whose generators make the given angle θ with the plane VTH. The horizontal trace of this cone is an ellipse of which $e''r''$ is major axis. The ellipse is shown by dotted line in the Figure, and the HT of the required plane must pass through h, the HT of AB, and be tangent to this elliptical trace. The HT can be drawn without constructing the ellipse by the second method of Art. 514.

H′ T′ V′ is a plane satisfying the conditions of the problem. As two tangents can always be drawn to an ellipse from an external point, a second plane can be found. Its HT must also pass through h and touch the ellipse. This plane is not shown in the diagram because the plan of the line AB is almost tangent to the ellipse, and is thus confounded with the HT of the second plane. Observe that, when the plan of AB is tangent to the ellipse, one plane is vertical, because only a vertical plane can contain a line in space and have the plan of the line for HT.

Special Case. Fig. 510. When the given line AB is in the given plane VTH, AB is the intersection of the required plane and the given plane VTH, and the problem is clearly the converse of Problem 523.

Construction. Obtain a new elevation $b\ a''$ of BA, using the plan as X_1Y_1. From any point s'' draw $s''o$ perpendicular to $a''b$, meeting ab in o, through o draw or perpendicular to ab, intersecting HT in r. Rabat the triangle SOR about R_1O into HP at Sor. At S draw Sh, making with Sr the given angle θ and intersecting ro in h, from b draw bh, meeting XY in t, join ta'. Then $a't$, tb are the VT and the HT of the required plane.

FIG. 509.

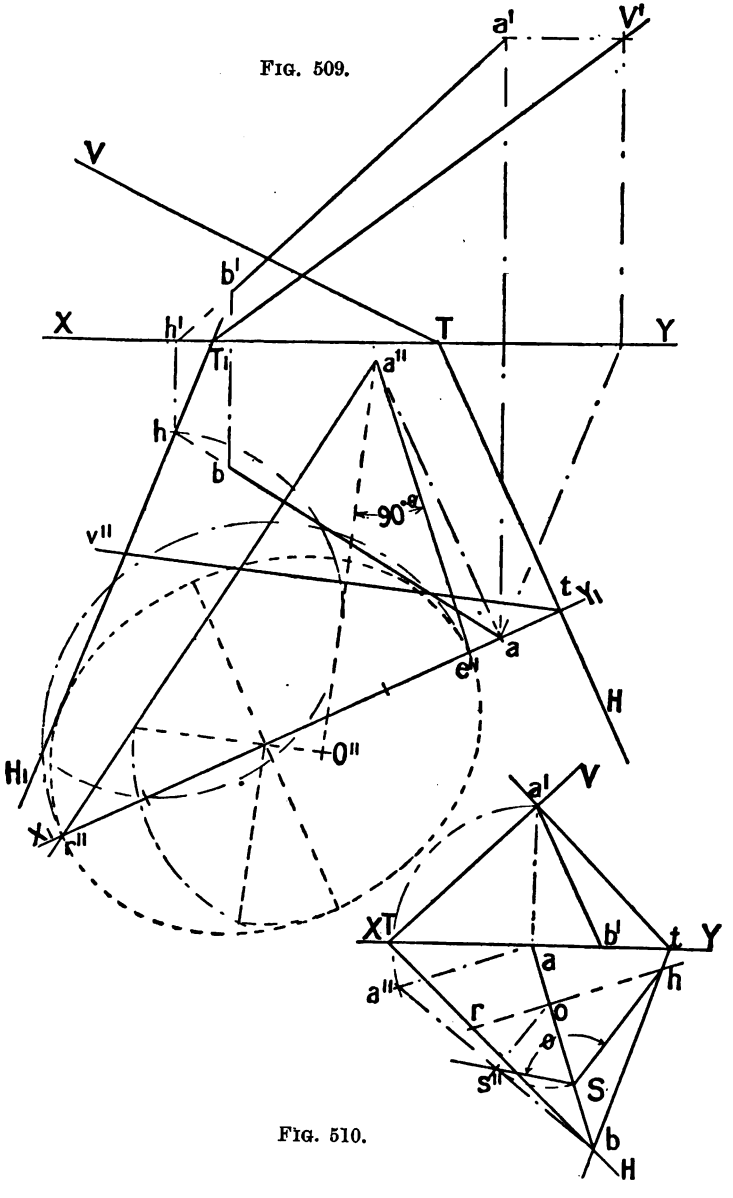


FIG. 510.

539. **Problem.** Fig. 511. *Determine a plane to contain a given point a, a^1 and make angles θ and ϕ respectively with* HP *and a given plane.*

(*Note.*—In the diagram, the plane is ⊥ to VP. Should the plane be given oblique, obtain a profile view, then proceed as in this case.)

Analysis. The required plane will be tangent to two cones having the point A for common apex. One cone having axis AO ⊥ to HP and base angle equal θ; the other with

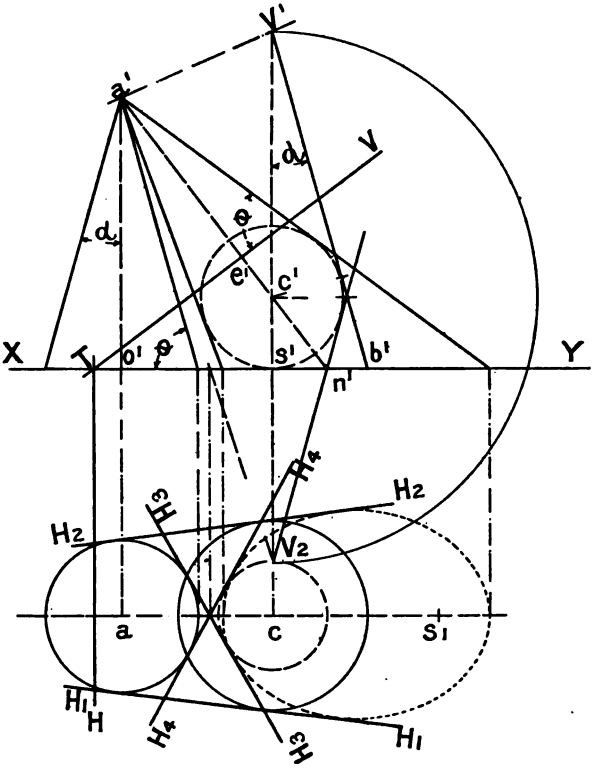


FIG. 511.

axis AE ⊥ VTH and base angle equal ϕ. The horizontal trace of the first cone will be a circle centre a, radius $o'r'$, and that of the other an ellipse. This ellipse may be determined as in Problem 561. It is shown by dotted line.

The problem may be solved without the ellipse if we inscribe any sphere, centre c', in the inclined cone and circumscribe the sphere with a cone similar to the cone which has base on HP, and make the axis $v'c'$ of this cone ⊥ to HP. The trace of this cone on HP is a circle centre c, radius $s'b'$. The plane which is tangent to the two cones with vertical axes will also be tangent to the inclined cone.

A line H_1H_1 tangent to the circle centres c and a, is therefore the horizontal trace of a plane satisfying the given conditions. The VT of the plane is readily found since the point A is in the plane.

Observe that the plane must contain the line AV which passes through the vertices of the two cones. If the HT of this line comes within the limits of the drawing, it may be used as a point in the HT of the plane.

In the Figure, the elliptical trace of the inclined cone does not intersect the circular trace of the cone which has axis AO, which shows that the two cones have nothing in common except the point A. In this case, four lines can be drawn to touch the ellipse and the circle centre a, whence there are four planes which satisfy the conditions of the problem. In this case, two HT's are common tangents to the circles, centre a, radius $o'r'$; and centre c, radius $s'b'$; and two are tangent to the circle centre a, and a circle centre c, of radius $s'n'$, which is obtained by inverting the cone that envelops the sphere centre c'. The elevation of the apex of the inverted cone is at v_2. Observe that the tangents H_3H_3, H_4H_4 pass through the HT of the line $a'v_2$.

If the inclined cone has line contact with the first cone, their traces on HP will be in contact, and *three planes* can be found touching both cones and satisfying the given conditions. One plane contains the line of contact of the two cones and is perpendicular to the common axial plane.

If the cones intersect, their traces intersect and *two planes only* can be found.

If one cone falls just within the other, so that they have line contact, the trace of the inner cone will lie inside the trace of the other and the two will be tangent. In this case, the common tangent at the point of contact will be the HT of the only plane that satisfies the conditions.

If one cone lies wholly inside the other, the trace of one is wholly inside the other and obviously no common tangent can be drawn. Consequently no plane can be found.

540. **Problem.** No Fig. *Determine a plane which shall*

FIG. 512.

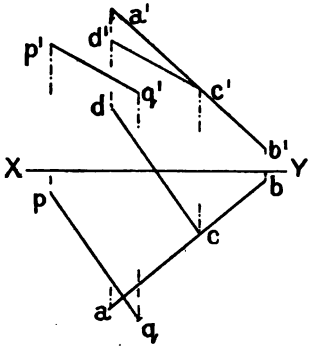
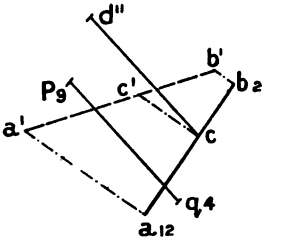


FIG. 513.



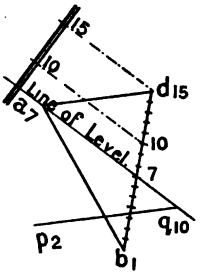


FIG. 515.

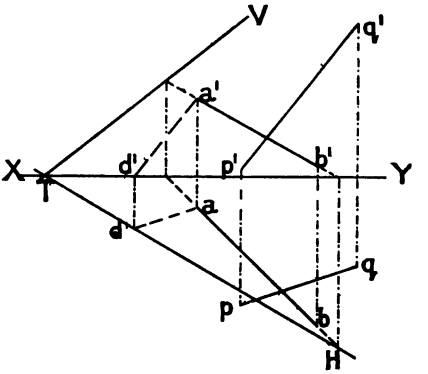


FIG. 514.

be perpendicular to a given oblique plane VTH; *pass through a given point aa', and have a given inclination θ to* HP.

This is a special case of the last problem.

Make the point the apex of a cone with the given base angle θ and standing on the ground, and draw from the point a perpendicular to the given plane VTH. Then find the plane containing the perpendicular from A and resting on the cone.

541. **Problem.** Fig. 512. *Given two non-parallel lines* AB, PQ *which do not intersect, draw through a point* C *in* AB *a line parallel to, and equal in length to,* PQ.

Parallel lines have parallel projections, and projections in the ratio of their lengths. Art. 433. Hence draw *cd*, *c'd'* respectively parallel to and equal to *pq*, *p'q'*.

In Fig. 513, the lines AB, PQ are given by indexed plans $a_{12}b_2$; p_9q_4.

Draw cd'' ∥ to, and equal in length to, q_4p_9. Then cd'' is plan of a line through *c* parallel and equal to *qp*. The point *d* is as much above *c* as *p* is above *q*, *i.e.*, 5 units. In order to index *d*, we must first ascertain the height of *c*. This is conveniently done by getting an elevation *a'b'* of AB on a plane which has the plan $a_{12}b_2$ for XY. From this view the height of C is obtained by measuring *cc'*. It scales 6 units. Hence *c* would be indexed 6 and *d''* would be indexed 6+5=11.

542. **Problem.** Fig. 514. *Given two non-intersecting and non-parallel straight lines* AB, PQ, *determine a plane to contain* AB *and to be parallel to* PQ.

A plane will be parallel to a given line if it contains a line which is parallel to the given line. Hence, through any point *a*, *a'* in AB draw *ad*, *a'd'* parallel to PQ (preceding problem), and find the plane to contain AB, AD (Problem 527). This plane will be parallel to PQ.

In Fig. 515, the lines are given by indexed plans a_7b_1; p_2q_{10}. Draw *ad* ∥, and equal to, *pq*. Index *d* 15, *i.e.*, 7+10−2. Since *d* is as much above *a* as *q* is above *p*. Divide b_1d_{15} into 14 equal parts. The sixth division from *b* in this line is level 7. A line from 7 through *a* is a line of level 7 in the plane of ABD. The scale of slope is shown at 10, 15.

543. **Problem.** No Fig. *To determine the angle between two non-intersecting straight lines ab, a'b' ; pq, p'q'.*

The angle between two non-intersecting straight lines is the angle between the projections of the lines on a plane parallel to both lines, or, what amounts to the same, the angle between two lines drawn from any point parallel to each of the non-intersecting lines.

From any point *aa'* in AB draw *ad*, *a'd'* parallel to *pq*,

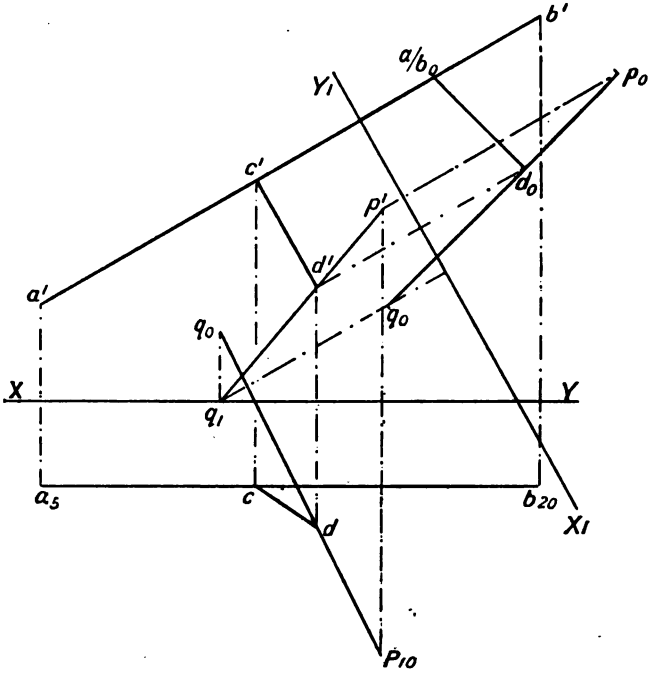


FIG. 516 (a).

p'q'. Determine the true shape of the angle BAD. This is the angle required.

544. **Problem.** *To determine a line* BCD *having an extremity in each of two non-intersecting lines* a_5b_{20}, $p_{10}q_0$, *and to be perpendicular to both lines.*

1st Method. Fig. 516 (a). If the given line be projected on to a plane perpendicular to one line, as AB, the projection of AB on this plane will be a point, and the distance of this point from the projection of the other line is the distance between the two lines, *i.e.*, the length of the common perpendicular.

Construction. Obtain an elevation $a'b'$, $p'q'$ of the

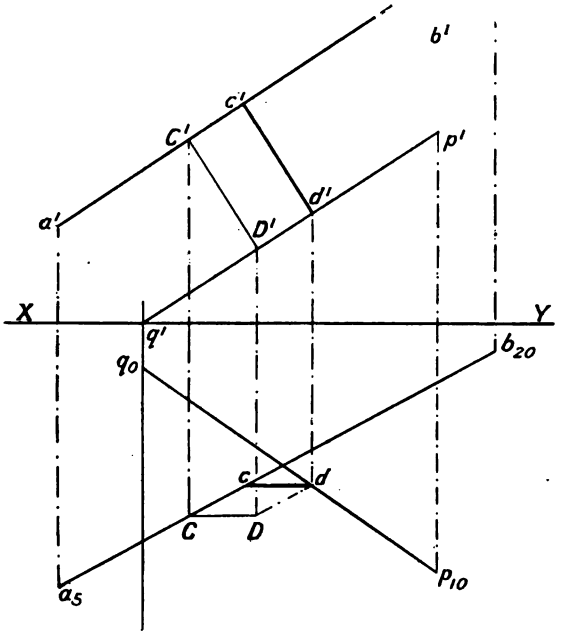


FIG. 516 (b).

given lines on any XY parallel to ab. Make a projection q_0p_0, $\frac{b}{a}$ of the lines on any X_1Y_1 perpendicular to $a'b'$. From the point projection of AB draw a perpendicular on to q_0p_0 intersecting same in d_0. Draw projector d_0d' meeting $p'q'$ in d', and draw $d'c'$ parallel to X_1Y_1. Then $d'c'$ is an eleva-

tion of the required line, its plan is dc found by projectors from the elevation.

2nd Method. Fig. 516 (b). If a plane be found which is parallel to one line and contains the other, and the two lines be projected on to a plane which is perpendicular to this first plane, the projections of the two lines will be parallel and the length of the common perpendicular is equal to the distance between these parallel projections.

Construction. Determine HT the horizontal trace of the plane which contains AB and is parallel to PQ (Problem 542). Take XY $\perp^r$ HT and make an elevation of the given lines at $a'b'$, $p'q'$. Draw C'D' in any position between $a'b'$, $p'q'$ and $\perp^r$ thereto, drop a projector from C' on to ab in C, draw CD $\parallel$ XY meeting the projector from D' in D. Then CD, C'D' is plan and elevation of a line parallel to and equal in length to the line required, but this line has not its extremity, D in the line PQ. The line may be moved parallel to itself until the end D falls on PQ. Thus, draw $Dd \parallel ab$ meeting pq in d; draw $dc \parallel DC$, then cd is the required plan. Its elevation on any XY is found by projectors perpendicular to such XY.

545. **Problem.** Fig. 517. *Two lines* AB, CD *and a plane are given by* $a_{10}b_{80}$; $c_{25}d_5$, *and scale of slope* 0·40. *Determine a line* EF *having an extremity in each of the given lines; to be parallel to the plane* 0·40 *and* M *distance therefrom.*

The line EF will lie in a plane which is parallel to the given plane and distant M therefrom. The ends E, F of the line are the points in which AB, CD penetrate this plane.

E and F are readily found by taking a profile view of plane 0·40; drawing an elevation of the two lines, as $a'b'$, $c'd'$, and drawing the edge view of a parallel plane at the distance M from plane 0·40 intersecting $a'b'$ in f', and $c'd$ in e'. Projectors from e', f' determine the plan $e_{12\cdot8}$, $f_{24\cdot8}$.

Exercises.

(1) A person is making a survey on ground which is a sloping plane. Starting from a place A he goes west until he is at 10 ft. higher level, and marks a point B; AB measures 42 ft. Returning

to A he goes north to C until he is at the same level as B ; AC measures 33 ft. In order to show these points on a plan or map, he determines the horizontal distances between AB and AC.

What are these distances ?

Show the points A, B, C in plan to a scale of 1 in. to 10 ft.

In this plan show the direction of a footpath which shall have the greatest upward slope. (B.E., 1905.)

(2) A line a_5b_{12}, 2 ins. long and inclined at 25° to xy, the end a being nearest to, and $\frac{3}{4}$ in. from, xy, is the plan of the base of an isosceles

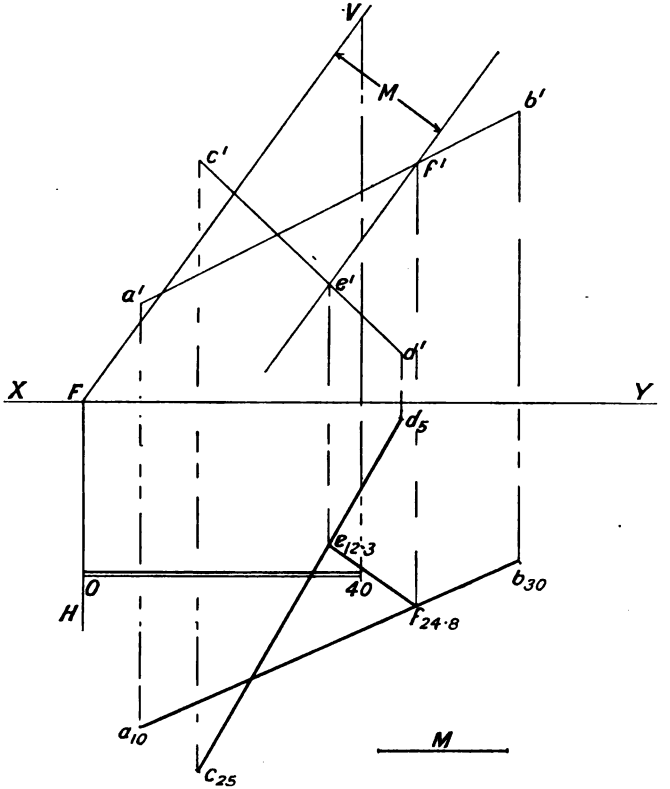


FIG. 517.

triangle, the apex of which is in xy. Draw plan and elevation of the triangle.

Hint. The apex of triangle is the point in which a plane $\perp^r$ to, and bisecting AB intersects xy.

(3) The horizontal and vertical traces of a plane make 60° and 45° with xy. A point p, distant 1 in. from the horizontal trace and 1½ ins. from xy, is the plan of a point P contained by the plane; draw the plan and determine the elevation of P.

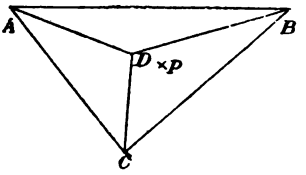


FIG. 517 (a).

Draw the projections of a line PM which is perpendicular to the plane and has its foot M in the horizontal plane. (B.E., 1908.)

(4) Fig. 514. Represent by a scale of slope a plane which shall contain the given line AB and be perpendicular to the given plane. Unit for heights 0·1 in. (B.E. (2), 1907.)

(5) Fig. 517 (a). ABC is the base of a tetrahedron, and is in the horizontal plane. D is at 2·5 ins. above it. (a) Find the values of the dihedral and plane angles round the vertex D. (b) P is a point in the base: find the points where perpendiculars from P meet the three other faces of the tetrahedron. (B.E. (2), 1895.)

(6) Determine a plane inclined to HP and VP at 60° and 52° and to contain a given point A which is 1½ in. in front of VP and 2 ins. above HP.

Hint.—Determine any plane having the given inclinations to HP and VP (Problem 492).

Then a plane through A parallel to that plane.

Or, make the point the common apex of two cones, one with base on HP and base angle 60°; the other with base on VP and base angle 52°.

(7) Determine a plane inclined at 65°, making 70° with the given plane, and ¾ in. distant from the given point Pp' (unit 0·1 in.). (1889 Hons.)

Hint.—Make P the centre of a sphere of radius ¾ in. and circumscribe the sphere with two circular cones, the base angle of one being 65° and base on HP; base angle of the other 70° and base on the given plane (Problem 539).

CHAPTER XXXI.

PLANE SECTIONS AND DEVELOPMENTS OF SURFACES.

Miscellaneous Examples.

546. **Right Circular Cylinder.** *The section of a cylinder by a plane* **perpendicular to the axis** *is a circle equal to the base circle.*

The section by a plane **parallel to the axis** *is two generating lines of the cylinder.*

The plane which contains the axis of the cylinder and is perpendicular to a cutting plane is called the **axial plane.**

547. If a plane VF, inclined to the axis of the cylinder (Fig. 518), intersect the axial plane in AA', the cylinder in APA', and is tangent at points S and S' to two spheres, centres O, O', inscribed in the cylinder, it can be shown that *the section is an ellipse, having* AA' *for* **major axis,** *foci at points* S, S', *and* **minor axis** *equal to the diameter of the cylinder.*

The inscribed spheres which touch the section plane at S, S' are called **focal spheres.**

548. It follows that all elliptical sections of a right circular cylinder have minor axes equal to the diameter of the cylinder; the major axes depend upon the angle which the cutting plane makes with the axis of the cylinder.

549. If we denote the diameter of the cylinder by D, and the angle between the cutting plane and a plane perpendicular to the axis by θ, and the major axis by AA', we have $AA' = \frac{D}{\cos \theta}$.

The area of a section on a plane inclined at θ to the plane of the base is $A = \frac{a}{\cos \theta}$,

where A = area of section, and a = area of base.

550. Any straight line on a curved surface is called a **rectilinear element** of the surface.

All rectilinear elements of a cylinder are parallel to the axis.

551. **Problem.** Fig. 518. *A right circular cylinder standing with its base on* HP *is cut by a plane* VTH ; *determine the true shape of the section and show the trace of the cut on the surface when developed.*

Divide the circular base into a number of parts—preferably equal,—at *a, b, c, d, e, f, g*. These points are plans of straight lines on the surface, shown in elevation at $a'1$, $b'2$, $c'3$, etc.

The elevation of each line is also the elevation of a corresponding line on the half of the surface behind the axial plane *ag*.

Since the section plane is perpendicular to VP, the points 1, 2, 3, etc., in which VT cuts the elevations of rectilinear elements, are elevations of points in the section. The true shape of the section is known to be an ellipse having major axis of length 1 – 7, and minor axis = *ag*. It is shown by the shaded figure in the diagram, projected from the VT on the auxiliary ground line X'Y'.

Development. *When any cylindrical surface is developed, rectilinear elements remain parallel, and the line in which a plane perpendicular to the axis cuts the cylinder develops into a straight line perpendicular to rectilinear elements. The traces of planes which are not perpendicular to the axis are curves when developed.*

To Show the Trace of the Plane VTH on the Developed Surface. Rectify the semicircle *a . . . g* at A, B, C, . . . G along XY extended.* At these points draw perpendiculars AI, BII, . . GVII, making AI = $a'1$, BII = $b'2$, etc. A fair line drawn through the points I, II, . . VII is the development of the plane trace on *one half* of the cylindrical surface. The line AI is a line of symmetry, and to show a complete development the curve I, II . . VII must be repeated on the left-hand side of AI.

552. **Practical Application.** If the portion of the cylinder above the cutting plane be turned through 180°, it will take up the position shown in the diagram by dotted lines,

* In practical work, the circle is rectified as the combined length of a series of short chords. The result obtained by this method is only approximately correct, but, provided the angle subtended by any chord does not exceed 15°, the error is smaller than the unavoidable errors of drawing. The correct values of an arc and chord of 15° in terms of the radius are : Arc 15° = 0·2610799 ; chord 15° = 0·2610524, which are seen to be the same for four decimal places.

and the two parts of the cylinder will "fit" each other along VT, because the major and minor axes of the ellipse

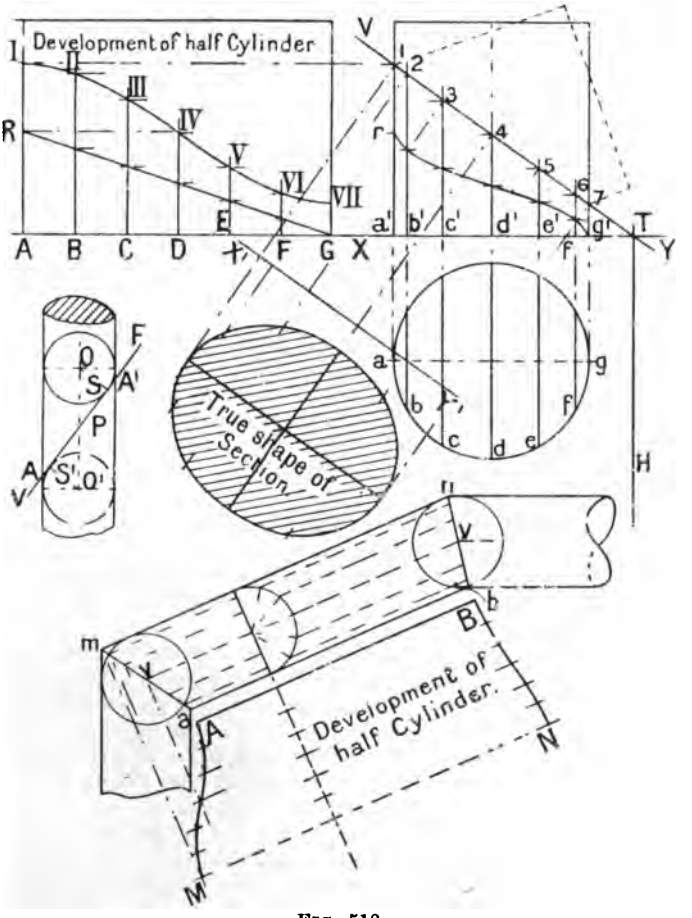


FIG. 518.

on both parts of the cylinder will coincide. The two parts arranged in this way form an "elbow-joint" in a cylindrical

pipe. Observe that VT, the trace of the joint plane on the axial plane of the pipe, bisects the angle between the axes of the two pieces of pipe.

Fig. 518 shows three pieces of round pipe assembled to form a continuous pipe, the axes all being in the same plane. This diagram also shows how to obtain the true shape of a piece of sheet metal to form the middle piece of pipe. The cylinder is unwound in a direction perpendicular to the axis VV. Only one half of the surface is developed; the line MN is an axis of symmetry.

553. **Problem of the Shortest Path.** We may wish to know the shortest distance between two given points on a surface. If the surface is developable, the distance is readily found as the straight line which joins the two points on a development of the surface.

Ex. *To determine the shortest path between two points* G *and* R *on the cylinder in Fig.* 518. Mark the position of the points GR on a development of the surface, and join them by a straight line, then RG is the shortest line that can be drawn from R to G. If folded back on to the cylinder, the line RG becomes a curve, as shown on the elevation. This curve is called a **helix.**

554. **Problem.** Fig. 519. *The regular octagon a, b, .. e .. h is plan of a domical surface; an elevation is given at a'v'e'. Horizontal sections of the dome are regular octagons. Complete the elevation, and draw a development of one of the eight curved pices.*

Solution.—As XY is parallel to *ave*, the elevation is the true shape of the section of the dome made by a VP on *ave*. As the octagon is regular, sections by vertical planes through the angles *b*, *e*, etc., are equal, and each of the eight parts of the surface are equal. The surface of a piece, as AHV, may be conceived as generated by a horizontal line sliding along the two curved edges AV, HV from bottom to top of the dome. If we take points as I, II, III in *av*, and draw parallels I1, II2, III3, parallel to *ah*, meeting *hv* in 1, 2, 3, and thence continue to complete the octagons through I, II, III, we have in plan a series of contour lines on the surface. The elevations of these lines are parallels to XY drawn through points on the curve *a'v'*, which are found by

projectors from I, II, III. Projectors from 1, 2, 3 intersect the elevations of the level lines in 1′, 2′, 3′. A fair line through h', 1′, 2′, 3′, v' is the elevation of HV. The elevations of the other edges are found in a similar manner. To develop the surface AHV, we cut the surface by a VP on $vs \perp ah$, and obtain the section shown at sv on the plan. Then rectify this curve along vs produced, and project out points I, II, III, on to their respective parallels in the development, and draw a fair line $aI_0II_0III_0V_0$. Observe that $V_0\ s$ is a line of symmetry.

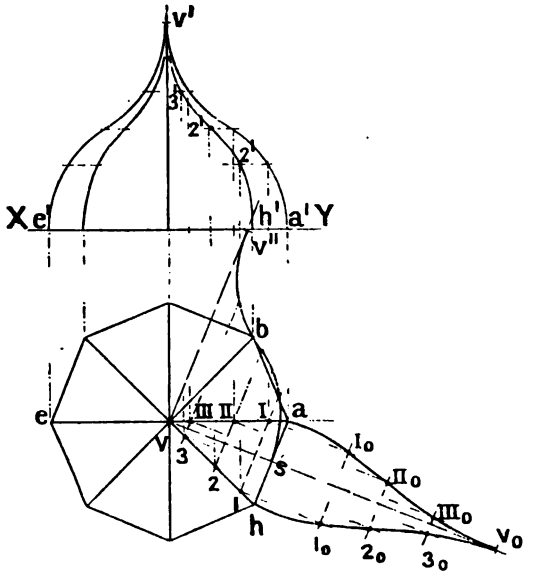


FIG. 519.

Exercise. Fig. 519. Make the vertical section of the dome on EVA a semi-circle : draw the elevation ; and a development of one of the eight curved faces of the dome. Find the number of square feet of sheet lead which would be required to cover the surface of the dome the scale of the drawing being 1 in. to 10 ft. (B.E. (3), 1904.)

555. **Sections of Prisms.** If a prism be cut by a plane perpendicular to its axis, the cut is called a **cross-section** of the prism. In the case of a *right* prism, the cross-section is the same as the base.

In *any* prism, the edges and faces are parallel to the axis, *i.e.*, perpendicular to the planes of cross-section; hence the area of a section on a plane inclined $\theta°$ to the cross-section is given by the formulæ

$$A = \frac{a}{\cos \theta}.$$

Where A = area of section, and a = area of cross-section.

Sections made by parallel planes inclined to the axis are equal in all respects.

556. **Problem.** Fig. 520. *A right prism given by plan and elevation is cut by a plane* VFH ; *determine the projections of the cut, its true shape, and a development of the frustrum below the cutting plane.*

It will shorten the labour somewhat to make use of an operation plane perpendicular to HF. Draw X'Y' ⊥r HF, thus fixing an auxiliary vertical plane. Determine OV_1, the trace of VFH on this plane, and a new elevation of the prism. The points in which the section plane cuts the vertical edges of the prism are shown at k'', m'', n'', p'', q'', on the auxiliary elevation. The plan of the section coincides with *abcde*, because the axis of the prism is vertical.

To find the elevation of the cut, make $a'k' = a''k''$, $b'm' = b''m''$, $c'n' = c''n''$, $d'p' = d''p''$, $e'q' = e''q''$, and join up the points k', q', p', n', m', k' in proper order.

The true shape of the section KQPNM may be found by rabatting the plane into HP.

When developed, the periphery of the base folds into one straight line perpendicular to the axis of the prism, and the edges in which the faces intersect.

To set out the development of the frustrum, mark off along a straight line CC_1 the lengths *cd*, *de*, *ea*, *ab*, *be* at $CDEABC_1$. Draw perpendiculars at these points, and mark thereon the length of the respective edges CN and $C_1N_1 = c'n'$, $DP = d'p'$, $EQ = e'q'$, etc., and join up by straight lines.

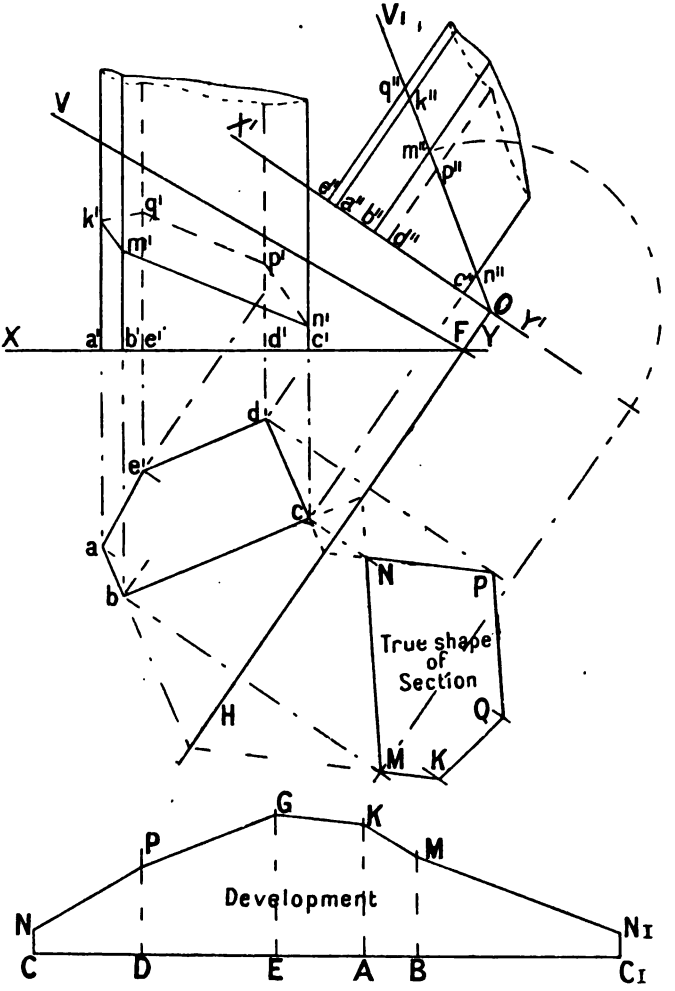

V1
V
q''
k''
m''
p''
e''
a''
b''
d''
c''
n''
O
Y1
k'
q'
m'
p'
n'
X
a'
b'
e'
d'
c'
F
Y
d
e
c
a
b
N
P
True shape
of
Section
Q
H
M
K
G
K
P
M
Development
N
NI
C
D
E
A
B
CI

FIG. 520.

557. **Sections of a Pyramid.** The sections of a pyramid made by parallel planes inclined to the axis of the pyramid are similar polygons.

For the sections are homothetic with respect to the apex of the pyramid.

558. **Problem.** Fig. 521. *An oblique pyramid on a square base, given by plan and elevation, is cut by the plane* VFH; *determine projections of the cut; its true shape, and show the trace of the cutting plane on a development of the surface.*

We make use of a vertical plane on X_IY_I perpendicular to HF, and project the plan of the cut *efghk* from this auxiliary elevation. In doing this, we find the projector from k'' is almost parallel to *vd*, the plan of the edge on which K is situated. In order to locate the plan of K with precision, we fold the edge VD into position parallel to the plane on X_IY_I, as shown in plan at vd_I, and elevation at $v''Dv$; and draw $k''Kv \parallel X_IY_I$ meeting $v''Dv$ in Kv; then project Kv on to the plan at k_I and make $vk = vk_I$.

(This expedient should be resorted to in all cases when the intersection of two lines cannot be determined accurately by the intersection of given projections.)

The elevation of the cut on XY is $e'f'g'h'k'$. This view, and the true shape of the cut *g*F*o*E*o*K*oh*, are found as in the preceding problem.

Note.—In obtaining the elevation on XY, the heights of points such as f', which cannot be determined accurately, as the point in which a projector from the plan meets the elevation of the line containing the point, are easily located by measuring their height from the elevation on X_IY_I.

To obtain a development of the Pyramid, we require to know the true shape of each triangular face, and must arrange these triangles in proper order and in the same plane. For example, the face VCB. Determine the length of VC, VB, CB, and set out the triangle VCB as shown. Similarly, determine the length of each edge, and construct the figure $VCBADC_I$.

To show the position of the lines in which the plane VFH cuts the surface, determine the true length of CG, VF, VE, VK, C_IH, and mark these lengths on the respective lines in development and join up as shown in the Figure.

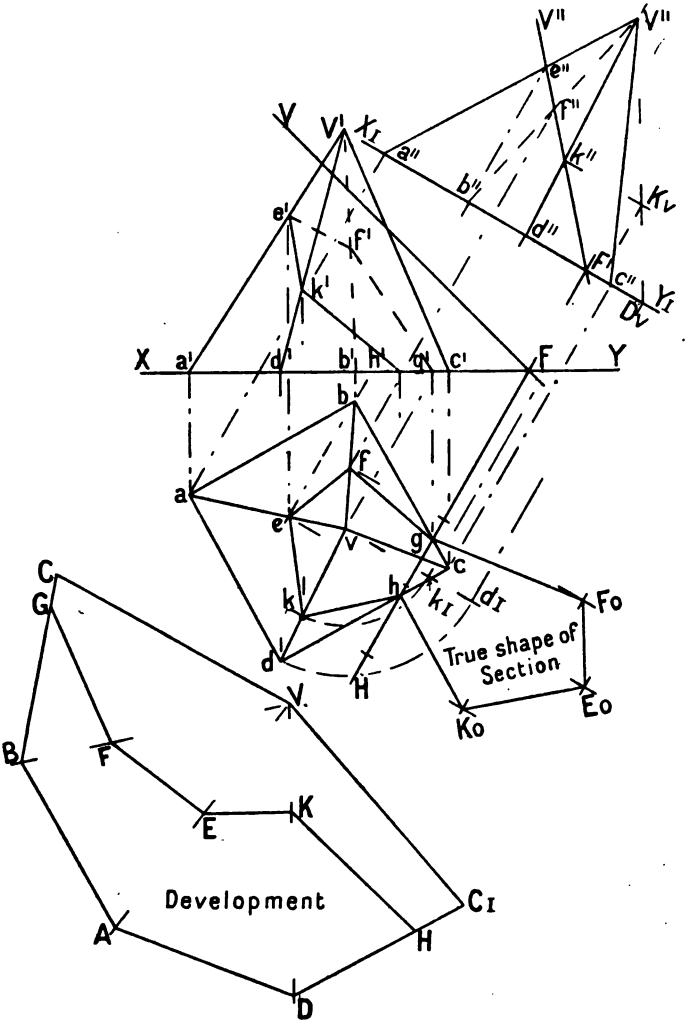

V''
V''
e''
f''
k''
K_V
d''
f'
c''
Y_I
D_V
V
V'
X_I
a''
b''
e'
f'
k'
X
a'
d'
b'
H'
g'
c'
F
Y
b
f
a
e'
g
c
C
G
k
h
k_I
d_I
Fo
d'
H
True shape of Section
V.
Ko
Eo
B
F
E
K
Development
C_I
A
H
D

FIG. 521.

559. **Application.** Fig. 522. A sheet metal tray, having a rectangular bottom and sloping sides, is shown in plan and sectional elevation. The shape of a piece of flat metal out of which this tray can be made may be found by folding the sloping sides into HP as shown.

The two short sides, or ends, have the same pitch and are both alike; the two long sides have also equal pitch, but we see from the plan that they are less steeply pitched than

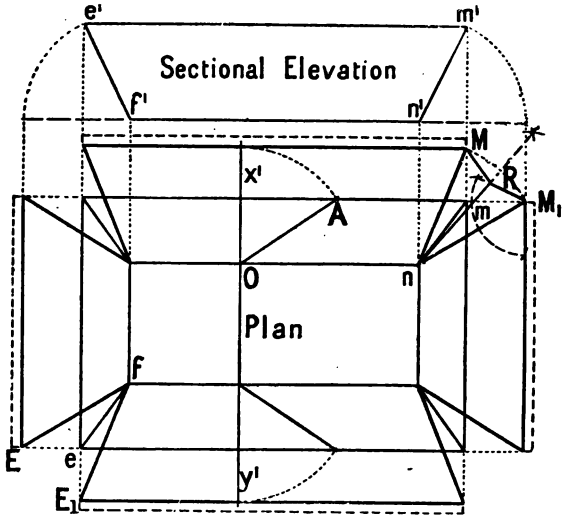


FIG. 522.

the ends. The width of the sides, measured on the slope, is found at OA by a section on $x'y'$.

Note.—In practice, allowance has to be made for laps at seams and joints, and, in an article of this type, the top edge would probably be stiffened by wiring. The dotted line in the diagram indicates an allowance of metal to wrap around such wire. A watertight joint at the corners can be made without solder, by bending (and not cutting) the metal along the developed mitre lines nM, nM$_1$, and the line nR, and folding the two triangles nRM, nRM$_1$ together and then back against either the side or end face when brought into correct position.

A *circular* **cone** is one in which the base is a circle. The line joining the vertex of the cone to the centre of the base is called **the axis.** The cone is said to be **right** or **oblique** according as the axis is, or is not, perpendicular to the base. The generating line of a cone extends indefinitely both ways, and thus the surface consists of two sheets united at the vertex.

560. **Sections of a Right Circular Cone.** The section of a cone by any plane which contains the axis is two straight lines which intersect in the apex.

A plane which contains the axis and is perpendicular to a cutting plane is called an **axial plane.**

The section by a plane perpendicular to the axis is a **circle.**

Fig. 523 (a). Let the two generating lines in the axial plane be intersected by a cutting plane AA_1, both on the same branch of the cone ; and let two spheres centre O, O_1 be inscribed in the cone, and touching the cutting plane at S, S_1. Let AL, A_1L_1 be diameters of circular sections of the cone through points A, A_1.

Then it can be shown that the section of the cone by the given plane is an *ellipse*, having major axis = AA_1 ; minor axis = $\sqrt{AL \cdot A_1L_1}$; and points S, S_1 for **foci.**

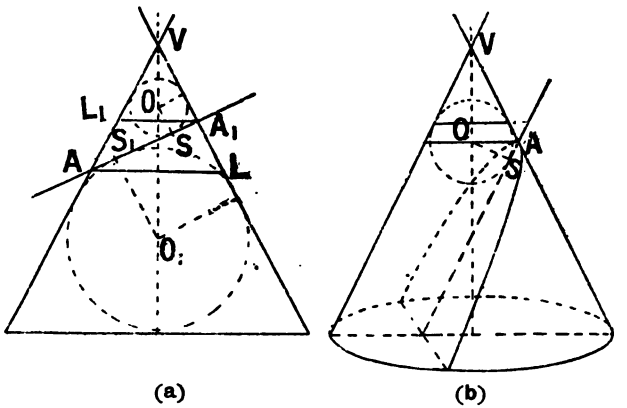


FIG. 523.

If the cutting plane intersects the two generating lines on different branches of the cone, as $a'a''$ (Fig. 524), the section is a *hyperbola,* which has major axis = $a'a''$; minor axis = $\sqrt{a'l' \cdot a''l''}$; and the points s', s'' for foci.

The asymptotes of the hyperbola are parallel to the two generating lines which lie in a plane parallel to the section plane, and passing through the apex of the cone.

Imagine a section plane which cuts a cone in a hyperbola, to move parallel to itself nearer and nearer to the apex of the cone; the vertices of the conic approach nearer and nearer to its centre, and ultimately when the plane passes through the apex of the cone the vertices and centre of the conic coincide, and the hyperbola degenerates into two generating lines of the cone, and the curve is confounded with its asymptotes.

If the cutting plane is parallel to one rectilinear element in the axial plane (Fig. 523 (b)), only one sphere can be inscribed in the cone to touch the plane, and the section is a *parabola,* having A for *vertex,* S for *focus,* and *latus rectum* = 4 AS.

Imagine a plane which cuts a cone in a parabola to move parallel to itself nearer and nearer to the apex of the cone; the focus approaches nearer and nearer to the vertex of the conic, and ultimately when the plane passes through the apex of the cone the parabola degenerates into a line generator of the cone.

Sections of a cone by parallel planes are homothetic curves.

561. **Problem.** Fig. 524. *A right circular cone standing with base on* HP *is cut by a plane* O.30. *Determine the* **elements** *of the section.*

Take XY ‖ to scale of slope, and obtain an elevation of the cone, and profile view of the plane. From this view we see that the plane cuts both branches of the cone, and consequently the section is a hyperbola. The points a', a'' are vertices of the conic. To find the foci s', s'', inscribe spheres in the cone to touch the plane, and determine their points of contact. The conjugate axis of the hyperbola bisects $a'a''$ at right angles in c' ; its length may be found as follows: Draw $a'l'$ intersecting the axis of the cone in G, make $GZ = \frac{1}{2}\ a''l''$. Find a mean proportional to $a'G$ and GZ, as shown at GB. Then GB = semi-conjugate axis. In plan the axes, projected from the elevation, are shown at aca_1, bcb ; the foci are not shown.

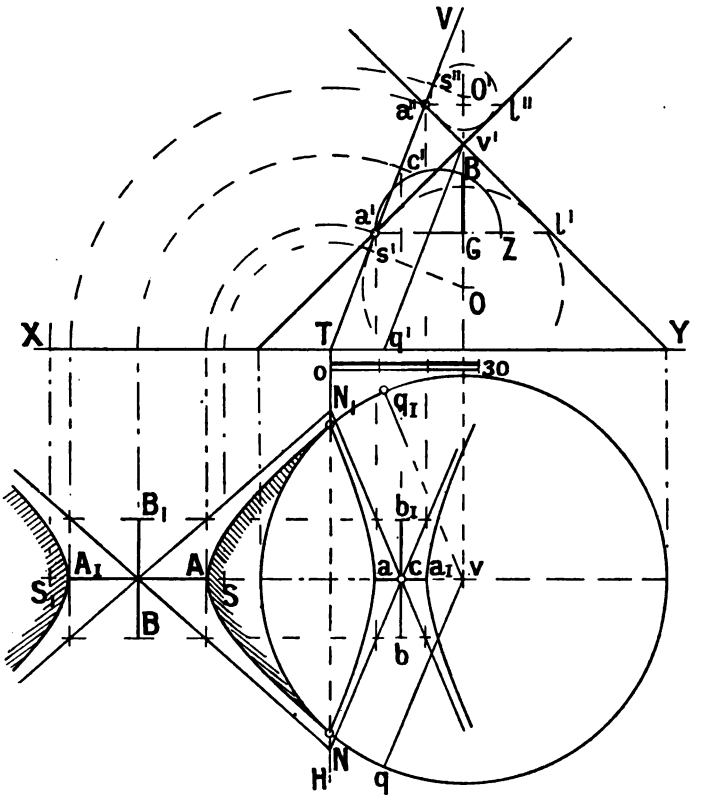


FIG. 524.

The asymptotes may be determined on the plan by drawing the diagonals of the rectangle about the axes ; or independently of the axes as follows : Draw $v'q'$ parallel to the VT of the plane, then $v'q'$ is elevation of two rectilinear elements of the cone parallel to the section plane. Draw vq, vq_1 the plans of those lines. Through c draw cN, CN_1 parallel to vq, vq_1. These parallels are asymptotes of the section. *Refer to* Art. 560. On the left of the plan the section plane is shown rabatted into HP, revealing the true shape of the *section*.

562. **Problem.** Fig. 525. *Determine the plan and the true shape of the section of the given cone made by the plane* VTH ; *and show the trace of the cut on a development of the cone.*

The first two parts of this problem can be solved exactly as in the preceding problem. The following solution is independent of properties of conic sections :—

If at a point between a' and a'', the cone be cut by a plane parallel to its base, such plane will intersect the cone in a circle, and the section plane in a straight line. These two sections, the circle, and line, have two points in common, and these points are points in the required conic section. Cutting planes in any position may be taken, but one should always be taken through b' the mid-point of $a'a''$, because this section determines the conjugate axis of the conic.

In Fig. 525, three horizontal cutting planes are taken, one of them through b'. In plan, the circular sections are circles described from o. The lines in which these planes intersect the plane VTH are perpendicular to XY and the points in which these projectors meet the circles are points in the *plan* of the required section. The **true shape of the section** may be found by rabatting the plane into HP, as shown at AB_1A_1B.

To Show the Trace of the Cut on Development of the Cone. If the conical surface be cut along the rectilinear element EO, and folded into a vertical plane along this element, it will form a sector of a circle, the radius of which is O, e', the length of *arc* being equal to the circumference of the circular base, *i.e.* $= \pi o$, e. This arc subtends at O, an angle which, measured in radians $= \frac{2\pi r}{EO}$ where r = radius of base of cone (Art. 135).

Similarly, the circles containing the points a'', c', d', f', a', when developed, are circular arcs described from O. Rectilinear elements of the cone through points K, L, M, N, P develop as radials from O, and determine by their intersection with the circular arcs points S, B, Q, A, Q_1, B_1, S_1, A_2, in the development of the cut.

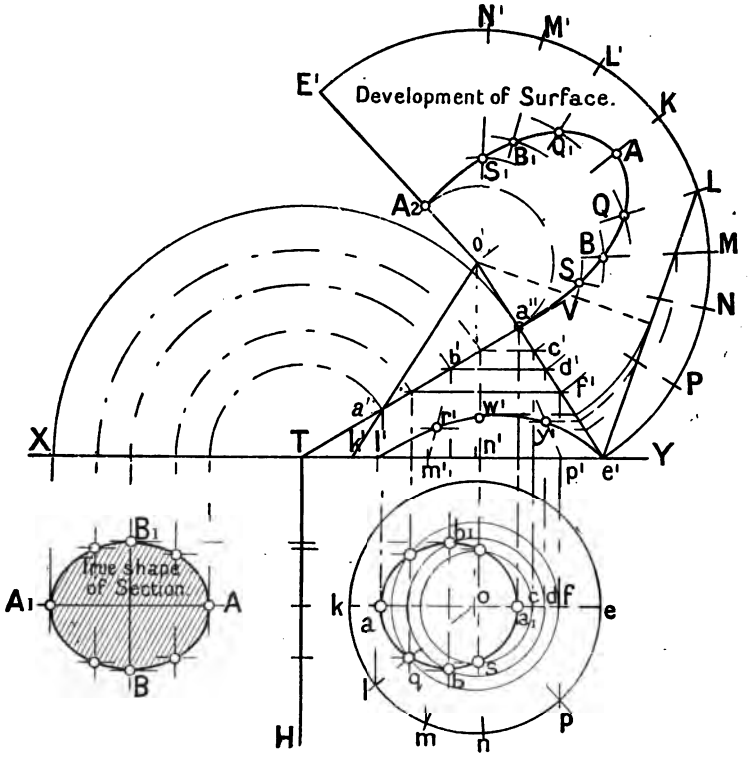


FIG. 525.

The arc $e'P$ = arc ep, arc PN = arc pn, etc. These lengths are best transferred as a series of short chords.

563. **Problem of the Shortest Path between Two Points on the Surface of a Cone.** If we wish to know the shortest path between two points on a conical surface we obtain a development of the surface and mark the points thereon. Join the points by a straight line and draw a few generators of the cone in position on the development inter-

secting the straight line. Then find the position occupied by these generators, and mark the positions of the points in which they intersect the path.

As an example, suppose we wish to trace the shortest path on the conical surface between two points EL on the base circle. Fig. 525. Make arc $e'L$ = arc el; draw $e'L$ which intersects the generators OM, ON, OP in R_1, W_1, Y (points not lettered in the diagram). From centre O mark off along Oe' the distances OR, OW, OY, and draw parallels to XY meeting the elevations of these generators in r', w', y'. A fair line through l', r', w', y', e' is an elevation of the shortest path between the two points. The plan is found by projectors from points in the elevation. It is not shown in the diagram.

An important point in the projection of the line EL is the highest point of the curve. This is readily found by a perpendicular from O on to the line in development at Z (point Z not shown). Mark off Oz from O along Oe', then a level line through z is tangent to the curve $l' \ldots e'$.

564. **A sphere** is the surface generated by a circle revolving about a fixed diameter.

A section by a plane which contains the axis is called **a meridian section.**

The portion of the surface included between two parallel planes is called a **zone.** The portion of the surface included between two meridian sections, that is, two planes containing the axis, is called a **lune.**

Section of a Sphere. All plane sections of a sphere are circles. Sections by planes equidistant from the centre are equal circles.

565. **Problem.** Fig. 526. *Determine the section of the sphere centre c, c', by the plane* VTH.

The section plane is given ⊥r to VP. If the section plane be oblique, obtain a profile view and proceed as in this example.

The plan of the section is an ellipse having major axis parallel to HT of the section plane and equal to $a'b'$, and minor axis ab obtained by projectors from $a'b'$ on to a line through e perpendicular to HT.

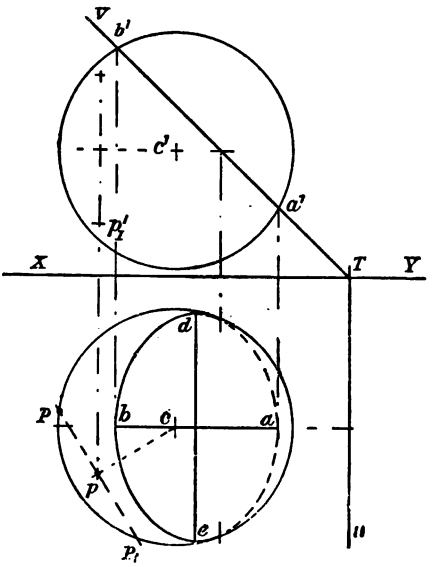


FIG. 526.

566. **Problem.** Fig. 526. *Given the projections of a sphere and one projection p of a point on the spherical surface, determine the other projection of the point.*

Analysis. If the sphere be cut by a vertical plane passing through its centre and containing the point P, the section will be a circle equal to the plan of the sphere, and the point P must lie on that circle. Imagine the circle of section turned about its centre into a horizontal position, then the point P will be found by drawing PP_I through $p \perp^r cp$. p is thus the plan of two points on the sphere, one above, and one below the centre. And $pP = pP_I$ is the distance of the point above, or below, the level of C.

Construction. Draw $pP \perp^r cp$ meeting the plan circle in P. On the projector from p make $op' = pP$ above, or below, the level of c', according as the point is to be on the upper, or lower, surface.

Exercise. Draw any square ABCD and draw the inscribed circle. Consider this circle as the plan of a hemisphere which is supported on a zone of a second hemisphere whose base or *meridian section* is parallel to the base of the first hemisphere and circumscribes the square ABCD. If vertical section planes stand on each side of the square; draw a sectional elevation on a XY, parallel to a diagonal of the square, showing the traces of the cutting planes.

This problem occurs where a spherical dome is placed over a square chamber. All that is left of the large sphere is the small spandril pieces which are called *pendentives.*

Fig. 527 shows a part of such a construction.

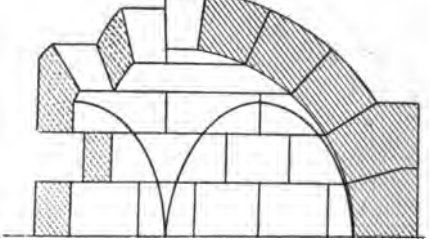


FIG. 527.

567. **Approximate Developments of Double-curved Surfaces of Revolution.** An approximate development of a double-curved surface of revolution may be obtained by either of two methods, which will be illustrated by their application to a sphere.

First Method. Fig. 528 (a). Let a quadrant of a sphere be given in plan and elevation. Divide the arc $a'n'$ into a number of equal parts—3 in the diagram. Draw the chord B_1D_1 and produce to meet the vertical axis in T. Then the cone that would be generated by the revolution of TB_1 about the vertical axis Tc' will intersect the sphere in parallel circles through B_1, D_1, and provided B_1, D_1 are not far apart, the surface of the frustrum of the cone lying between the parallels will not differ much from the surface of the spherical zone. A development of the frustrum is shown at B_1BDD_1.

Second Method. Fig. 528 (b). Divide the arc $a'n'$ as in the preceding. Draw the plans of the parallels through B_1, D_1. Divide the surface into a number of lunes, one of

which is shown in plan at *cef*. Make $c'N$ = arc $a'n'$ and divide $c'N$ into the same number of equal parts as $a'n'$. Draw verticals from 1, 2, 3, 4, meeting horizontals through the points of division on $c'N$. And draw a fair curve through these points, which is a development of the lune.

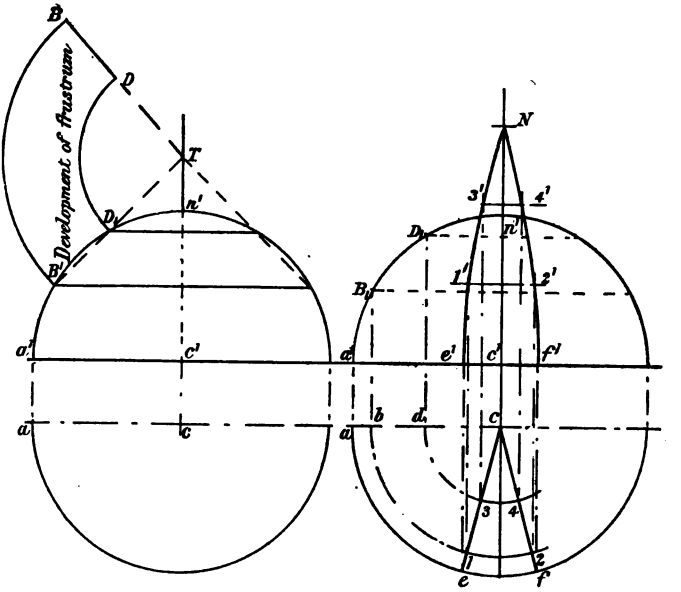


FIG. 528.

568. **Problem.** Fig. 529. *Inscribe a sphere in a given right cone to pass through a point p, p' on the conical surface.*

Analysis. The point P must lie on the circle of contact of sphere and cone; this circle will show as a straight line perpendicular to the axis on a plane parallel to the axis.

Construction. Obtain an elevation of the cone and point on X'Y' parallel to the plan of the axis. Through p'' draw $q'r'$ perpendicular to $a''n''$, the elevation of the axis. From q' draw $q'c' \perp$ to $a''d''$, meeting $a''n''$ in c''. Then c'' is the centre, and $c''q'$ is radius of the required sphere. The plan

of the centre is at *c*. Its elevation on XY is on the axis *a'n'*, and may be projected from the plan *c*. As a check remember *c'* must be the same height above XY as *c''* is above X'Y'.

569. **Problem.** Fig. 530. *Given the projections of a cone and one projection of a point on its surface, determine the other projection of the point.*

Let the cone be given by the projection of its apex and an inscribed sphere, centre C, and let the plan *p* of the point be given.

Analysis. All generating lines of the cone touch all inscribed spheres, hence the generator AP touches the sphere centre C. If the cone and sphere be cut by a vertical plane on *ap* the line AP will be tangent to the circular section of the sphere; and an elevation on a plane parallel to the section plane will disclose the height of the point of contact.

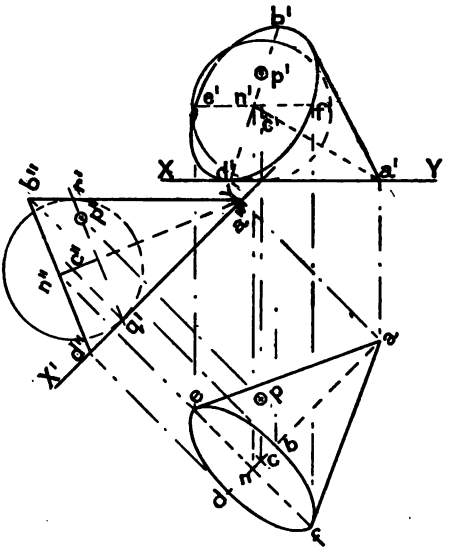


FIG. 529.

Construction. Draw ap intersecting the circle centre c in s, s_1. Take X'Y' $\parallel$ ap and get elevations of A and C on this plane at a'', c''. With centre c'' and radius os, or os_1, describe a circle. This circle is the elevation of the section which AP touches. Draw from a'' lines tangent to this circle. There are two such lines, $a''q'$, $a''d'$, one above, and one below, the axis. Draw projector from p intersecting $a''q'$ in p'', and $a''d'$ in p^2. Then either point may be taken as the elevation of P, since both are on the cone. Their elevations on XY are at p', p^2.

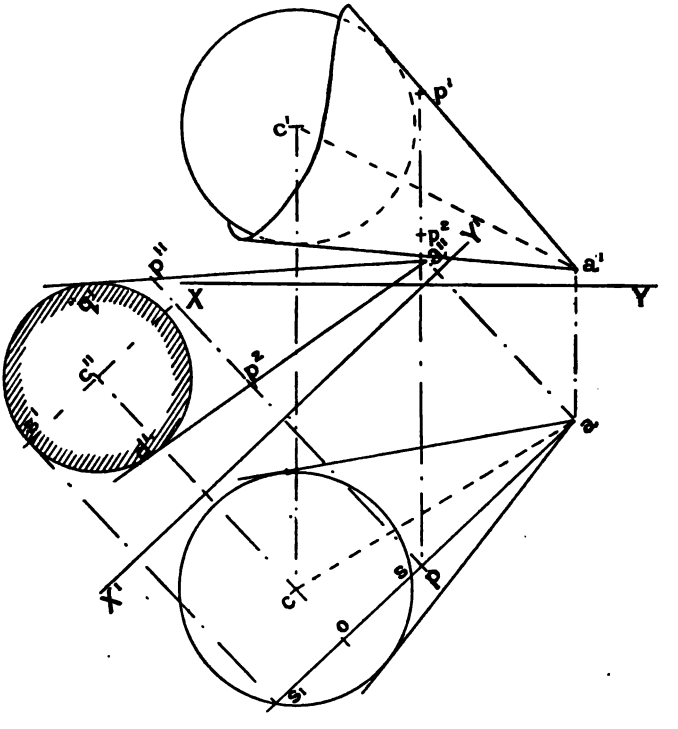


FIG. 530.

570. **Plane Sections of Mouldings.** A straight length of moulding of uniform cross-section may be conceived as generated in either of two ways :—

(1) The surface traced out by a plane figure moving parallel to itself with two fixed points in the figure always in fixed parallel straight lines.

(2) The surface traced by a straight line moving parallel to itself and in contact with a director curve.

A plane section perpendicular to rectilinear elements is called a **cross-section**, or a **profile.**

Sections by parallel planes are congruent figures.

This is manifest from conception (1) above.

571. **Problem.** Fig. 531. *A moulding running horizontally is shown in plan and cross-sectional elevation; determine the shape of the section made by a vertical section plane* SS'.

Fold the section plane SS' into position parallel to XY about a vertical axis BA. The points d', e', etc., move perpendicular to $b'a'$ into positions D, E, etc.

If the moulding be cut by a second vertical plane TT', which is inclined to rectilinear elements at the same angle θ as the plane SS', but in a contrary direction, the sections made by SS' and TT' will be congruent, and, if the wedge-shaped portion of the moulding between the two planes be removed, the end pieces can be placed "*cut to cut*," as shown by dotted lines, and will fit exactly in a **mitre joint.**

From this we conclude that a straight moulding running horizontally on two adjacent walls of a room will mitre correctly in a vertical plane which bisects the angle between the walls.

Suppose two lengths of horizontal moulding M, N, mitred together in a square corner, as shown in plan and cross-sectional elevation at Fig. 531. Then *ab*, *cd*, the two top outer edges of the two pieces are level and meet at *bc*. (This is true also for any pair of corresponding lines on the two surfaces.) Suppose the length N rigidly fixed to the wall W, and let the end A of the piece M be raised into position shown in elevation at a'', whilst the corner O remains fixed. Then the point B moves upwards to b'', and the edge AB no longer meets DC in the plane of the mitre joint. Nor will they intersect in any vertical plane through O. *Thus, if two lengths of moulding of the same cross-section be placed one horizontally and the other in an inclined position on surfaces which form an angle, the two mouldings will not mitre in a vertical plane.*

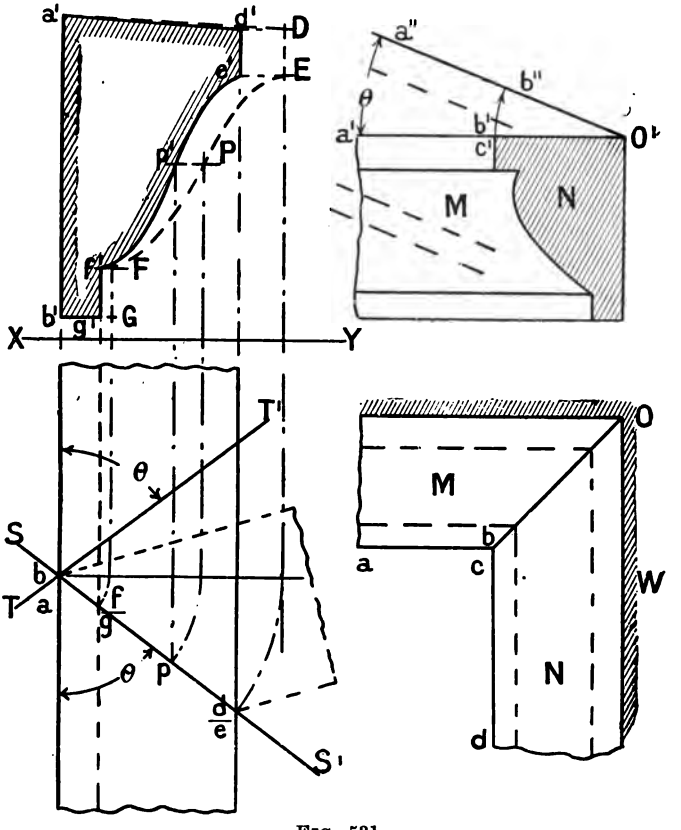


FIG. 531.

In order to make a moulding, corresponding with M and inclined upwards or downwards, mitre with the level moulding N, the inclined, or raking piece, must have a particular cross-section which will depend on the angle of inclination θ.

Similarly, if the cross-section of the raking piece be given, the cross-section of the horizontal piece will be wholly dependent thereon, and must be determined from the given data.

572. **Problem.** Fig. 532. *Two pieces of moulding* N, M, *the former horizontal, the latter inclined at* 32°, *are shown in plan at* (a). *A cross-section of* N *is shown at n ; determine the cross-section of* M *so that the two pieces shall mitre in a vertical plane. Show also a section of* M *by a vertical plane on* HT.

Take XY parallel to the wall *W*, and make an elevation of the level moulding and its section on the mitre plane. This is shown at *a'*, *b'*, *c'*, etc., heights of points being measured from the given cross-section *n*. Through critical points in the elevation of the section draw lines inclined at 32° to XY (the given inclination of the piece M).

To Determine the Cross-section of M. Take a section plane SS' ⊥r to rectilinear elements in elevation cutting same in O',Ō', etc. Projectors from these points on to the plan determine the plan of the section. The true shape of the section is found by folding about SS' into the plane of elevation at A, B, R, etc.

The section by the plane on HT is also found by folding into the plane of elevation.

573. **Example of Curved Moulding.** Fig. 533 shows part plan and elevation of a curved pediment on a straight wall face F. At the base of the pediment the upper part of the moulding has to mitre with a moulding which runs horizontally on the return wall face E. A cross-section of the curved moulding is shown at A ; this section is obtained by cutting the moulding by a plane perpendicular to the wall face F and normal to the direction of the moulding.

The mitre plane is a vertical plane bisecting the angle between the two wall faces ; its horizontal trace is OH.

To Determine the Intersection. Draw the plan of the curved moulding on the face F and show thereon all salient lines. Projectors from the points in which these lines meet OH will intersect the elevations of the respective lines in points on the required intersection.

Observe that the elevation of the mitre is also a cross-section of the horizontal moulding on the return ; because this view is an *end view* of that moulding.

The converse problem, viz., to determine the cross-section of the curvilinear moulding, when the straight moulding is given, will present no difficulty. *Remember, the cross-section is a normal section.*

FIG. 532.

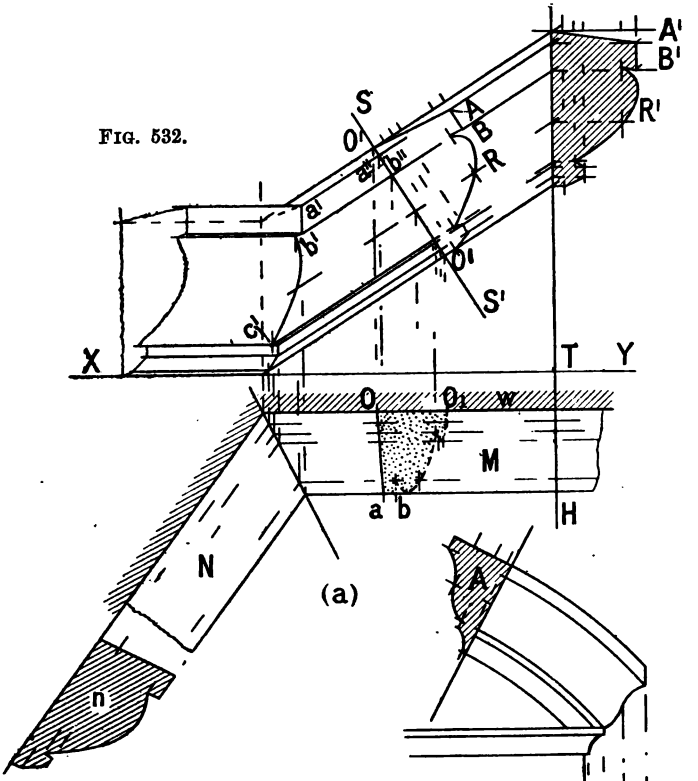
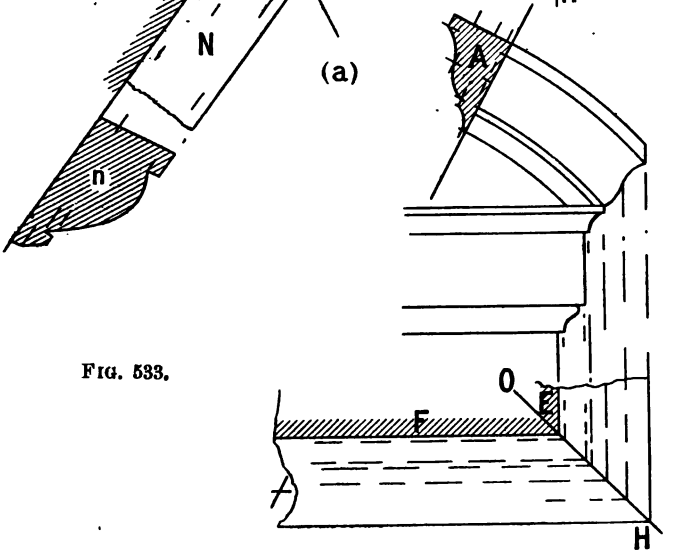


FIG. 533.



Exercises.

(1) A conical pail is 10 ins. diameter at the top, 6 ins. at the bottom, and its depth is 8 ins. Draw its plan and elevation, scale $\frac{1}{2}$.

Draw a development of the piece of sheet metal forming its curved surface, omitting all allowances for seams, etc. Give the area of the piece in square inches. (B.E. (2), 1905.)

(2) Fig. (*e*). In the given Figure A is the base, and B, C are the developments of two of the sloping faces of the frustrum of a triangular pyramid. Draw the development of the other sloping face. Show the plan of the frustrum before development. From this plan project a sectional elevation on xy, taken parallel to the vertical section plane ss. (B.E., 1909.)

(3) Fig. (*c*). The Figure is the elevation of a solid composed of two truncated cones. Draw its plan, and show upon the given elevation the section made by a vertical plane passing through the axis AB of the solid, and inclined at 45° to the vertical plane of projection. (B.E., 1895.)

(4) Fig. (*d*). Bell buoy. Draw a sectional elevation of the buoy on $x'y'$, taken parallel to the vertical section plane ss, the portion A of the buoy in front of ss being supposed removed. (B.E. (2), 1908.)

(5) Fig. (*a*). The Figure shows an arch ring of seven stones, built to pass obliquely through a wall. (The wall is omitted.) Project an elevation on $x'y'$. Develop the soffit or inner curved surface of the arch, showing the coursing lines. Draw a development of the second ring stone, enlarged to double size, thus exhibiting the shapes of the templates for its seven faces. (B.E. (3), 1910.)

(6) Fig. (*b*). AVB is the elevation of a right cone. A circle, whose centre is O, is drawn on the elevation, tangent to VB, and two tangents to the circle, CD, DE. Develop the cone, opening it along the line VB, and determine the developments of the circle and lines. (B.E. (2), 1895.)

Hint. First find the plan of the lines CD, CE and the circle centre O. This may be done by taking a number of horizontal cutting planes as in Problem 562. Remember the curves CD, CE when developed will touch the development of the circle.

(7) Fig. (*f*). You are given the plan and elevation of the frustrum of a square pyramid. Set out the true shape of one of the sloping faces of the solid.

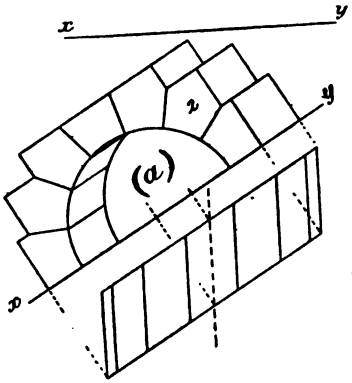
Draw the plan of the mid-section ss, and find its area a, and the areas a_1 and a_2 of the ends of the solid. Measure the height and calculate the volume of the frustrum by using the prismoidal formula,

$$\text{Volume} = \text{height} \times \frac{a_1 + 4a + a_2}{6}.$$

(B.E., 1908.)

(8) Fig. (*g*). You are given the plan and elevation of the frustrum of a cone. Set out the development of a symmetrical half of the curved surface.

Draw the plan of the mid-section ss, and find its area a, and the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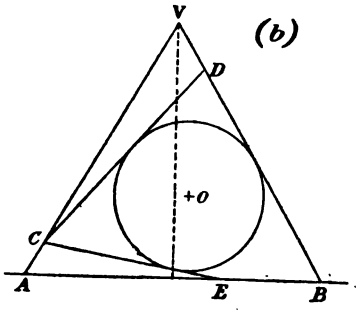
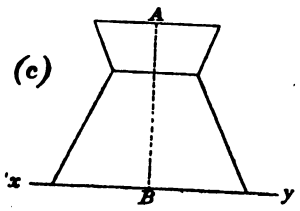
y
z
(a)
p


V
(b)
D
O
C
A
E
B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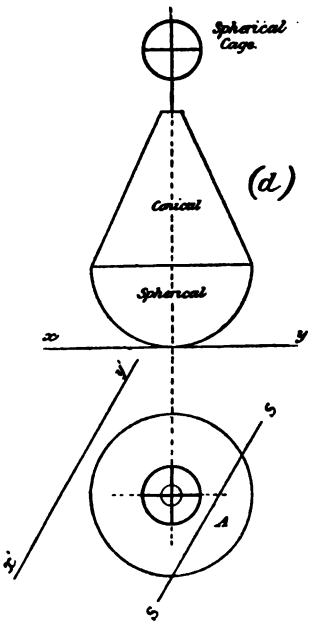
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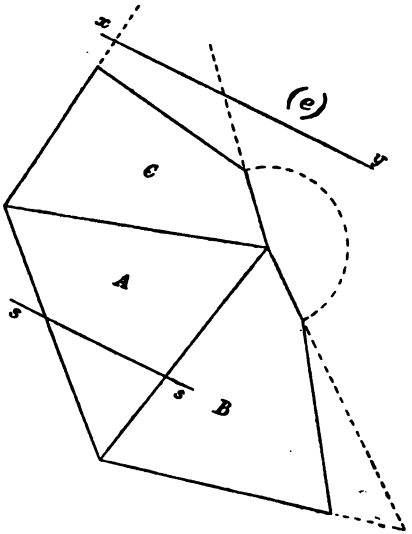
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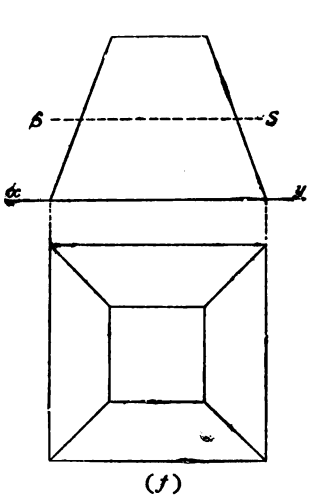
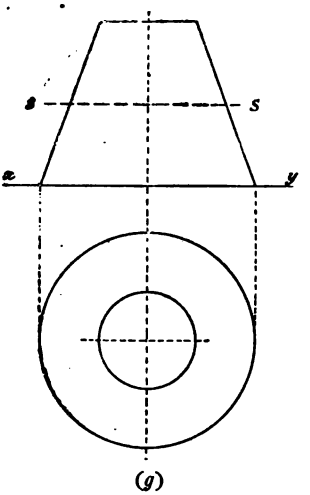
areas a_1 and a_2 of the ends of the solid. Measure the height and calculate the volume of the frustrum by using the prismoidal formula, as above. (B.E. (2), 1908).

(9) Fig. (*h*). Two projections of a prismatoid are given. Show the development, in a single piece, of one of the symmetrical halves of the sloping surface.

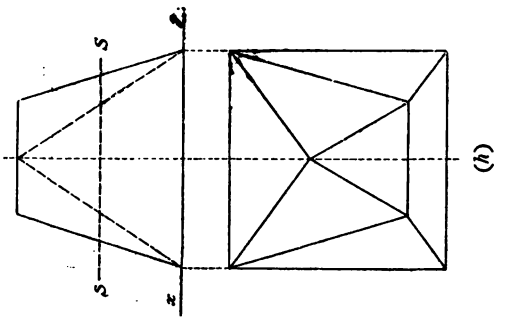
Draw the plan of the mid-section *ss*, determine its area *a*, and the areas of a_1 and a_2 of the ends of the solid. Measure the height and calculate the volume of the solid by using the prismoidal formula, as above. (B.E. (3), 1908.)



(*f*)



(*g*)



(*h*)

(10) Draw a quadrilateral *abcd* making ab = 3 ins.; bc = 2 ins.; cd = 2·25 ins.; ad = 2·5 ins.; angle abc = 105°. Index the corners as follows: a_0, b_8, c_2, d_{12}. Then

(a) Draw the plan of a sphere of 3 ins. radius having the points BCD on its surface. How many solutions are there? *Answer*, 2.

Hint. The centre of any sphere which has three points on its surface lies on the perpendicular drawn from the plane of the three points and through the centre of the circle which circumscribes the triangle BCD.

(b) In Exercise 10 (a) denote the centres of the two spheres by O and O′. By joining these points to the corners CDB a double pyramid is formed. Show a development of this double pyramid and determine its volume.

(c) One end A of a line AF 2½ ins. long has one end in a_0; the line moves so that the end F is always on the surface of one or other of the spheres of 10 (a). Draw plan of the locus of F.

(d) Determine the sphere which contains the four points a_0, b_8, c_2, d_{12} on its surface. (Refer to the hint given on Exercise 10 (a).)

(11) A right circular cone, axis 3 ins.; diameter of base 1¾ ins., rests on HP, as in Fig. 529. Take a point O in HP and 3 ins. distant from the plan of the apex A and on a line the plan of which makes 55° with the plan of the axis. Show a line OF resting on the cone and (1) inclined to HP at 40°; (2) making 75° with AO; (3) having the end F 2 ins. above HP; (4) having the end F 2¼ ins. from A.

(12) A sphere 2 ins. diameter rests on HP, and a point P is 3 ins. above HP, and has its plan 2¼ ins. from the plan of the sphere's centre. Draw a line PQ penetrating the surface in Q, R, such that PQ · PR = constant. Draw the locus of Q, and of R. What is the value of the constant PQ · PR? *Answer.* The square on the tangent from P to the sphere.

CHAPTER XXXII.

INTERPENETRATION OF SOLIDS.

574. If two solids are so placed that one falls in part within the other, the intersection of the two surfaces depends on the nature of the surfaces and also on their relative position.

If both surfaces consist wholly of plane faces, the line of interpenetration will be a rectilinear figure, which may be a plane figure ; but in general it is not a plane figure. Fig. 534, which is a sketch of two hollow interpenetrating pyramids, may help the student to understand the nature of such interpenetration.

If a polyhedron penetrates a curved surface, each plane face will cut the surface in a plane curve (unless the surface is *ruled* and the trace of the plane face on the surface coincides with a rectilinear element), and the complete interpenetration is a figure comprised of arcs of plane curves.

The line of interpenetration of two curved surfaces is, in general, a curve, which may be plane but is usually a tortuous curve.

Fig. 535 shows a cylinder penetrating a sphere.

Fig. 536 shows a cone penetrating a second cone.

Fig. 537 shows a cylinder penetrated by three cylinders.

These diagrams will help the student to get a clear idea of the nature of lines of interpenetration. They also serve to show that when one solid completely penetrates the other, the interpenetration has two branches—one corresponding to the entry and one to the exit of the penetrating body. When the two surfaces touch at a point (see B, Fig. 537) the two branches unite and form a *node* at the point of contact of the surfaces. When only a portion of the penetrating surface passes through the other surface, the line

of interpenetration is one closed figure, which will be curved or rectilinear according as the surfaces are curved or plane surfaces.

575. Points in the interpenetration of two solids, given by their projections, may be located by cutting the surfaces with a series of section planes or other convenient surfaces. This is called a *method of sections*. The cutting surfaces may be taken in any position so long as both surfaces are intersected; but in all cases there are some critical points in the interpenetration, which, if accurately located, are of great assistance in drawing the complete intersection. These important points ought always to be determined, and then such intermediate points as will suffice to enable the complete interpenetration to be drawn correctly. The example problems which follow will serve to explain the application of the method.

FIG. 535.

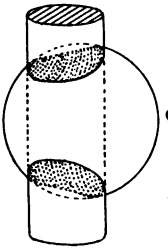
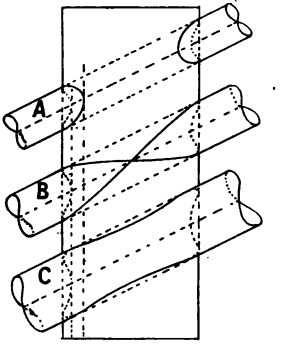


FIG. 537.



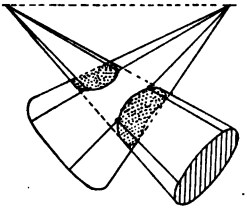


FIG. 536.

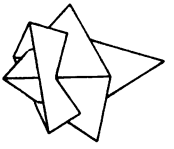


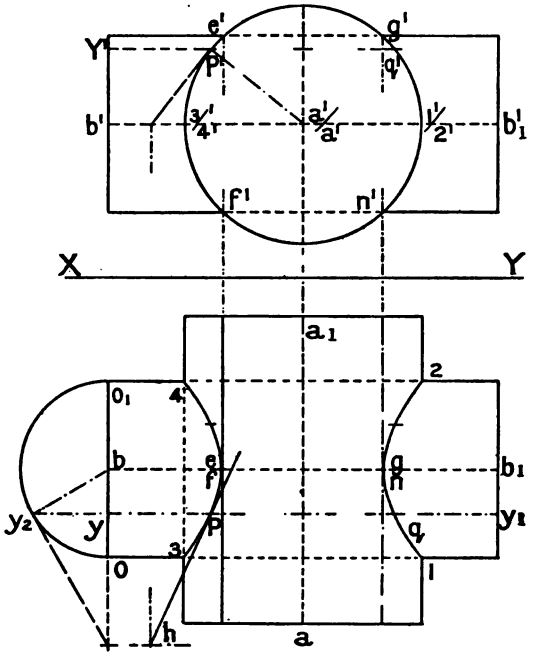
FIG. 534.

576. **Ex. 1.** Fig. 538 *shows the interpenetration of two cylinders of revolution when their axes* AA, BB *intersect at right angles.*

Some important points in the interpenetration are found on the plan by inspection at 1, 2, 3, 4. These points are all in the common axial plane of the surfaces, and their elevations at $^{1'}/_{2'}$, $^{3'}/_{4'}$, on the elevation $b'b'_1$. Other important points are the highest and lowest points in the intersection; these are shown in elevation at $e'g'$, $f'n'$. They are all in the vertical axial plane BB, and their plans are at e/f, g/n. If we imagine a horizontal plane tangent to the cylinder (axis BB) along its highest generating line, this plane will cut a in two lines which are tangents to the lines of interpenetration at the points EG. Similarly a plane tangent to the surface (axis BB) along the lowest generator cuts the other cylinder in two lines which are tangent to the interpenetration at the points FN. (In plan the two lines in the lower plane coincide with those in the upper plane.) These tangents show the direction of the curve of interpenetration at *critical* points. Intermediate points may be found by taking vertical section planes parallel to the axis BB_1. Each such plane will intersect the surface (axis BB) in two generators, and the other surface in a circle. The intersection of these straight lines and circular sections give four points in the interpenetration. As illustration, take a vertical section plane on yy_1. To find the level of the lines in which this plane intersects surface BB, describe a semicircle on oo_1, as diameter. This may be conceived as a cross-section of one-half the cylinder, and yy_2 is a measure of the distance of the lines of intersection, one above and one below the axis BB_1. Make $b'y' = yy_2$ and draw $y'q'$ parallel to $b'b'_1$, meeting the circle centre a' in p', q', which are elevations of points in the interpenetration. The plans of the points are on yy_1. There are two corresponding points on the underside of the solids. These are *not* shown in the diagrams.

By taking other planes parallel to yy_1 any number of points in the interpenetration can be determined, and the fair line through these points is the required interpenetration.

FIG. 538.



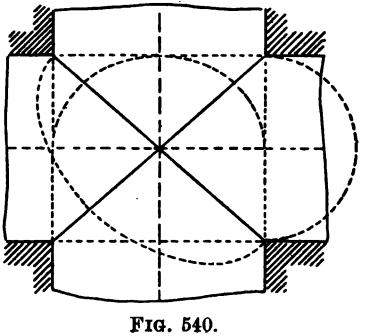


FIG. 540.

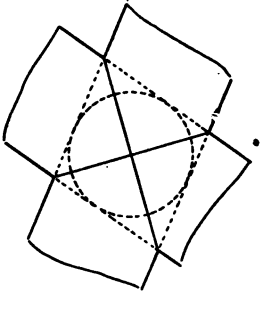


FIG. 539.

577. If two cylinders (Fig. 539) whose axes intersect have equal diameters they will intersect in two ellipses. For they will circumscribe a common sphere, Art. 580.

Two cylinders whose axes intersect will intersect in ellipses provided the diameter conjugate to the common axial plane is equal in both cylinders. Fig. 540 shows a circular cylinder and an elliptical cylinder which intersect in ellipses. The shape of the semi-elliptic intersection is shown by dotted lines.

This arrangement occurs when arched ceilings over two halls of different widths are required to be level at the springing and crown.

578. **Ex. 2.** Fig. 541 *shows a vertical cylinder pierced by a smaller cylinder axis ob inclined.*

The line of interpenetration is found by taking section planes parallel to VP. The lines in which these planes intersect the small cylinder are accurately located by folding the cross section of this surface into a horizontal position, *e.g.*, $b'e' = n\text{E}$. The chain dotted line on the left of the Fig. shows a development of the hole pierced in the vertical cylinder. At PQRS is shown a development of the front half of the small cylinder jutting out from the larger.

These developments are found by rectifying the cross sections of the cylinders and setting off lengths on the different generators as in Prob. 562.

579. **Ex. 3.** Fig. 542 *shows a cylinder penetrating a cone of revolution, both axes vertical.*

NOTE.—Only one-half the cone is shown in plan. In this example horizontal section planes, which cut both surfaces in circles, are convenient.

The trace of the cylinder on the developed conical surface is shown. This is readily obtained by developing that part of the cone lying between the generators ON, OM, which touch the cylinder. Observe that ON, OM are also tangents to the development of the intersection and the generator OP is a line of symmetry.

On the left a development of one-half the upper part of the cylinder is shown.

580. **THEOREM.** *If two cones, or two cylinders, or a cone and cylinder, circumscribe the same sphere, the two surfaces intersect in two conics, the planes of which are perpendicular to the common axial plane of the two surfaces.*

FIG. 541.

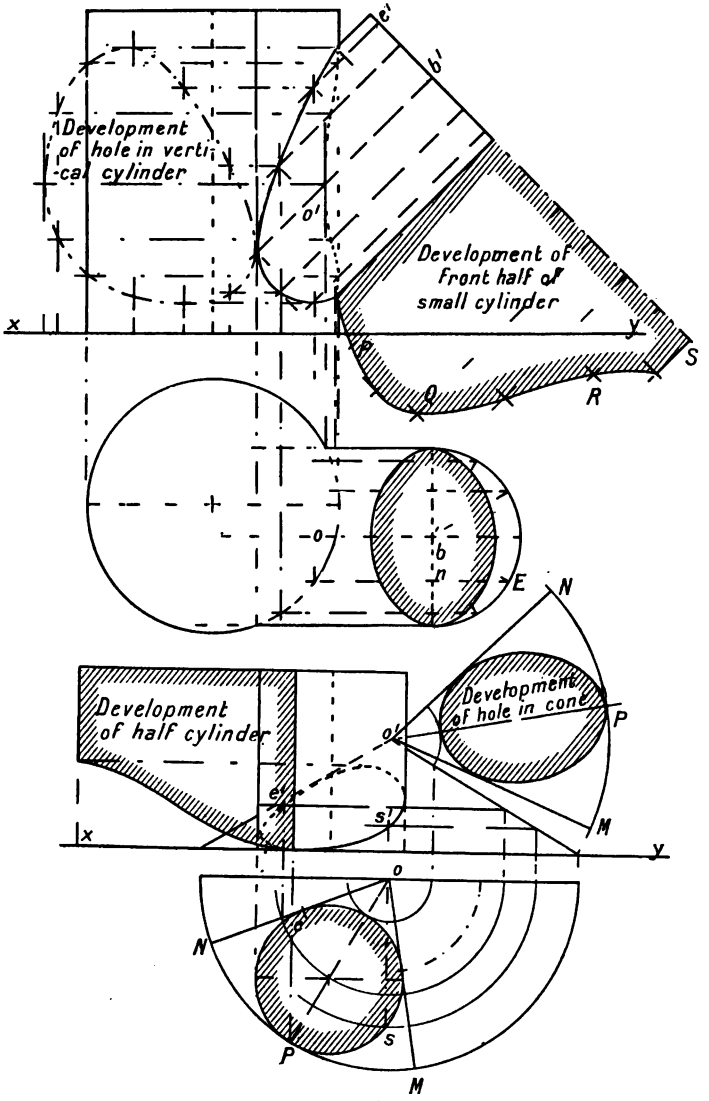


FIG. 542.

The three cases are illustrated in Fig. 543 *a*, *b*, *c*.

If we conceive each cylinder in *b* and *c* as a cone whose vertex lies at infinity the generators of the surface will be conceived as meeting at infinity, *i.e.*, will be parallel to the axis of the surface, and the same demonstration can be applied to all three cases.

Let the axes of the circumscribing surfaces lie in the plane of the paper. Let the circles in which the circumscribing surfaces touch the sphere be denoted by S and Z respectively. The common chord of S and Z is PQ. The trace of each surface on the common axial plane is two straight lines intersecting in the respective vertices, and shown in the diagrams at AD, AT and VD, VR, together with their points of intersection R, T, B, D.

Join RT and take a plane ϕ to contain RT and be perpendicular to the axial plane. Then, the trace of each cone on the plane ϕ is an ellipse, and the longest diameter in each section is the line RT in which the axial plane intersects the cutting plane, and therefore RT is the major axis of both sections. Now, P (an end of the common chord of the circles of contact S and Z) is obviously on the surface of the sphere and both cones. If P A, P V, P C be joined, we have lines PA, PV, one on each cone and both touching the sphere at P, and the plane containing the two lines AP, VP will be tangent to both cones and the sphere, and its trace (HH) on the cutting plane ϕ will be tangent to the traces of the two cones.

Thus, the plane ϕ intersects both cones in ellipses which have a common major axis RT, and a common tangent HH ; and since only one conic can be drawn on a given axis to touch a given line, the traces of the two cones on the cutting plane ϕ must coincide, which proves the proposition.

In a similar manner, it may be shown that the two circumscribing surfaces have a common elliptical trace on the plane containing BD and perpendicular to the common axial plane ; the major axis of the section being BD.

We may remark that the axial plane of a cylinder intersects the surface in a pair of lines parallel to the axis. The two circles of contact on the sphere in Fig. *c* are both meridian sections perpendicular to the axial plane, and therefore their common chord is a diameter of the sphere perpendicular to that plane.

SPECIAL CASES. If the vertices V, A of the two cones be equidistant from the centre of the sphere, the apex angles will be equal.

(1) If the cone VC be rotated so as to bring the element VD ∥ to AB, these two elements being parallel cannot meet on either branch of the cones, hence the point T is at infinity and the intersection RT becomes a parabola.

(2) If the cone VC be rotated beyond the position in which VD is parallel to AB, the point T in which these elements intersect will be on the side of R remote from the sphere and the intersection RT will be a hyperbola the centre of which is determined in the point where a line AV meets the intersection plane RT.

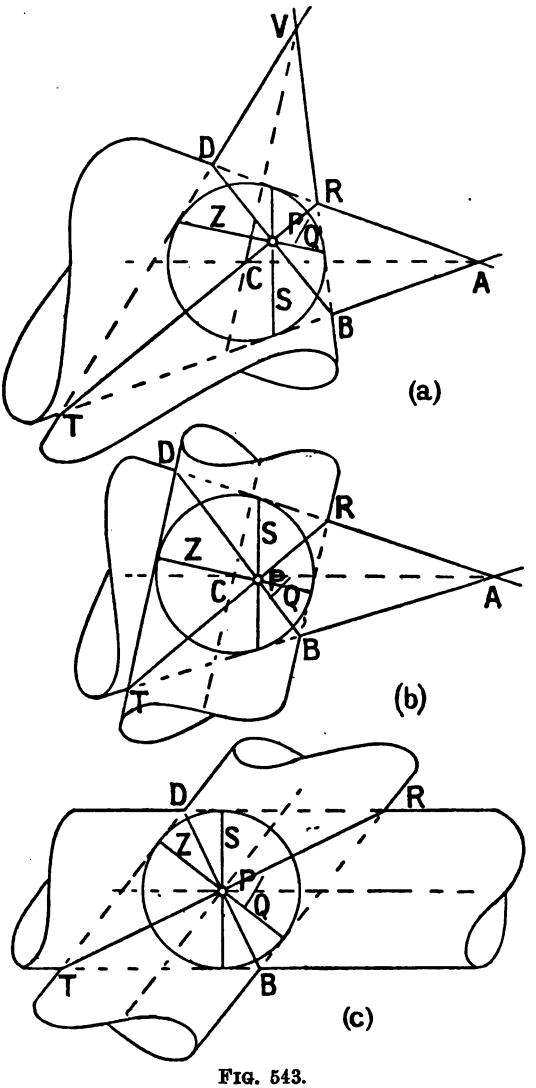


FIG. 543.

581. **Ex. 4.** *Fig.* 544 *shows two cones vertices* A, V. *The axis of* A *is vertical whilst that of* V *is obliquely inclined. The trace of this cone on* HP *is the ellipse centre* O.

If we join the vertices A, V by a straight line and prolong this line to meet HP in *h*, and draw from *h* any line *h*Z in

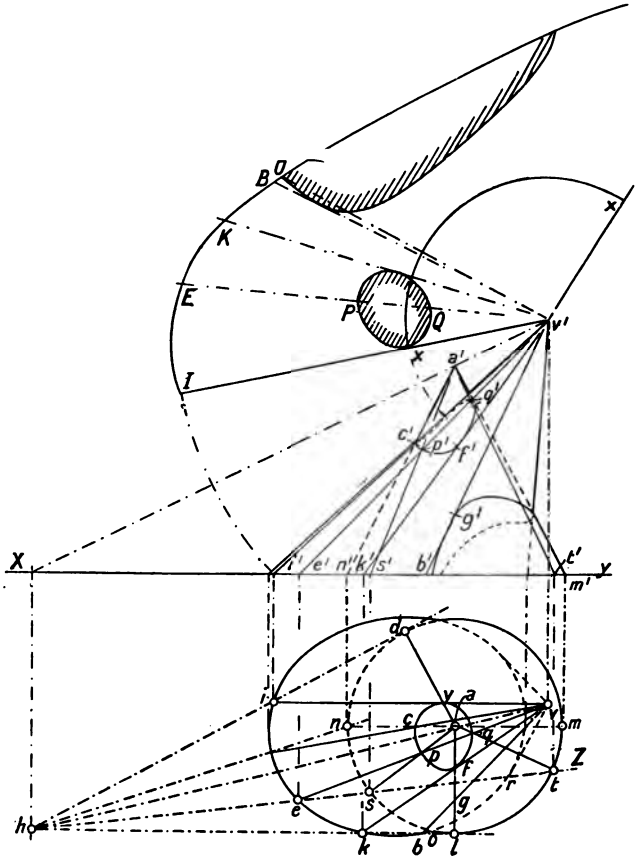


FIG. 544.

HP to cut the circular base of cone A and the elliptical trace of V, the plane of V*h*Z will cut each cone in two rectilinear elements and the points in which these straight lines intersect will be points in the interpenetration of the surfaces.

Hence, draw $v'a'$, va and determine its HT in h; draw hZ intersecting the circle centre a in s, t, and the elliptical trace in r, e. Draw the elevations $a's'$, $v'e'$. These meet in p', which is the elevation of a point in the interpenetration. The plan of this point is on as. Similarly, draw the elevation $a't'$, meeting $v'e'$ in q', which is the elevation of another point; its plan is q on ve.

By taking other planes containing Vh and cutting the bases, any number of points in the interpenetration may be found.

There are certain critical points which should be determined. Thus, $a'n'$, the extreme visible element of cone A, is tangent to the elevation of the interpenetration; the point of contact of this element is found by a section plane containing the element $a'n'$, an. Similarly the other extreme element $a'm'$, am is also tangent to the elevation of the curve; and its point of contact is similarly found.

The elements $v'k'$, $v'b'$, in which the plane that touches the cone A cuts the cone V, are also tangents to the elevation of the curve at f, f'; g, g'.

Another tangent vi, $v'i'$ is determined by the plane whose HT is hd tangent to the base of A.

Critical points in the plan are those in which the plan of the curve is tangent to the extreme visible elements of the plan of the inclined cone.

These are not shown in the diagram, nor is the plan of the lower branch of the interpenetration, which is omitted to avoid overcrowding.

A development* of the inclined cone and the interpenetration is shown in the upper part of the diagram, the surface being opened along VI.

To set out the development, we know (560) that the section of the cone by any plane ⊥r to the axis is a circle, and this circle will develop into an arc of a circle centre v' and of radius equal to the slant height of the cone,

* The development is incomplete as it extends beyond the limits of the page.

measured from the section plane. (The radius corresponding to any section plane is readily obtained from an elevation of the inclined cone on a plane parallel to its axis.) The arc xx is a development of one such section. Having a circular section of the surface developed, generators of the surface may be placed thereon as explained in Art. 562. As the plane of the base is not $\perp^r$ the axis of the cone, generators of the surface vary in length according to the position occupied. In development, each line has its true length. The interpenetration line is set out by taking different generating lines on the surface which intersect the interpenetration and marking thereon the point (or points) of intersection. Thus, for example, on development elements VI, VE, VK, VB, VO are shown in true length and correct position. Points in the interpenetration lying on VE are P, Q, the positions of which are readily determined from their projections p, p' ; q, q'.

Observe that tangents in projection are tangents in development.

582. Should the relative positions of the cones be such as to render it difficult to obtain the trace of one on the plane of the base of the other we may take a new plane which will cut both cones in convenient sections. Alternatively, the use of conics may be avoided altogether by an expedient illustrated in Fig. 545, where a cone of revolution axis vn, $v'n'$ intersects a cone the axis AC of which is vertical.

The axis VN is $\parallel$ the VP on xy,* hence any plane V'FH $\perp^r$ to the axis will be $\perp^r$ to VP. The line of vertices AV penetrates HP in h, and V'FH in $t't$.

If any plane contain AV and cuts both cones, the HT of such plane will pass through h and cut the circular trace of the vertical cone. Similarly, the trace of the section plane on V'FH will pass through T and cut the circular trace of the inclined cone in that plane. If the trace of the cone, and the point T be rabatted into HP (or VP), the points in which the cone trace is cut by the section planes will be located as easily as are the corresponding points in HP.

* It is always possible to obtain projections of the two cones on planes respectively $\perp^r$ to the axis of one cone, and $\parallel$ the axes of both.

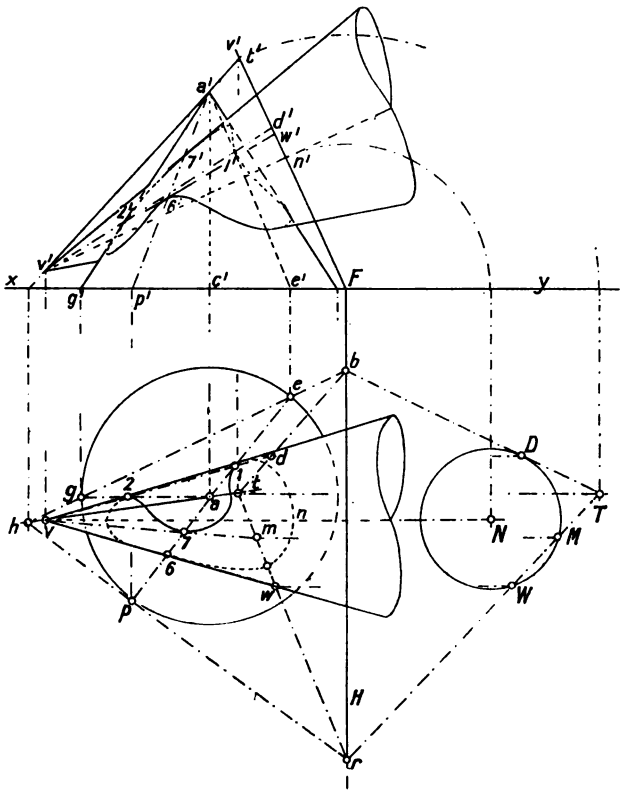


FIG. 545.

Thus, draw T*b* tangent to the circle centre N at D and join *bh*. Then clearly the plane *hb*T is tangent to the inclined cone along the generator VD, and cuts the other cone in lines AG, AE. Fold D back into the plane V'FH at *d'd* and join *v'd'*, *vd*. Then points 1 and 2 are in the required interpenetration. Similarly the section plane which has *hr* for HT intersects V'FH in *r*T. Rabatting

points W, M and drawing projections of the lines VW, VM, AP we obtain points 6, 7.

In a similar manner additional points in the interpenetration may be located. Observe that the generators AG, AE are tangents to the interpenetration, as also are the generators VW, VM.

Exercise. 2 equal cones of revolution axis 3 in. diameter of base $2\frac{1}{2}$ in. stand on HP; their axes are 2 in. apart. Determine their intersection.

NOTE.—The intersection is a hyperbola.

583. **Problem** (Fig. 546). *Determine the interpenetration of a cylinder and sphere.*

In the figure, the axis of the cylinder is parallel to a plane of projection VP. Should the cylinder be inclined to both HP and VP

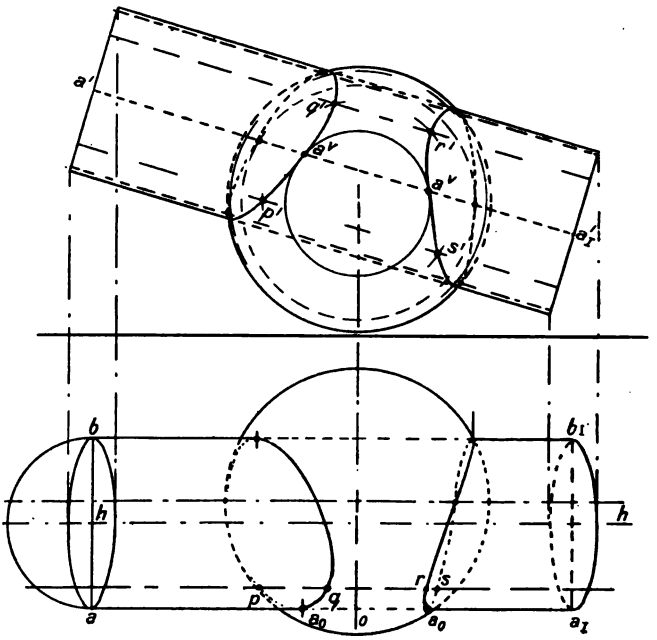


FIG. 546.

make a projection of the surfaces on a plane parallel to the axis of the cylinder, then proceed as in this example.

Take section planes vertical, and parallel to the axis of the cylinder. The plane tangent to the cylinder along aa_1 cuts the sphere in a circle, the elevation of which intersects $a'a'_1$ in points a^v, a^v. Tangents to the circle of section at these points are also tangents to the elevation of the interpenetration, and the plan $a \,.\, a_1$ is tangent to the plan of the interpenetration $a_0 a_0$.

Similarly, the plane tangent to the cylinder along bb_1 cuts the sphere in a circle the elevation of which intersects $a'a'_1$ in points b^v, b^v. The interpenetration and circle of section have common tangents in elevation, and the plan bb_1 is tangent to the interpenetration. The plane through the centre of the sphere determines the points in which the elevation of the interpenetration and sphere coincide.

The plane containing the axis of the cylinder determines the highest and lowest points in the interpenetration.

The foregoing are the more important points in the interpenetration. Intermediate points, as necessary, are found by taking other planes. The plane on hh, for example, determines points p', q', r', s'; p, q, r, s.

Exercises.

(1) A sphere of $2\frac{1}{8}$ in. diameter is pierced by a circular cylinder diameter $1\frac{1}{2}$ in. Show the interpenetration and development of same, if, (*a*) the axis of the cylinder passes through the centre of the sphere; (*b*) the axis of the cylinder is $\frac{1}{2}$ in. from the centre of the sphere.

(2) For the cylinder in Exercise 1, substitute a circular cone, axis 3 in., diameter of base $1\frac{1}{2}$ in.

(3) For the cylinder in Exercise 1, substitute (1) a square pyramid, (2) a triangular pyramid, (3) a hexagonal pyramid, making in each case the axis of the pyramid 3 in., and edge of base $1\frac{1}{4}$ in.

(4) Select any example of a chimney stack passing through a sloping roof, and show the trace of the chimney on a development of the roof surfaces.

(5) Determine the interpenetration of a cone and cylinder.

CHAPTER XXXIII.

SHADOWS.

584. If an opaque object be placed in the path of luminous rays, the surface on which the rays impinge will be illuminated, whilst the remainder will be in shade. The line which separates the illuminated from the shaded parts of the surface is called the **line of shade.**

585. If an opaque object be placed between a surface and the source of light, the object will intercept some of the rays and cast a shadow on the surface. The boundary of this shadow is called the **line of shadow.**

586. If the light emanates from a point, as, for example, a small arc lamp, the line of shade on a surface is a director curve (or polygon) of a cone (or pyramid) which has the luminous point for vertex; and the line of shadow is the trace of this cone (or pyramid) on the shadow surface. Rays of light emanating from the sun are practically parallel, and the enveloping tangent cone and pyramid of the last case become respectively a cylinder and prism.

587. The geometrical problem of determining the shadow of an object on a plane or other surface is thus, that of finding, in the first place, the line of contact of the circumscribing cylinder (or prism) or cone (or pyramid), according as the source of light is remote or near. Then finding the line in which the tangent cylinder, or cone, intersects the plane or other surface.

In many problems we require the trace of a small part only of the complete tangent surface, and this is most easily determined by taking lines through a few important points. This is exemplified in some of the examples which follow.

Shadows on Planes.

Ex. 1. Fig. 547. *Determine the shadow cast by the given prism on the* HP *when the rays of light are parallel to r, r'.*

Shade lines on solids the surfaces of which consist of plane faces always coincide with some edges of the solids.

Take lines parallel to R through the top corners of the prism and determine their HT's at A, B, C, D. Join *c*C, *a*A, AB. Then *a*ABC*c* is the shadow cast on HP. The top face and the sides under AD, DC are illuminated, whilst the faces AB, BC are in shade.

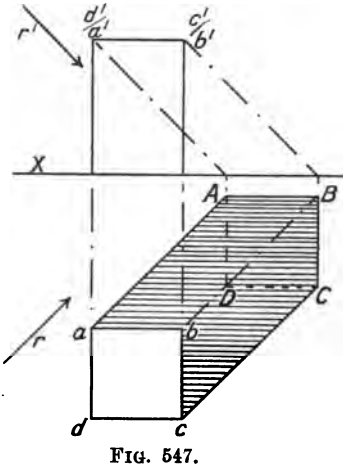


FIG. 547.

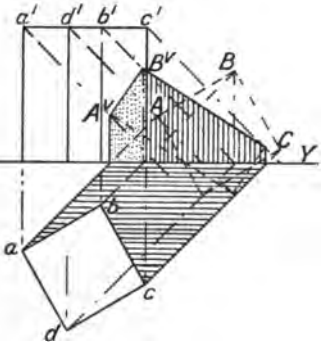


FIG. 548.

Ex. 2. In Fig. 548 a square prism is placed so that the VP intercepts some of the rays. In this case the shadow on the HP is determined in precisely the same manner as though the VP were not in position. Then the shadow on the VP is determined by finding the points in which the rays through A, B, C penetrate that plane. Only that part of the shadow which comes in front of VP and above HP are shown. The shadow lines on the two planes always intersect in XY. Advantage should be taken of this fact to shorten the labour. The elevation of the prism and its shadow on VP overlap, as shown by the dotted shading.

Ex. 3. Fig. 549. *Determine the line of shade on a cone and the shadow of the cone on the planes of projection, rays of light being parallel to the direction of the arrow.*

The two planes that can be drawn tangent to the cone and parallel to the given arrow will determine by their contact with the cone the line of shade on the surface, and by their traces on the HP and the VP the shadow cast by the cone on HP and VP respectively. Such planes must contain a

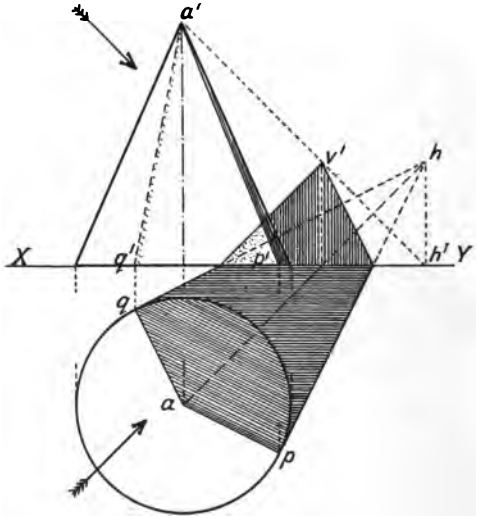


FIG. 549.

line through A parallel to the arrow. Hence, draw *ah*, *a'h'* respectively parallel to the plan and elevation of the light rays and penetrating HP in *h*. Draw *hp*, *hq* tangents to the circular base of the cone. Join *aq*, *ap*. Then *ap*, *a'p'*, *aq*, *a'q'* are shade lines on the cone and *hqp* is the shadow cast on HP. To determine the shadow on VP find *v'*, the vertical trace of the line *ah*, *a'h'*, and join *v'* to the points where *hq*, *hp* cross XY.

Ex. 4. Fig. 550. *Determine the shade line and the shadow of a cylinder on the planes of projection, rays of light being parallel to the given arrow.*

If vertical planes parallel to the arrow be placed against the cylinder, such planes will touch the cylinder along vertical generators shown in plane at *a*, *b*. The top of the cylinder and all the surface between *a* and *b* on the light side will be illuminated, whilst the remainder will be in

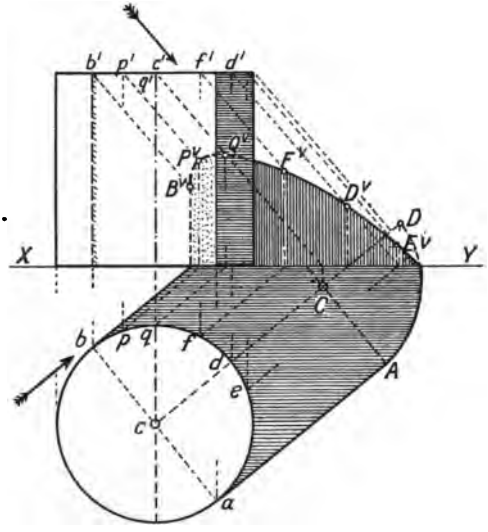


FIG. 550.

shade. The shade line is thus the arc *ab* and the vertica generators under *a* and *b*. The shadow of the arc *ab* on HP will be a semicircle equal and parallel to *ab*, its centre *c* being the HT of a line through *c* drawn parallel to the light rays. In the diagram AD is one half of the semi-circular shadow cast by *ab*. The shadow of the arc *ab* on VP is elliptical. Points as E^v, D^v, F^v, etc., in this ellipse may be found by finding the vertical traces of lines drawn

parallel to the arrow through a series of points in the top edge of the cylinder. Observe that a vertical tangent to the elliptical shadow on VP meets the HT of the tangent plane at b in XY.

Ex. 5. *In Fig.* 551 *the cylinder rests on* HP *with its axis* AB *inclined to* VP.

To determine the lines of shade on the cylinder take $X_1 Y_1$ perpendicular to the plan of the axis, and get a new elevation

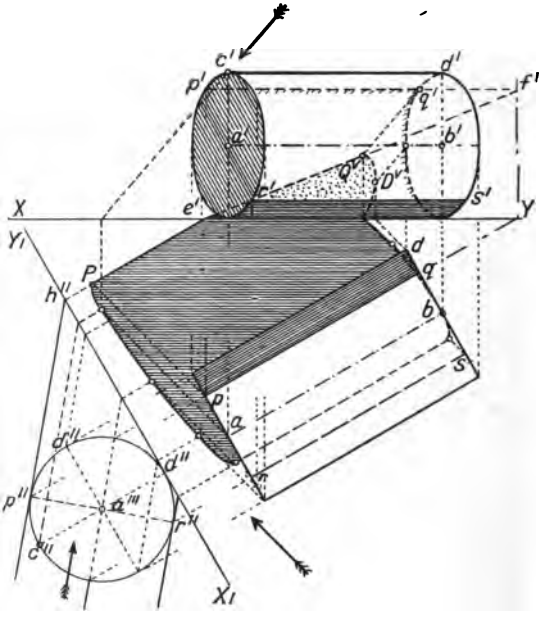


FIG. 551.

of the cylinder and arrow on this plane. Draw parallels to the arrow tangent to the circular elevation of the cylinder at p'', r''. The tangents at p'', r'' are profile views of planes tangent to the cylinder, and the points p'', r'' are end views of the generators which mark the lines of shade on the

surface. One end of the cylinder is in shade and one illuminated.

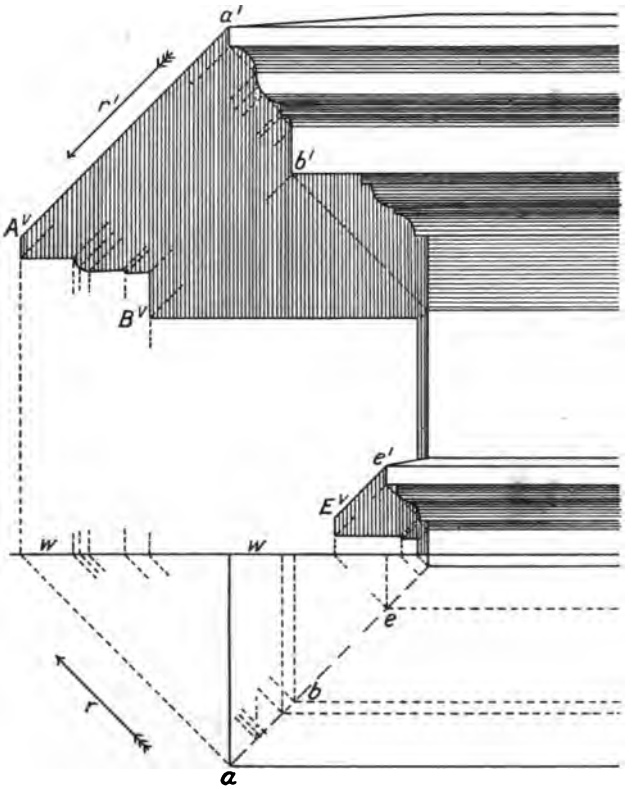


FIG. 552.

The shadow on HP lies between the horizontal traces of the planes which touch the cylinder along pq, $p'q'$ and rs, $r's'$. The circular ends cast elliptical shadows, which may be determined as in the preceding example.

Observe that there is some shadow on the VP behind the elevation of the cylinder. This is indicated by the dotted shading. The line $e'f'$ is the VT of the tangent plane touching the cylinder along PQ.

The shadow is determined as previously explained.

Ex. 6. Fig. 552 shows the shadow cast by the end of a moulded cornice on a vertical wall, the face of which is WW,

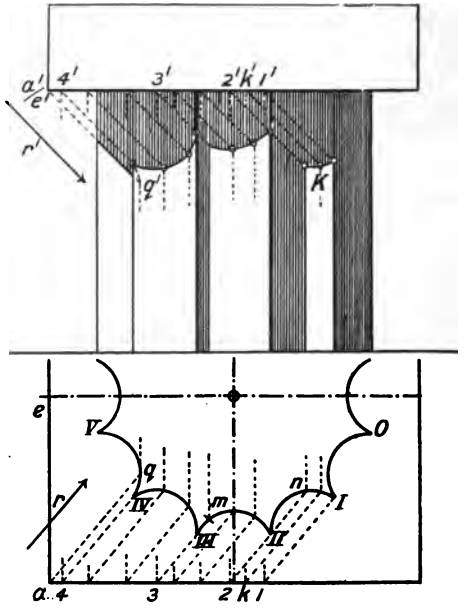


FIG. 553.

rays of light being parallel to r, r'. WW is plan of the face of the wall. The drawing is self-explanatory.

Ex. 7. Fig. 553. *Determine the shadow cast by the square abacus on the fluted column, rays of light being parallel to* r, r'.

The surface of the column consists of a number of concave cylindrical surfaces, rectilinear elements of which are vertical.

We will first consider the shadows which the column receives from its own elements.

If lines be drawn parallel to the plan of the ray through points I, II, III, IV, in which two concavities intersect, these lines will be horizontal traces of vertical planes parallel to the light rays.

We see from the plan that the plane on I1 does not intersect the surface of the column again; hence the vertical arris on I is a line of shade on the column, and all the surface OI is in shade. The plane on II2 intersects the plan again in *n*; hence that part of the surface I II which lies between verticals on II and *n* is in shade, whilst the remainder of that surface is illuminated.

Similarly the plane III3 intersects the surface II III in *m*, and that part of the surface II III which lies between III and *m* is in shade. All the surface between III and V is illuminated.

Next consider the shadow of the abacus. If through the elevation 1′ of the point plan 1 a line be drawn parallel to the elevation of the ray, this line will intersect the vertical from I in a point, and all on the vertical from I above this point will be shadowed. Similarly, a parallel to *r′* from 2′ meets the verticals from II and *n* each in a point, and all above these points will be shadowed.

To determine a point between *n* and I, draw plan and elevation of a ray through a point *kk′* and find its trace K on the surface.

A fair curve through the three points is the shadow of the abacus on the surface I II.

In a similar manner the whole shadow may be determined.

The shadow on each concave surface is part of an ellipse, since it is the line in which a plane, containing the bottom edge of the abacus, cuts the surface.

All of the shadow to the left of the ray *aq*, *a′q′* is due to the edge *ae* and appears in elevation as a straight line, since the plane of rays through points in *ae* is seen in profile.

Ex. 8. Fig. 554. ***Determine the shade line on a sphere and the shadows cast by the sphere on the planes of projection. Rays of light are parallel to the given arrow.***

In all problems on shadows in which a sphere is involved, take a new ground line XY, parallel to the plan of the light ray, and obtain a new elevation of the sphere and ray on this plane.

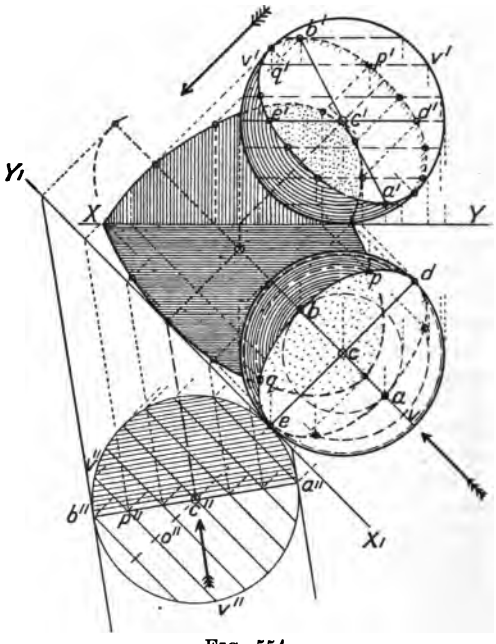


FIG. 554.

The tangent surface to the sphere is a right cylinder which touches the sphere in the meridian circle $b''a''$. This circle of contact is the line of shade on the surface.

The plan of the line of shade is an ellipse having major axis ed and for minor axis ab, determined by projectors from $a''b''$.

The elevations of these axes on the original VP are $e'd'$, $a'b'$, which are conjugate diameters of the elliptical elevation of the line of shade. Having these elements the ellipse may be drawn as in Prob. 247. In the diagram points in the plan and elevation of the shade line have been determined as follows:—

Suppose a number of horizontal section planes taken through the sphere. The sections by these planes are circles, and plans of the sections are also circles. Their elevations on XY are horizontal lines.

As an illustration, consider the plane through $v''v''$ which intersects $b''a''$ in p''. The plan of this section is a circle of diameter $v''v''$ centred in c.

This circle is cut by a projector from p'' ⊥r to XY, in points q, p, which are points in the plan of the shade line. The elevation of the circular section on the original VP is $v'v'$.

Projectors from the plans pq on to $v'v'$ determine points $v'q'$ in the elevation of the shade line.

In this manner any number of points in the plan and elevation of the shade line may be found.

The shadows cast by the sphere on HP and VP are the elliptical traces of the enveloping tangent cylinder on these planes.

These ellipses may be constructed by the methods of plane geometry, or points may be located as desired by finding the traces of lines taken parallel to the ray through points in the shade line.

The latter method has been employed in the diagram.

Exercises.

(1) A cylinder lies on HP with axis horizontal, and a circular cone stands on HP and near to the cylinder. Show plan and elevation (on any VP) of the shadow cast on the cylinder by the cone.

(2) A square rod 6 ft. long and 1 in. cross section stands vertical and 9 ins. away from the centre of a 3-in. sphere which rests on HP. Determine plan and elevation (on any VP) of the shadow cast on the sphere by the rod.

Ex. 9. *In Fig.* 555 *is given the plan and elevation of a cylindrical niche in a vertical wall terminated by a spherical hood and an elevation of the shadow cast on the niche by its own contour, rays of light being parallel to the given arrow.*

First obtain, on a VP parallel to the light ray, a new elevation of the ray and the sphere which forms the surface of the hood. Draw ab, $a'b'$ parallel to the plan and elevation of the ray. This penetrates the cylindrical surface of the niche in b, b', which marks the lowest point in the shadow cast by the front edge of the hood, and clearly all that part of the cylinder lying between the elements aa' and bb' is in shade.

The point bb' is a critical point in the shadow line; a second critical point is ff', which marks the shade line on the front edge of the hood. The point F is readily found thus: Draw $e''f''$ parallel to the light ray and tangent to the new elevation of the sphere at f''. Project f'' on to the plan at f, and from f project on to the original elevation at f'.

(The student will observe that tangents to the sphere parallel to the light ray form a cylinder which has circle contact with the sphere, shown in part elevation and plan by $c'f''$ and the ellipse nfm.)

Other points in the required boundary line may be found as follows: Let os, parallel to the plan of the arrow, meet the wall face in r and cylinder in s. Then os is radius of a circular section of the sphere by a VP on os. Project r on to the elevation of this section at r''. Through r'' draw a parallel to the elevation of the arrow intersecting the circle again in h''. Then h'' reveals the height at which a ray through r, r'' penetrates the sphere. This height is transferred to the original elevation at h', and h' is the elevation of a point in the shadow line. In a similar manner, any number of points in the line may be found.

Exercises.

(1) Determine the shadow cast on its own surface by the upper edge of a hemispherical basin which rests on a level table. Show also the shadow cast on the table.

(2) Place a small cylinder with axis vertical in the basin of Exercise (1), and determine the shadow of the cylinder on the inside of the basin.

(3) A cylinder stands on its end on the HP and is surmounted by a cone of half the height of the cylinder. Rays of light inclined at 30° to HP cast a shadow of cylinder and cone on a plane inclined at 30° to HP. Determine the shadow on the plane.

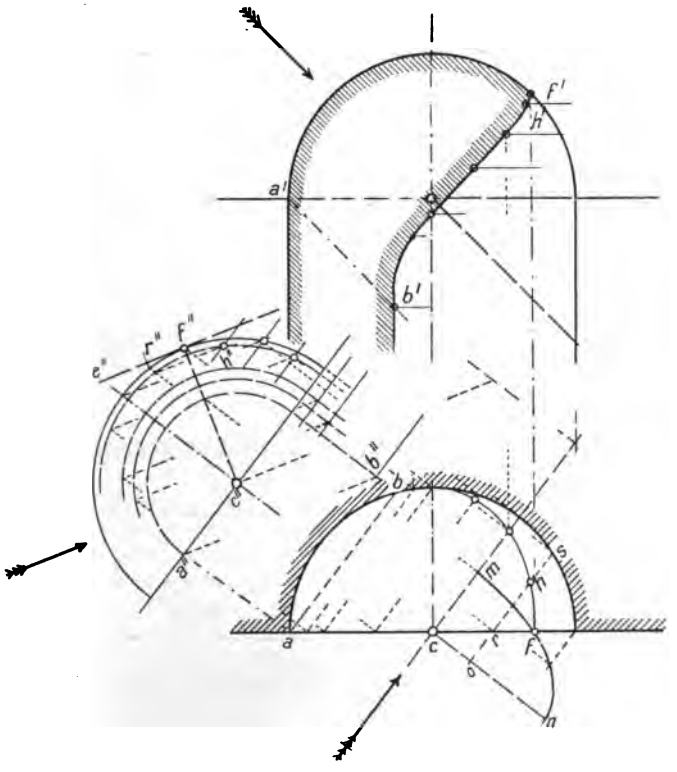


FIG. 555.

Ex. 10. In Fig. 556 is given an elevation of the shadow cast by a "flying buttress" on a sloping roof and a vertical wall. The outline of the shadow is determined by taking

lines parallel to the light ray r, r' through various points in the contour of the buttress and finding where such lines penetrate the wall and roof surface. By the use of a side elevation of the buttress the points in which the rays penetrate the wall and roof surface are easily located. The shadows cast by the curved edges of the arch are elliptical arcs on both roof and wall.

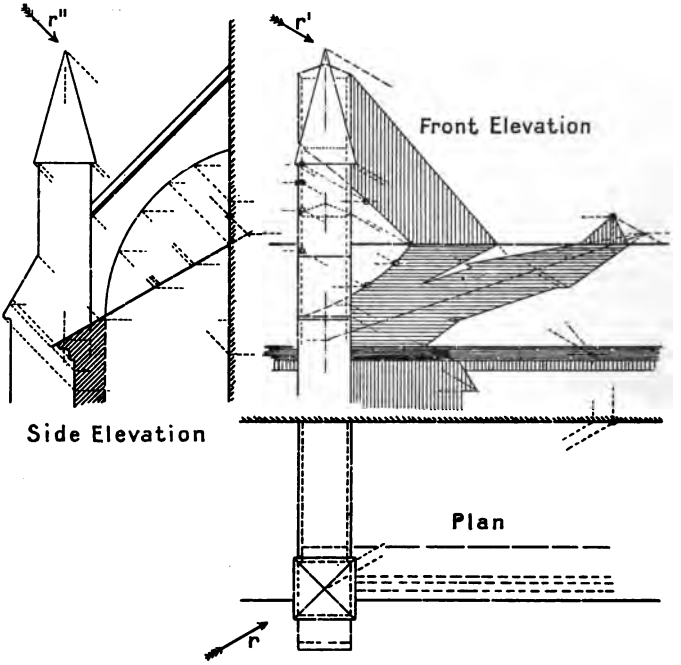


FIG. 556.

Exercise.

Determine the shadow cast by the "flying buttress" of Fig. 556, upon the roof and wall, if the rays are inclined at 30° to the horizontal and 40° to the wall.

CHAPTER XXXIV.

CONTOURS, TRIHEDRAL ANGLES, AND PICTORIAL PROJECTION.

588. **A Contour Line** is a line drawn on a surface through points all at the same level. Thus a contour line is, in all cases, the line in which a horizontal plane intersects some surface. If the surface is a plane, the contour will be a straight line; if the surface is that of a polyhedron, the contour line will be a polygon; if the surface is irregular and curved, the contour line will be an irregular curve.

589. Assuming a series of horizontal planes at known levels and finding the traces of these planes on a given surface, we have a series of contour lines on the surface, and the *plans* of these lines represent the surface without ambiguity. Fig. 557 shows a series of contour lines traced on a hillside at levels of 5-ft. intervals as indicated by the numbers 15, 20 . . 55. In Art. 497, it is shown that the distance between the plans of level ines in a *plane* depends on the *pitch* or *slope* of the plane, and the steeper the pitch the nearer together are the plans. The same holds for any surface whatever, consequently

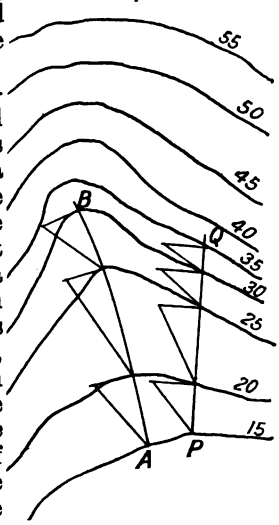


FIG. 557.

the nearer together the contour lines appear on the plan the steeper the slope of the ground. By inspection, we see that if AB and PQ are two paths on the hillside,

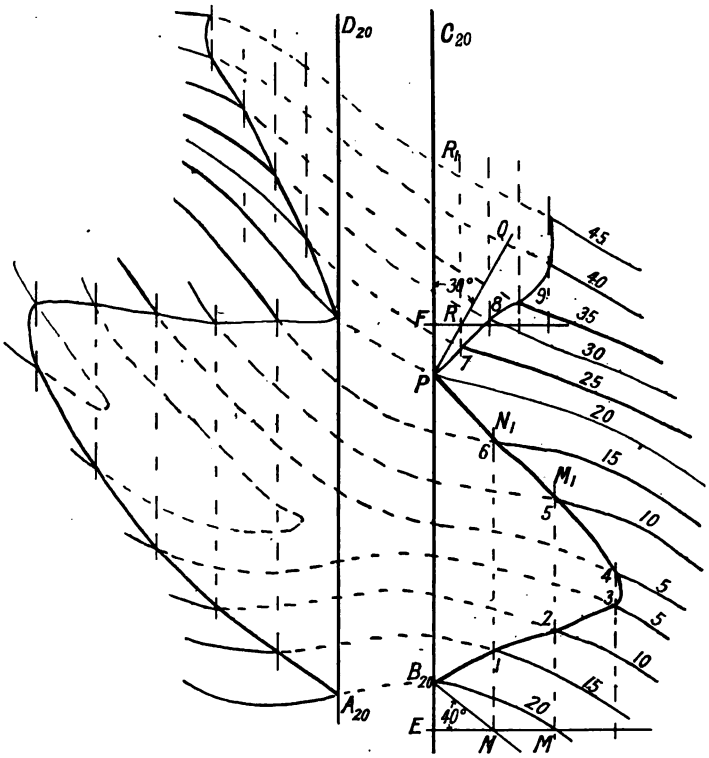


FIG. 558.

the gradient of the former is much less than that of the latter. If we imagine the slope of either path to be uniform between consecutive levels, the gradient of any segment is found by constructing a right-angled triangle

on the plan; making the altitude of the triangle equal to the difference in level, viz., 5 ft. to scale. In the Fig., the gradient of each portion of the two paths is shown in this manner.

Contour lines are extensively used on maps to show the form of the surface of the country. They may also be used to advantage in the making of plans of earthworks, as illustrated in the following examples.

Ex. 1. *Fig.* 558 *shows at* ABCD *plan of a level railway across a valley. The surface of the ground is indicated by contour lines indexed in feet. If the sides of the road embankment and cutting are plane surfaces, and slope, the former at an angle of* 40°, *and the latter at an angle of* 60° *to the horizontal, trace the plan of the railway on the surface of the ground. Scale of plan* 1 *in.* = 10 *ft.*

To Determine the Bottom Edge of the Bank. Draw BN making $90°-40°$ with CB, make BE = ½ in., *i.e.*, 5 ft., the distance between consecutive contour levels. Draw EN ⊥ᵣ BE meeting BN in N. Then $NN_1 \parallel BC$ is a level line on the bank surface at level 15. This line intersects the contours indexed 15 in points 1, 6, which lie in the base line of the bank. Similarly $MM_1 \parallel BC$ and distant from $NN_1 = EN$ is plan of a line at level 10 and determines by its intersections with contours 10 other points 2, 5 in the base line. In a similar manner, points 3, 4 are determined. A fair line passing through points B, 1, 2, 3, 4, 5, 6, P is the plan of the bottom edge of the bank.

To Determine the Top Edge of the Cutting. Draw PQ making $90°-60°$ with CP. Make PF = ½ in., *i.e.*, 5 ft., the distance between consecutive contour levels. Draw FR ⊥ᵣ PC meeting PQ in R. Then $RR_1 \parallel CP$ is a level line in the surface of the cutting at level 25, and determines by its intersection with contour 25 a point 7 in the edge of the cutting. Other parallels to BC at successive equal distances FR determine other points 8, 9, etc., in the plan of the edge of the cutting. The plan of the bank and cutting on the other side of the roadway is found in a similar manner.

Ex. 2. *In Fig.* 559, *a piece of ground contoured at* 5*-ft. levels is given. A plane surface abcd is to be formed partly by*

cutting and partly by embankment. The side slopes are 45° *in cutting and embankment. Draw the plan of the completed earthwork. Scale* 1 *in.* = 20 *ft.* (B. E., 1888.)

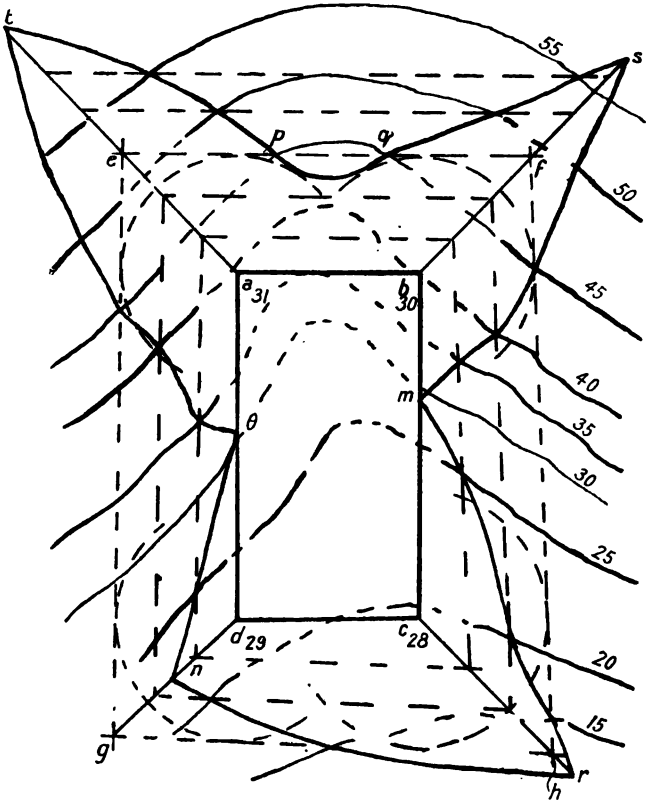


FIG. 559.

The contour lines are indexed 15 . . 55. The corners of the plane surface *a, b, c, d* are indexed 31, 30, 28, 29. We must first determine the direction of level lines in each

slope. Consider the sketch Fig. 560. Let ABCD represent the plane surface, and let HP be a horizontal plane taken through the lowest corner C. Imagine the plane supported on circular cones, having base angles of 45°, placed under the corners DAB and the corner C in contact with HP. Let a line x from C in HP be tangent to the base of the cone having apex at D. Then x is a line at level C in the plane of DCx. Let a line y be tangent to the circular bases of cones which support corners D and A. Then y is a line at level C in the plane of ADy. Similarly line m drawn from C tangent to base of cone B, and n tangent to the bases of

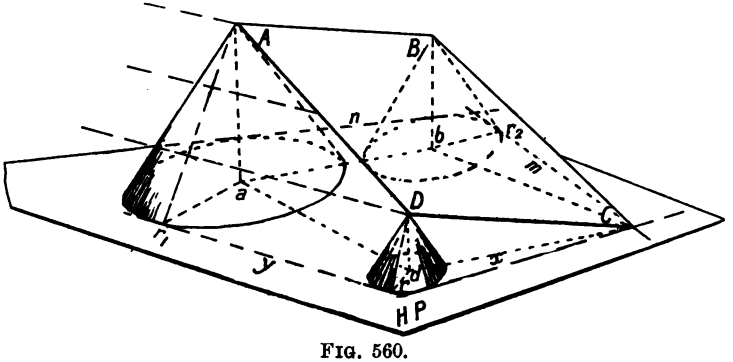


FIG. 560.

cones A and B are lines at level C in the slopes BCm, and ABn respectively. All level lines and the plans of such in the slope DCx will be parallel to x, and corresponding lines in the slopes ADy, BCm, ABn will be parallel to y, m and n respectively. Now, as the base angle is the same in each case the radii of the bases are in the ratio of the axes. Hence $ar_1 = 3\ dr$, and $br_2 = 2\ dr$, since aA $= 3\ d$D and bB $= 2\ d$D.

From the foregoing, we see that the plan of a level line at any height h in a slope as ab is found by describing circles from centres a, b having radii proportional to $h \pm$ index a, and $h \pm$ index b.

In our problem, the slope of each surface is 45°, *i.e.*, one horizontal to one vertical. Let us find a line at level 45 ft.,

say, in each slope. The corners *a* and *b* are indexed 31 and 30, hence *a* is 14 ft. and *b* is 15 ft. below level 45. With centre *a* and radius 14 ft. (to scale) describe a circle; with centre *b* and radius 15 ft. describe a circle. The line *ef* tangent to these two circles is a line of level 45 in the slope of surface *ab*, and determines by its intersections with contour 45 two points *pq* in the trace of the slope on the hillside. Other points are found by drawing parallels to *ef* at 5-ft. intervals corresponding to the contour lines on the ground.

Again, the corners *c* and *d* are indexed 28 and 29, hence *c* is 17 ft. and *d* is 16 ft. below the 45-ft. level. Make *c* and *d* centres of circles of radii respectively 17 ft, and 16 ft. (to scale). Then a line *gh* tangent to these two circles gives the direction of level lines in the slope *de*. Similarly *ge* tangent to circles from centres *a* and *d* gives direction of level lines in slope *ad*, and *fh* tangent to circles from centres *b*, *c* gives direction of level lines in the slope *bc*. The trace of the earthwork on the hillside is shown at *onrmsqpt*. The portion *onrm* is an embankment, whereas *msto* is cut in the hillside.

590. **Trihedral Angles.** Fig. 561. If through a point O three straight lines OA, OB, OC be drawn—not all in one plane—these lines form the edges of a solid angle formed by the three planes AOB, AOC, BOC. This solid angle is called a *trihedral angle*.

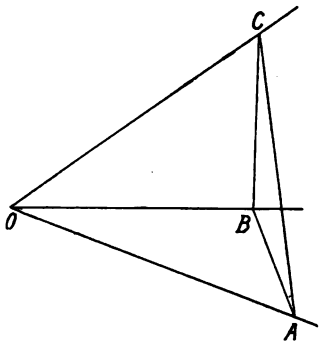


FIG. 561.

In a trihedral angle there are six elements, viz., the three dihedral angles formed by the intersection of the plane faces at the edges of the trihedral, and the three plane angles at O one in each face between the edges. These latter are called **face angles** of the trihedral.

If two of the faces of a trihedral be perpendicular to each other, the trihedral is called a **right trihedral.**

It is usual, and convenient, to denote the dihedral angle at the edge OA by a capital letter A, and the face angle opposite to that dihedral by a small letter a; similarly the dihedral angles at the edges OB, OC are lettered respectively B and C, and the face angles opposite to them by b and c. If any three of these elements be known, it is a determinate problem to find the other three, and make projections of the trihedral except in the case of a right trihedral where there are only five variables, since one dihedral is always 90°, and *two* of these elements are necessary and sufficient to render the problem determinate.

591. **The Right Trihedral.** If a and b denote the perpendicular faces of the right trihedral, and c the oblique face, the fixed dihedral will be represented by C, and those opposite to a and b by the corresponding capitals A, B. The different cases that may arise are as follows :—

Given (1) a, b, (2) a, C, (3) a, A, (4) a, B, (5) c, A, (6) c, B, (7) B, A.

If we assume the faces b and a to be horizontal and vertical respectively, and their common edge OC as a ground line, different cases may be solved, as in Chap. XXVII.

Thus in Case 1 apply problem 487.

,,	,,	2	,,	,,	488	then	487 (b).
,,	,,	3	,,	,,	490	,,	487.
,,	,,	4	,,	,,	489	,,	487.
,,	,,	5 & 6	,,	,,	491	,,	487 (b).
,,	,,	7	,,	,,	492	,,	487 (a).

592. **Spherical triangles.** Fig. 562. Let O, the apex of a trihedral angle, be the centre of a sphere of radius OA. Let A, B, C be the points in which the edges of the trihedral angle pierce the spherical surface, then the figure formed by the arcs AB, AC, BC, which mark the intersection of the faces of the trihedral angle on the spherical surface, is called a *spherical triangle.*

The angle at B of a spherical triangle ABC is the angle between BE, BR, tangents to the circular arcs AB, CB at the point B. And since these tangents are ⊥r OB it follows that the angle EBR = the dihedral angle between the faces AOB, COB of the trihedral angle.

Thus if A°, B°, C° represent the angles of the spherical

triangle, and $a°$, $b°$, $c°$ represent the angles which the arcs subtend at the centre of the sphere, we have A°, B°, C° and $a°$, $b°$, $c°$, respectively, equal to the corresponding *dihedral angles* and *face angles* of the trihedral angle.

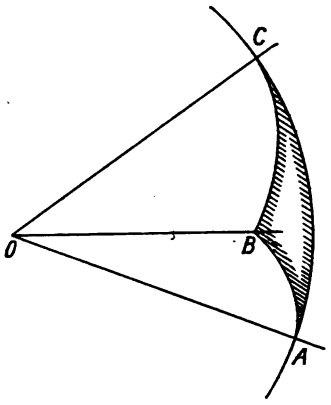


FIG. 562.

The Polar Triangle. If lines OA_1, OB_1, OC_1, be drawn through the centre of the sphere and respectively perpendicular to the faces OBC, OCA, OAB, and all directed *outwards*, or all *inwards*, meeting the spherical surface in A_1, B_1, C_1, we have a second trihedral angle with its corresponding spherical triangle $A_1°B_1°C_1°$. The triangle $A_1°B_1°C_1°$ is called *the polar triangle* of the triangle ABC. Reciprocally, the triangle A°B°C° is the polar of the triangle $A_1°B_1°C_1°$. It can be shown that the angles of a spherical triangle and the face angles of the trihedral angle corresponding to its polar triangle are supplementary. Thus if $A_1°B_1°C_1°$, $a_1°b_1°c_1°$, respectively denote the spherical and face angles of a polar triangle, we have

$$A_1° = 180° - a° \qquad a_1° = 180° - A°$$
$$B_1° = 180° - b° \qquad b_1° = 180° - B°$$
$$C_1° = 180° - c° \qquad c_1° = 180° - C°.$$

593. It will be clear from the foregoing that the solution of a problem on spherical triangles involves the solution of the corresponding trihedral angle, and conversely. As stated in Art. 590, any three of the six elements may be given, and the solution of the problem consists in finding the remaining three. By aid of the polar triangle, the number of possible cases is reduced by one-half. The following cases occur :—

Case 1. Fig. 563. *Given three face angles $a°$, $b°$, $c°$.*

Set out the three face angles in development at b_2OC, COA, AOb_1. With centre O and any radius OA describe a

circle intersecting the edges of the developed trihedral angle in b_2, C, A, b_1. Draw b_2b, b_1b, respectively $\perp^r$ OC and OA intersecting in b. Join bO which is plan of the third edge of the trihedral angle.

Dihedral angles A°, B°, C° are found as explained in 523. The plan of the circular arcs CB, AB of the spherical triangle A°B°C° are arcs of ellipses into which the great circles of the sphere of radius OA project. The plan of these arcs is given, but construction lines are omitted to avoid crowding.

Case 2. *Given the dihedral angles* A°, B°, C°.

Subtract each angle from 180° and we obtain corresponding face angles a_1°, b_1°, c_1° of the polar triangle. Then by the construction above we can determine the dihedral angles of the polar triangle. Subtracting each of these angles from 180° we obtain the required face angles a°, b°, c°.

Case 3. *Given two face angles and the included dihedral angle, say* a°, b°, C°.

Set out the given face angles in development and rabat the face Ob_2C into correct position, carrying the point b_2 to $b'b$. The other elements are found as in **Case 1.**

Case 4. *Given two dihedral angles and the adjacent face angle, say* A°, B°, c°.

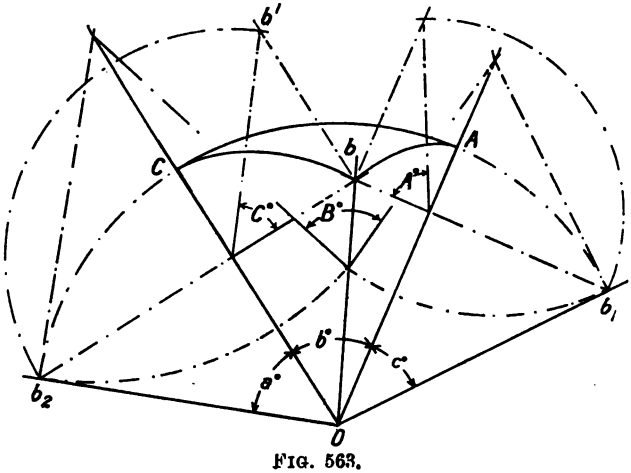


FIG. 563.

This problem may be solved directly by 538 (Special Case). Alternatively, the supplements of A°, B°, are the face angles of the polar triangle, and the supplement of $c°$ is the corresponding dihedral angle of the polar triangle.

Case 5. Fig 564. *Given two face angles and a dihedral angle opposite to one of them, say* $a°$, $b°$, A°.

From b_2 draw b_2E $\perp^r$ OC intersecting OC in f and OA (produced if necessary) in E. Draw fR $\perp^r$ OA and draw Rf', making the given angle B° with fR and meeting ff' drawn $\perp^r$ fR in f'. With centre f and radius fb_2 describe a circle and make fF on OC produced equal ff'. Join FE which intersects the circle centre f, radius fb_2 in points $b'b''$. Project these points on to b_2E at b, b_0. Then Ob or Ob_0 is the plan of the third edge of the trihedral angle. There are thus two solutions if EF cuts the circle of radius fb_2, one solution if EF is tangent, and no solution if the circle and line EF have no point in common.

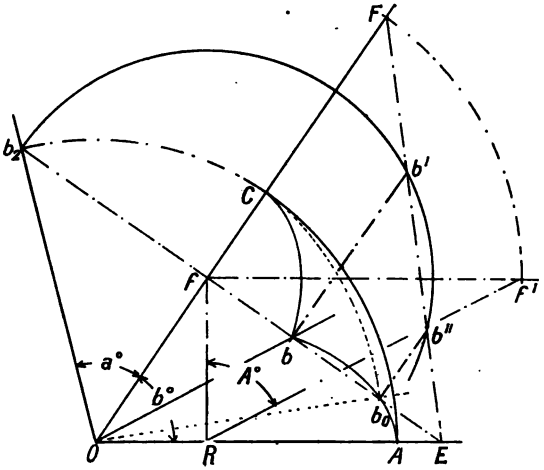


FIG. 564.

Case 6. *Given two dihedral angles and a face angle opposite to one of them, say* A°, B°, $a°$.

This may be solved by means of the polar triangle.

594. **Pictorial projection.** Fig. 565. Let OX, OZ ⊥r to each other, and OY inclined at equal angles to OX and OZ be the lines in which a horizontal plane x and vertical planes z and y intersect. Let A be a point in space, and lines drawn from A parallel respectively to OX, OZ, OY meet y in E, x in a, and z in F. Then lines from E and a respectively parallel to OZ, OX intersect in e on OY, and lines from F

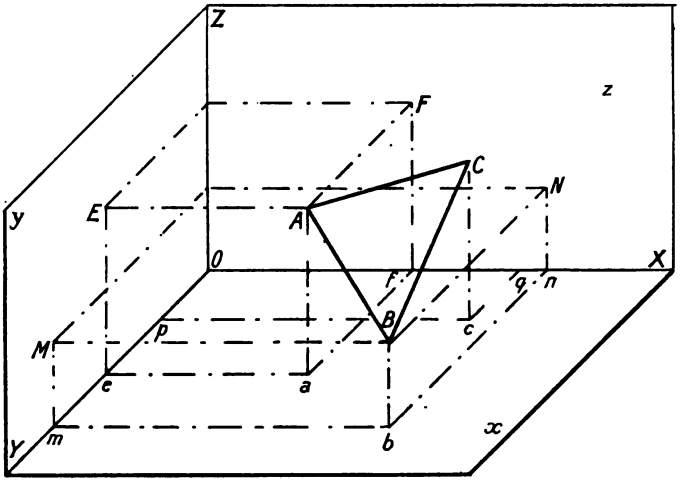


FIG. 565.

and a respectively parallel to OZ, OY intersect in f on OX. Clearly aA is the height of point A above plane x, and ae = AE, af = AF; hence the lines aA, ae, af give the position of point A relative to the fixed planes x, y, z. Similarly bB, bm, bn parallel to OZ, OX, OY give the position of a point B in space. Also, the position of a point C is given by cC, cp, cq.

It will be observed that the distances of the point A from planes y, z are not *equal* to ae, af, since the angle between

planes y, z is not a right angle, but the position of the point is nevertheless definitely given by the intercepts ae, af of lines parallel to OX, OY, and if we agree to measure the distance between points and the three planes x, y, z, along lines parallel to the intersections OZ, OX, OY, we can employ this method to represent an object of three dimensions by one view in which dimensions parallel to each of three mutually perpendicular planes can be measured. This method of representing objects is called *pictorial projection*. The intersection of the planes OX, OY, OZ are called the *axes of projection*, or simply the *axes*.

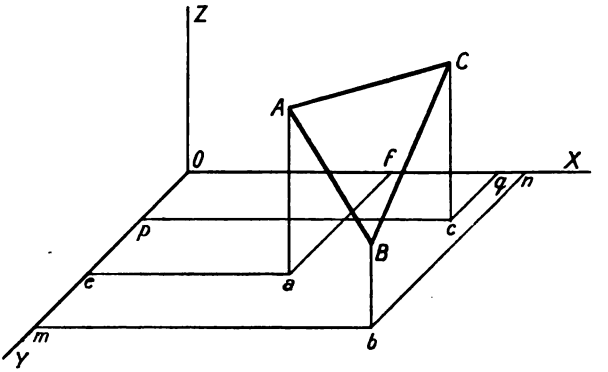


FIG. 566.

In Fig. 566, the triangle ABC and the three axes are shown without planes.

A pictorial projection is a distorted view, and in order to lessen the distortion, distances parallel to OY are usually made to *half* the scale used for distances parallel to OZ and OX.

From Fig. 566, it will be evident that, if given the position of each corner of the triangle ABC relative to each of three mutually perpendicular planes, a pictorial projection of the triangle is found thus: Draw OX and OZ $\perp$ OX; draw OY making 135° with OX (use a 45° set square).

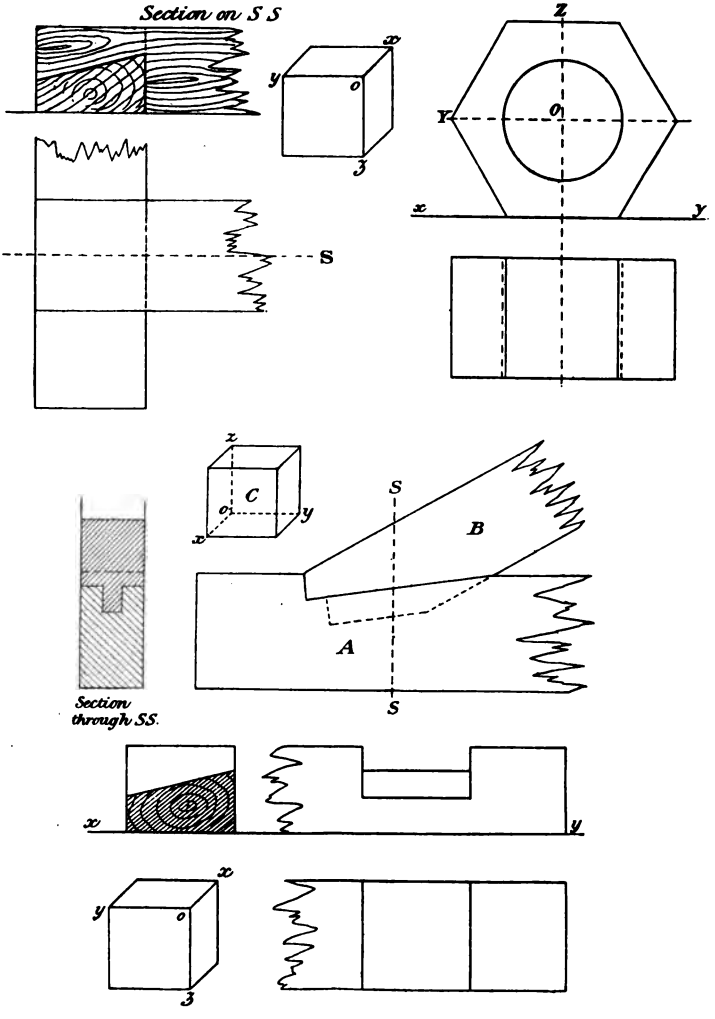
Heights are measured along OZ, distances from back to

front are measured along OY, and distances from left to right are measured along OX. Thus for the point A, make Of = the given distance of A to the right of a fixed plane YOZ (to any convenient scale). Make Oe = the distance of the point in front of plane XOZ measured to *half* the scale used for Of. Draw from f and e parallels to OY, OX meeting in a. Draw $aA \parallel OZ$ and equal to the given height of the point, using the same scale as for OF. Then A is the pictorial projection of the point A. Points B and C are located in a similar manner. Joining points we obtain the pictorial projection of the triangle.

To familiarise himself with the method, the student should work the following exercises. The labour is reduced often by aid of the fact that parallel lines project into parallels.

Exercises.

Make pictorial views of the objects shown by plans and elevations on page 606, taking axes parallel to the edges of the cubes and dimensions parallel to OX being measured to half the scale used for OZ and OY.


Section on S S
x
y
o
z
S
S
Y
o
x
y
z
C
o
y
x
S
B
A
S
Section through SS.
x
y
x
y
o
z

APPENDIX I.

The following Tables are those supplied to candidates at examinations conducted by the Board of Education.

EXAMINATION TABLES.

USEFUL CONSTANTS.

1 inch = 25·4 millimetres.
1 gallon = ·1604 cubic foot = 10 lb. of water at 62° F.
1 knot = 6080 feet per hour = 1 Nautical mile per hour.
Weight of 1 lb. in London = 445,000 dynes.
One pound avoirdupois = 7000 grains = 453·6 grammes.
1 cubic foot of water weighs 62·3 lb.
1 cubic foot of air at 0° C. and 1 atmosphere, weighs ·0807 lb.
1 cubic foot of hydrogen at 0° C. and 1 atmosphere, weighs ·00559 lb.
1 foot-pound = $1{\cdot}3562 \times 10^7$ ergs.
1 horse-power-hour = 33,000 × 60 foot-pounds.
1 electrical unit = 1,000 watt-hours.
Joule's equivalent to suit Regnault's H, is { 774 ft.-lb. = 1 Fah. unit. / 1,393 ft.-lb. = 1 Cent. unit.
1 horse-power = 33,000 foot-pounds per minute = 746 watts.
Volts × ampères = watts.
1 atmosphere = 14·7 lb. per square inch = 2,116 lb. per square foot = 760 mm. of mercury = 10^6 dynes per square centimetre nearly.
A column of water 2·3 ft. high corresponds to a pressure of 1 lb. per square inch.
Absolute temp., t = θ° C. + 273° or θ° F. + 460°.
Regnault's H = 606·5 + ·305 θ° C. = 1,082 + ·305 θ° F.
$p\, u^{1{\cdot}0646} = 479$.

$$\log_{10} p = 6{\cdot}1007 - \frac{B}{t} - \frac{C}{t^2},$$

where $\log_{10} B = 3{\cdot}1812$, $\log_{10} C = 5{\cdot}0882$.

p is in pounds per square inch, t is absolute temperature Centigrade, u is the volume in cubic feet per pound of steam.

$\pi = 3{\cdot}1416$.
One radian = 57·3 degrees.
To convert common into Napierian logarithms, multiply by 2·3026.
The base of the Napierian logarithms is $e = 2{\cdot}7183$.
The value of g at London = 32·182 ft. per second per second.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
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·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
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25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	Degrees.
								Angle.	

APPENDIX II.

BOARD OF EDUCATION.

PRACTICAL GEOMETRY AND GRAPHICS.

LOWER EXAMINATION.

*1. The given figure, bounded by concentric circular arcs and radial lines, is the development of one plate of a conical vessel built up of metal plates. Draw the figure to the given dimensions. Scale, 1 cm. to 1 inch. What are the radii of the arcs?

Note.—First draw the radial ends. Then find the middle points C, C of the arcs. Find additional points on the arcs by using tracing paper or otherwise. Through these points sketch in the curves freehand. (25)

*2. Masonry arch. Set out the arch ring to the scale of 1 cm. to 1 ft., working to the following dimensions:—

Span, 8 ft.; rise, 3 ft.; curve, a semi-ellipse. The points *a*, *a* . . . divide the curve into nine equal parts. The joints *ab*, *ab* . . . are normals to the ellipse, and each 1 ft. long. (30)

*3. The shaded trapezoid shows in section the winding of a portion of the secondary wire of an induction coil. Set out the section to the given dimensions. Determine the centre of area G of the section. Find the volume of the winding, this being equal to the area of the section multiplied by the circumference of the circle, radius OG. (30)

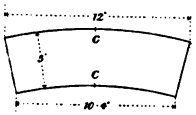
*4. The load W, acting on the given crane, causes the tie AC to lengthen by the amount 0·076″ and the jib AB to shorten 0·042″. Find the displacement of A and measure its vertical component.

What is the angular displacement of AC if its actual length is 100 inches? (25)

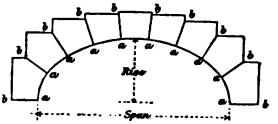
*5. In the given mechanism, the crank BC and lever AD are centred at A and B as shown, and are connected at their ends by the rod CD, which is extended to E. Set out the figure to the given dimensions, scale ½. Plot the curve traced by E during one complete rotation of BC. Show the angle through which AD oscillates. (30)

*6. A piece A of a certain machine has an oscillating motion. For successive positions of A the displacements y at the times t are noted, and a graph of the results is given.

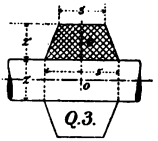
The shape of the curve indicates that the motion is nearly simple harmonic, and analysis gives as a first approximation the sine curve



Q. 1.



Q. 2.



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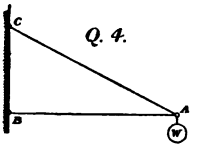

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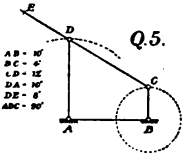

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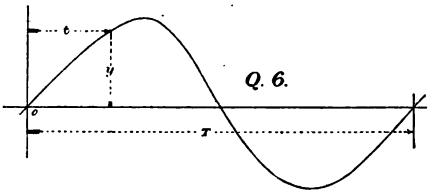

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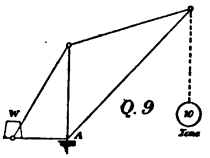

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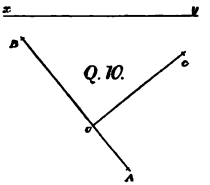

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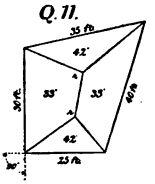

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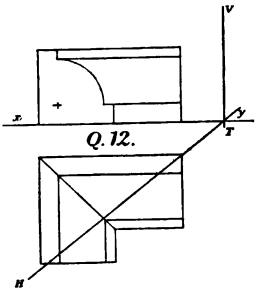

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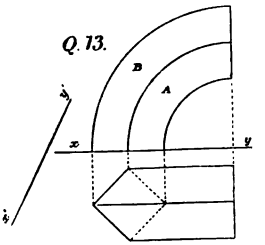

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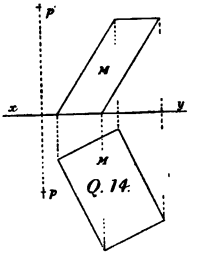

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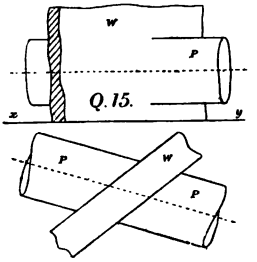

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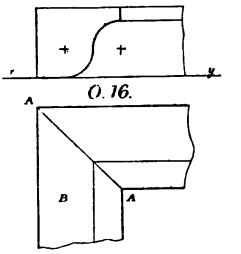

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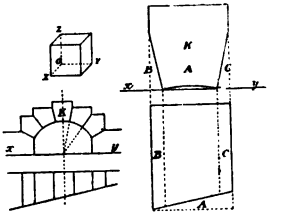

Qs 17 & 18.

$$y = a \sin \frac{t}{T} 360°$$

where the amplitude $a = 1{\cdot}29''$. The periodic time T is $\frac{1}{5}$ second.

Plot the sine curve on the same base as the given curve, and measure the greatest vertical deviation between the two curves. (25)

7. The position R of the centre of mass of a number of bodies situated in a plane is given by the vector equation

$$\mathrm{R} = \frac{m_1\mathrm{R}_1 + m_2\mathrm{R}_2 + m_3\mathrm{R}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

Find R for the following case of three masses :—

$$m_1 = 2{\cdot}5\ ;\ m_2 = 3{\cdot}5\ ;\ m_3 = 4.$$
$$\mathrm{R}_1 = 2_0°\ ;\ \mathrm{R}_2 = 3_{70}°\ ;\ \mathrm{R}_3 = 4_{180}°. \quad (30)$$

8. If a jet of water strikes a fixed vane and is deflected, the force F acting on the vane, being equal to the loss of momentum per second, is given by the vector equation

$$\mathrm{F} = \frac{m}{t}(\mathrm{V}_1 - \mathrm{V}_2).$$

Find F for the case in which

$\frac{m}{t} = 4$, the mass of water flowing per second ;

$\mathrm{V}_1 = 50_{15}°$ ft./sec., the velocity of the water as it enters :
$\mathrm{V}_2 = 7_{80}°$ ft./sec., " " " leaves. (30)

*9. In the given crane, the weight W just balances about A the load of 10 tons. Find W. Determine the forces in the five bars of the frame, writing the answers on the bars themselves. Use + for a pull and − for a thrust. (30)

*10. *A model in wire is made of three mutually perpendicular lines* OA, OB, OC. *A plan is given with the ends* A, B *of the model resting on the ground, the plane of* AOB *being vertical.*

Find and measure the true lengths of OA *and* OB *and the height of* O *above the ground. Draw an elevation of the model on* xy. (20)

*11. In the given roof plan the lengths of the horizontal eaves and the inclinations of the roof surfaces are written on the figure. Set out the plan of the eaves to the scale of 1 in. to 10 ft. Then determine the plan of the rafters and ridge, *i.e.*, of the lines in which the roof surfaces intersect. Figure the heights of the two ends of the ridge RR above the level of the eaves. (30)

*12. Determine the elevation and true shape of the section of the given moulding by the given vertical plane VTH. (25)

*13. The figure shows a quarter bend for a sheet metal square pipe. Draw an elevation of the pipe on $x'y'$. Set out patterns for making the bend, that is, develop the conical surfaces A and B. (30)

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